SOLVING FOURIER PHASE RETRIEVAL WITH A REFERENCE IMAGE AS A SEQUENCE OF LINEAR INVERSE PROBLEMS

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ABSTRACT

Fourier phase retrieval problem is equivalent to the recovery of a two-dimensional image from its autocorrelation measurements. This problem is generally nonlinear and nonconvex. Good initialization and prior information about the support or sparsity of the target image are often critical for a robust recovery. In this paper, we show that the presence of a known reference image can help us solve the nonlinear phase retrieval problem as a sequence of small linear inverse problems. Instead of recovering the entire image at once, our sequential method recovers a small number of rows or columns by solving a linear deconvolution problem at every step. Existing methods for the reference-based (holographic) phase retrieval either assume that the reference and target images are sufficiently separated so that the recovery problem is linear or recover the image via nonlinear optimization. In contrast, our proposed method does not require the separation condition. We performed an extensive set of simulations to demonstrate that our proposed method can successfully recover images from autocorrelation data under different settings of reference placement and noise.

Index Terms— Sequential recovery, linear inverse problem, reference-based phase retrieval, holographic phase retrieval.

1. INTRODUCTION

Fourier phase retrieval problem appears in diverse applications such as X-ray crystallography, diffraction imaging, ptychography, and astronomical imaging [1–6] as the only measurable quantity in these systems is the squared magnitude of the Fourier transform. The Fourier phase retrieval problem can be viewed as the recovery of an unknown signal or image from the amplitude of its Fourier transform, which is also equivalent to the recovery of a signal/image from its autocorrelation [2, 7].

Classical phase retrieval methods, such as error reduction and alternating minimization, exploit prior knowledge about the nonnegativity and support of the target image [4, 8]. Convex relaxation-based methods, such as PhaseLift [9, 10] and PhaseCut [11], lift the problem to a high-dimensional space. Other methods solve a convex problem for phase retrieval without lifting the problem into higher dimensions [12, 13]. Methods proposed in [14–18] directly solve the nonconvex optimization problem and require careful initialization to avoid local minima. Alternating minimization methods require prior knowledge about the target signal. The most popular signal domain constraints in 2D phase retrieval include positivity, support, and sparsity constraints [19–22]. In addition, adding a known reference signal has been shown to improve the Fourier phase retrieval performance [23–27].

Our work builds upon recent work on holographic phase retrieval [24,25,28], in which the presence of a known reference signal in the image makes the recovery problem tractable. In [24, 25], the Fourier phase retrieval problem becomes a linear problem if we place a reference image at large distance from the target image. In our previous work [28], we demonstrated that adding a reference image at arbitrary locations in the image can significantly improve the recovery performance even if the separation condition is not satisfied. The resulting Fourier phase retrieval problem remain nonlinear, which we solved using gradient descent and alternating minimization. In this paper, we propose sequential algorithm that recovers the target image by solving a sequence of (small) linear deconvolution problems.

In this paper, we propose to solve Fourier phase retrieval with side information as a sequence of deconvolution problems. We assume that some parts of the image are known as a reference that can be used to estimate the unknown parts. Given the location of known pixels, we can find which samples in the autocorrelation are linear measurements of the unknown parts. In general, we may not have enough linear measurements in the autocorrelation to compute the entire unknown part in one step. We show that, under certain mild condition, we can estimate a small fraction of the image by solving a linear deconvolution problem and use that estimate in subsequent steps to estimate other unknown parts of the image. We conducted several simulation experiments to evaluate the performance of our proposed method under different conditions of reference signal and noise.

Figure 1 illustrates our proposed method using a “pinhole” reference image. Figure 1 shows the autocorrelation of the image after adding a pinhole reference to the right side of the image in two scenarios. In the first scenario, as shown in Fig. 1 (a), the separation condition is satisfied and the Fourier phase retrieval problem turns into a linear inverse problem. The target image can be directly estimated from a selected region in the autocorrelation. Note that in this specific example, the linear measurement region of autocorrel-
lation (highlighted in a green box) is the target image itself. In a general setting, the linear measurements correspond to the autocorrelation of the target and reference images. In the second scenario, as shown in Fig. 1 (b), the pinhole reference is closer to the target image and the separation condition is not satisfied. Existing linear methods such as [24] cannot estimate the entire image because very few entries in the autocorrelation represent linear measurements of the target image (shown in a green box). In this situation, we can recover the target image either by solving the standard nonlinear phase retrieval problem or by using our proposed sequential method. In our proposed method, instead of estimating the entire unknown image in one step, we use the available linear measurements to estimate a small patch of the unknown image (e.g., two columns at a time). Then we remove the nonlinear contributions of the estimated patch from the correlation data, which provides us linear measurements of another patch of the unknown image. We repeat this procedure until we recover the entire unknown image.

2. FOURIER PHASE RETRIEVAL AS A SEQUENTIAL DECONVOLUTION PROBLEM

2.1. Problem formulation

In this paper, we focus on reconstructing an image from its autocorrelation measurements assuming that some parts of the image are known and can be used as a reference. In general, we can formulate the Fourier phase retrieval as the following nonlinear deconvolution problem:

\[
\min_{X} ||R - X \ast X||^2_2,
\]

where \( R \in \mathbb{R}^{(2p-1) \times (2q-1)} \) denotes the observed autocorrelation, \( X \in \mathbb{R}^{p \times q} \) denotes the unknown 2D image, and \( \ast \) represents the 2D cross-correlation operator.

The optimization problem in (1) is nonlinear and nonconvex in general. However, if some parts of the signal \( X \) are known, then some entries in the autocorrelation can be represented as linear measurements of the unknown parts [24]. We first formulate the problem as a blind deconvolution problem and then show how the known reference signal can convert it into a linear problem under some conditions. The autocorrelation of \( X \) can be written in a matrix form as

\[
vec(R) = C_X vec(X),
\]

where \( C_X \) is block Toeplitz matrix created using \( X \), \( vec(X) \) and \( vec(R) \) denote vectorized version of \( X \) and \( R \), respectively. Let us also assume that \( X \) has \( q \) columns that can be written as \( X = [x_1 \ x_2 \ x_3 \ldots \ x_q] \) and \( R = [r_1 \ r_2 \ r_3 \ldots \ r_{2q-1}] \), where \( x_i \) and \( r_i \) are the \( i \)th columns of \( X \) and \( R \), respectively. Note that due to the symmetry in the autocorrelation, half of the columns in the autocorrelation are redundant; therefore, we only keep the first half of the columns in the autocorrelation. The equation in (2) can be written as

\[
\begin{bmatrix}
1 & 2 & \cdots & \cdots & \cdots \\
C_{x_1} & 0 & \cdots & \cdots & \cdots \\
C_{x_2} & C_{x_1} & 0 & \cdots & \cdots \\
C_{x_3} & C_{x_2} & C_{x_1} & \cdots & \cdots \\
\vdots & \vdots & \vdots & \ddots & \ddots \\
C_{x_q} & C_{x_{q-1}} & C_{x_{q-2}} & \cdots & C_{x_1}
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
\vdots \\
x_q
\end{bmatrix}
\approx
\begin{bmatrix}
r_1 \\
r_2 \\
r_3 \\
\vdots \\
r_{2q-1}
\end{bmatrix},
\]

where \( C_{x_i} \) is a Toeplitz matrix created using column \( x_i \). For example, if \( z \) is an arbitrary column vector, we can write \( C_{x_i} z = x_i \ast z \). We can also write a similar equation based on the rows in the image and their autocorrelation. In this paper, we focus on formulating the Fourier phase retrieval problem as a sequential recovery of the columns of the unknown image. Based on (3), the problem is nonlinear. However, if some of the columns are known, we can linearize the problem in a sequential manner.

2.2. Sequential deconvolution method

In this section, we will discuss how we can solve the problem in (3) as a sequence of small (linear) deconvolution problems. If a small number of columns on left and/or right side of the image are known as a reference, then instead of solving a nonlinear phase retrieval problem to recover the unknown image, we can sequentially recover a few columns of the unknown image at every step.

2.2.1. Reference as known columns on both sides

Let us assume that the first and last columns of \( X \) (i.e., \( x_1 \) and \( x_q \)) are known apriori and we want to estimate other columns \( (x_2, \ldots, x_{q-1}) \) from \( R \). The known variables in (3) are highlighted in green. Although the overall system in (3) is jointly nonlinear for all the columns in \( X \), but if we know the green parts, we can recover \( x_2, x_{q-1} \). Subsequently, we can recover \( x_3, x_{q-2} \). In other words, we can develop a forward substitution method by which we can recover two columns using a linear system of equations at every step.

We discuss the steps to estimate all the columns of the image in a sequential manner below.

**Step 1:** The first row in (3) does not provide any new information as \( x_1, x_q \) are known. The second row in (3) gives us the following equation:

\[
r_2 = C_{x_1} x_{q-1} + C_{x_q} x_2 = \begin{bmatrix} C_{x_1} & C_{x_q} \end{bmatrix} \begin{bmatrix} x_{q-1} \\
x_2 \end{bmatrix},
\]

where we define the left correlation matrix as \( C_{x_i} \), \( z = x_i \ast z \).

We define the system matrix as \( H = \begin{bmatrix} C_{x_1} & C_{x_q} \end{bmatrix} \), This matrix needs to be well-conditioned, otherwise we will have large error in estimation of \( x_{q-1} \) and \( x_2 \). In this step, we recover \( x_{q-1} \) and \( x_2 \) by solving the linear inverse problem in (4). Let us denote the recovered columns in this step as \( \hat{x}_{q-1} \) and \( \hat{x}_2 \).

**Step 2:** In this step, we use the equations in the third row in (3) assuming that we know the estimated columns from the previous step shown as \( \hat{x}_{q-1} \) and \( \hat{x}_2 \). Based on the definition of the left correlation, we can write

\[
r_3 - \hat{C}_{x_{q-1}} \hat{x}_2 \approx \begin{bmatrix} C_{x_1} & C_{x_q} \end{bmatrix} \begin{bmatrix} x_{q-2} \\
x_3 \end{bmatrix}.
\]

By solving the linear inverse problem in (5), we estimate the two other columns of the image as \( x_{q-2} \) and \( \hat{x}_3 \).

**Step k:** The number of steps we need to recover the image depends on the width of the image. However, we generally estimate one column from the right and one column from the left side of the image. At any step \( k \), by solving the linear inverse problem in (6), we estimate two columns \( \hat{x}_{q-k} \) and \( \hat{x}_{k+1} \).

\[
r_{k+1} = \sum_{i=1}^{i=k-1} \hat{C}_{x_{q-k+i}} \hat{x}_{i+1} \approx \begin{bmatrix} C_{x_1} & C_{x_q} \end{bmatrix} \begin{bmatrix} x_{q-k} \\
x_{k+1} \end{bmatrix}.
\]

A pseudocode of the proposed method is presented in Algorithm 1. In our experiments, \( K \) is computed based on the total number of columns in the image (\( q \)) and the number of known columns. We observed that if a small number of columns from both sides of an image are known as a reference, the conditioning of the overall system in (3) improves, which results in stable recovery. SVD or
We performed a number of simulations to evaluate the performance of our proposed method in different settings for reference and measurement noise levels. The main motivation behind our experiments comes from “looking around the corner problem,” in which we estimate the reflectivity of the target objects in the scene that are hidden from the camera view. The imaging system captures correlation of the entire scene that includes objects within the direct line of sight and those hidden around the corner. In our experiments, we assume that some parts of the scene, such as the background or border around the object, are known a priori. Another motivation is related to the Fourier phase retrieval for a video sequence in which we may perfectly know parts of the scene that are static (e.g., background) and can be incorporated as side information. The reported SSIM and PSNR are calculated only based on the unknown parts of the image.

3. SIMULATION RESULTS

3.1. Sequential recovery with a known border

In this experiment, we evaluate the performance of our method assuming that a border around the image is known.

In the first experiment, we assume we can design such a border to make sure that the system is well-conditioned. One example of such a reference border is using two pinholes on both sides of the image. Sample reconstructions using our proposed method in this scenario are shown in Fig. 2. In Fig. 2, the border width is set to 8 pixels and the experiment is performed in the presence of different amounts of Gaussian measurement noise. Figure 3 presents PSNR and SSIM curves to show the performance of our proposed method in the presence of different amounts of measurement noise. Figure 4 shows a comparison to our proposed method with classical alternating minimization method with and without reference for a sample image.
3.2. Sequential recovery with a known patch

In this section, we evaluate the performance of our proposed sequential method when a small patch is known and the Fourier phase retrieval problem is considered nonlinear. Figure 6 shows sample reconstructions for the scenario where the known patch is concatenated to the right side of the image. The known patch contains a pinhole and multiple all zero columns. The number of columns in the known patch which also defines the separation between the pinhole and the unknown image is shown by \( h \). Figure 7 summarizes the performance of the proposed method for this scenario. The SSIM and PSNR values for recovered images confirm that the proposed method performs well even if the separation condition is not satisfied. Figure 4 shows a comparison with the existing methods in reconstruction of the baboon image in this scenario where \( h = 16 \).

4. CONCLUSION

We proposed a sequential method to solve Fourier phase retrieval problem with a known reference. Existing methods either assume a minimum separation between reference and unknown image that converts phase retrieval into a linear problem or solve a nonlinear problem to recover the entire image at once. Our method solves the (nonlinear) phase retrieval problem using a sequence of small linear deconvolution problems over small parts of the unknown image. Our simulation results demonstrate that our method can reliably recover images from Fourier amplitude measurements under different settings for reference and measurement noise levels.
5. REFERENCES


