SOLVING FOURIER PHASE RETRIEVAL WITH A REFERENCE IMAGE AS A SEQUENCE OF LINEAR INVERSE PROBLEMS

Fahimeh Arab and M. Salman Asif

Department of Electrical and Computer Engineering, University of California Riverside

ABSTRACT

Fourier phase retrieval problem is equivalent to the recovery of a two-dimensional image from its autocorrelation measurements. This problem is generally nonlinear and nonconvex. Good initialization and prior information about the support or sparsity of the target image are often critical for a robust recovery. In this paper, we show that the presence of a known reference image can help us solve the nonlinear phase retrieval problem as a sequence of small linear inverse problems. Instead of recovering the entire image at once, our sequential method recovers a small number of rows or columns by solving a linear deconvolution problem at every step. Existing methods for the reference-based (holographic) phase retrieval either assume that the reference and target images are sufficiently separated so that the recovery problem is linear or recover the image via nonlinear optimization. In contrast, our proposed method does not require the separation condition. We performed an extensive set of simulations to demonstrate that our proposed method can successfully recover images from autocorrelation data under different settings of reference placement and noise.

***Index Terms*—** Sequential recovery, linear inverse problem, reference-based phase retrieval, holographic phase retrieval.

1. INTRODUCTION

Fourier phase retrieval problem appears in diverse applications such as X-ray crystallography, diffraction imaging, ptychography, and astronomical imaging [1–6] as the only measurable quantity in these systems is the squared magnitude of the Fourier transform. The Fourier phase retrieval problem can be viewed as the recovery of an unknown signal or image from the amplitude of its Fourier transform, which is also equivalent to the recovery of a signal/image from its autocorrelation [2, 7].

Classical phase retrieval methods, such as error reduction and alternating minimization, exploit prior knowledge about the non-negativity and support of the target image [4, 8]. Convex relaxation-based methods, such as PhaseLift [9, 10] and PhaseCut [11], lift the problem to a high-dimensional space. Other methods solve a convex problem for phase retrieval without lifting the problem into higher dimensions [12, 13]. Methods proposed in [14–18] directly solve the nonconvex optimization problem and require careful initialization to avoid local minima. Alternating minimization methods require prior knowledge about the target signal. The most popular signal domain constraints in 2D phase retrieval include positivity, support, and sparsity constraints [19–22]. In addition, adding a known reference signal has been shown to improve the Fourier phase retrieval performance [23–27].

Our work builds upon recent work on holographic phase retrieval [24, 25, 28], in which the presence of a known reference signal in the image makes the recovery problem tractable. In [24, 25], the

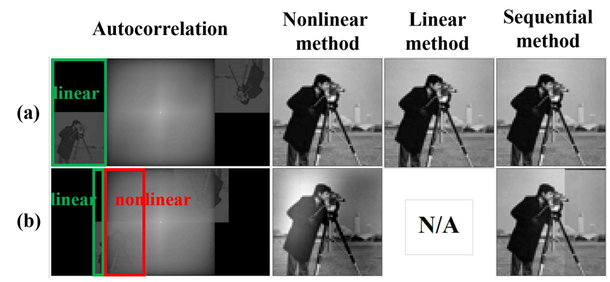

Fig. 1: (Left to right): Autocorrelation of cameraman image with a reference "pinhole" image placed at a distance; recovered images using nonlinear method in [28], linear method in [24], and the proposed sequential method. (Top row) Separation condition between target and reference image is satisfied, which provides sufficient linear measurements in the autocorrelation (shown by a green box). (Bottom row) Separation condition is not satisfied and the linear method in [24] cannot recover the image, but the nonlinear method in [28] and the proposed method can recover the target image.

Fourier phase retrieval problem becomes a linear problem if we place a reference image at large distance from the target image. In our previous work [28], we demonstrated that adding a reference image at arbitrary locations in the image can significantly improve the recovery performance even if the separation condition is not satisfied. The resulting Fourier phase retrieval problem remain nonlinear, which we solved using gradient descent and alternating minimization. In this paper, we propose sequential algorithm that recovers the target image by solving a sequence of (small) linear deconvolution problems.

In this paper, we propose to solve Fourier phase retrieval with side information as a sequence of deconvolution problems. We assume that some parts of the image are known as a reference that can be used to estimate the unknown parts. Given the location of known pixels, we can find which samples in the autocorrelation are linear measurements of the unknown parts. In general, we may not have enough linear measurements in the autocorrelation to compute the entire unknown part in one step. We show that, under certain mild condition, we can estimate a small fraction of the image by solving a linear deconvolution problem and use that estimate in subsequent steps to estimate other unknown parts of the image. We conducted several simulation experiments to evaluate the performance of our proposed method under different conditions of reference signal and noise.

Figure 1 illustrates our proposed method using a "pinhole" reference image. Figure 1 shows the autocorrelation of the image after adding a pinhole reference to the right side of the image in two scenarios. In the first scenario, as shown in Fig. 1 (a), the separation condition is satisfied and the Fourier phase retrieval problem turns into a linear inverse problem. The target image can be directly estimated from a selected region in the autocorrelation. Note that in this specific example, the linear measurement region of autocorrelation (highlighted in a green box) is the target image itself. In a general setting, the linear measurements correspond to the autocorrelation of the target and reference images. In the second scenario, as shown in Fig. 1 (b), the pinhole reference is closer to the target image and the separation condition is not satisfied. Existing linear methods such as [24] cannot estimate the entire image because very few entries in the autocorrelation represent linear measurements of the target image (shown in a green box). In this situation, we can recover the target image either by solving the standard nonlinear phase retrieval problem or by using our proposed sequential method. In our proposed method, instead of estimating the entire unknown image in one step, we use the available linear measurements to estimate a small patch of the unknown image (e.g., two columns at a time). Then we remove the nonlinear contributions of the estimated patch from the correlation data, which provides us linear measurements of another patch of the unknown image. We repeat this procedure until we recover the entire unknown image.

2. FOURIER PHASE RETRIEVAL AS A SEQUENTIAL DECONVOLUTION PROBLEM

2.1. Problem formulation

In this paper, we focus on reconstructing an image from its autocorrelation measurements assuming that some parts of the image are known and can be used as a reference. In general, we can formulate the Fourier phase retrieval as the following nonlinear deconvolution problem:

$$\min_{\mathbf{X}} \ \|\mathbf{R} - \mathbf{X} \star \mathbf{X}\|_2^2, \tag{1}$$

where $\mathbf{R} \in \mathbb{R}^{(2p-1)\times(2q-1)}$ denotes the observed autocorrelation, $\mathbf{X} \in \mathbb{R}^{p\times q}$ denotes the unknown 2D image, and $\star$ represents the 2D cross-correlation operator.

The optimization problem in (1) is nonlinear and nonconvex in general. However, if some parts of the signal $\mathbf{X}$ are known, then some entries in the autocorrelation can be represented as linear measurements of the unknown parts [24]. We first formulate the problem as a blind deconvolution problem and then show how the known reference signal can convert it into a linear problem under some conditions. The autocorrelation of $\mathbf{X}$ can be written in a matrix form as

$$\text{vec}(\mathbf{R}) = C_{\mathbf{X}}\text{vec}(\mathbf{X}), \tag{2}$$

where $C_{\mathbf{X}}$ is block Toeplitz matrix created using $\mathbf{X}$, $\text{vec}(\mathbf{X})$ and $\text{vec}(\mathbf{R})$ denote vectorized version of $\mathbf{X}$ and $\mathbf{R}$, respectively. Let us also assume that $\mathbf{X}$ has q columns that can be written as $\mathbf{X} = \begin{bmatrix}\mathbf{x}_1 & \mathbf{x}_2 & \mathbf{x}_3 & \dots & \mathbf{x}_q\end{bmatrix}$ and $\mathbf{R} = \begin{bmatrix}\mathbf{r}_1 & \mathbf{r}_2 & \mathbf{r}_3 & \dots & \mathbf{r}_{2q-1}\end{bmatrix}$, where $\mathbf{x}_i$ and $\mathbf{r}_i$ are the ith columns of $\mathbf{X}$ and $\mathbf{R}$, respectively. Note that due to the symmetry in the autocorrelation, half of the columns in the autocorrelation are redundant; therefore, we only keep the first half of the columns in the autocorrelation. The equation in (2) can be written as

$$\begin{bmatrix}\mathbf{r}_1\\ \mathbf{r}_2\\ \mathbf{r}_3\\ \vdots\\ \mathbf{r}_q\end{bmatrix} = \begin{bmatrix} C_{\mathbf{x}_q} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0}\\ C_{\mathbf{x}_{q-1}} & C_{\mathbf{x}_q} & \mathbf{0} & \dots & \mathbf{0}\\ C_{\mathbf{x}_{q-2}} & C_{\mathbf{x}_{q-1}} & C_{\mathbf{x}_q} & \dots & \mathbf{0}\\ \vdots & \vdots & \vdots & \dots & \vdots\\ C_{\mathbf{x}_1} & C_{\mathbf{x}_2} & C_{\mathbf{x}_3} & \dots & C_{\mathbf{x}_q}\end{bmatrix} \begin{bmatrix}\mathbf{x}_1\\ \mathbf{x}_2\\ \mathbf{x}_3\\ \vdots\\ \mathbf{x}_q\end{bmatrix}, \tag{3}$$

where $C_{\mathbf{x}_i}$ is a Toeplitz matrix created using column $\mathbf{x}_i$. For example, if $\mathbf{z}$ is an arbitrary column vector, we can write $C_{\mathbf{x}_i}\mathbf{z} = \mathbf{z} \star \mathbf{x}_i$. We can also write a similar equation based on the rows in the image and their autocorrelation. In this paper, we focus on formulating the Fourier phase retrieval problem as a sequential recovery of the columns of the unknown image. Based on (3), the problem is nonlinear in general. However, if some of the columns are known, we can linearize the problem in a sequential manner.

2.2. Sequential deconvolution method

In this section, we will discuss how we can solve the problem in (3) as a sequence of small (linear) deconvolution problems. If a small number of columns on left and/or right side of the image are known as a reference, then instead of solving a nonlinear phase retrieval problem to recover the unknown image, we can sequentially recover a few columns of the unknown image at every step.

2.2.1. Reference as known columns on both sides

Let us assume that the first and last columns of $\mathbf{X}$ (i.e., $\mathbf{x}_1$ and $\mathbf{x}_q$) are known apriori and we want to estimate other columns $(\mathbf{x}_2, \dots, \mathbf{x}_{q-1})$ from $\mathbf{R}$. The known variables in (3) are highlighted in green. Although the overall system in (3) is jointly nonlinear for all the columns in $\mathbf{X}$, but if we know the green parts, we can recover $\mathbf{x}_2, \mathbf{x}_{q-1}$. Subsequently, we can recover $\mathbf{x}_3, \mathbf{x}_{q-2}$. In other words, we can develop a forward substitution method by which we can recover two columns using a linear system of equations at every step.

We discuss the steps to estimate all the columns of the image in a sequential manner below.

Step 1: The first row in (3) does not provide any new information as $\mathbf{x}_1, \mathbf{x}_q$ are known. The second row in (3) gives us the following equation:

$$\mathbf{r}_2 = C^l_{\mathbf{x}_1}\mathbf{x}_{q-1} + C_{\mathbf{x}_q}\mathbf{x}_2 = \begin{bmatrix}C^l_{\mathbf{x}_1} & C_{\mathbf{x}_q}\end{bmatrix}\begin{bmatrix}\mathbf{x}_{q-1}\\ \mathbf{x}_2\end{bmatrix}, \tag{4}$$

where we define the left correlation matrix as $C^l_{\mathbf{x}_i}\mathbf{z} = \mathbf{x}_i \star \mathbf{z}$.

We define the system matrix as $H = \begin{bmatrix}C^l_{\mathbf{x}_1} & C_{\mathbf{x}_q}\end{bmatrix}$. This matrix needs to be well-conditioned, otherwise we will have large error in estimation of $\mathbf{x}_{q-1}$ and $\mathbf{x}_2$. In this step, we recover $\mathbf{x}_{q-1}$ and $\mathbf{x}_2$ by solving the linear inverse problem in (4). Let us denote the recovered columns in this step as $\hat{\mathbf{x}}_{q-1}$ and $\hat{\mathbf{x}}_2$.

Step 2: In this step, we use the equations in the third row in (3) assuming that we know the estimated columns from the previous step shown as $\hat{\mathbf{x}}_{q-1}$ and $\hat{\mathbf{x}}_2$. Based on the definition of the left correlation, we can write

$$\mathbf{r}_3 - \hat{C}_{\mathbf{x}_{q-1}}\hat{\mathbf{x}}_2 \approx \begin{bmatrix}C^l_{\mathbf{x}_1} & C_{\mathbf{x}_q}\end{bmatrix}\begin{bmatrix}\mathbf{x}_{q-2}\\ \mathbf{x}_3\end{bmatrix}. \tag{5}$$

By solving the linear inverse problem in (5), we estimate the two other columns of the image as $\hat{\mathbf{x}}_{q-2}$ and $\hat{\mathbf{x}}_3$.

Step k: The number of steps we need to recover the image depends on the width of the image. However, we generally estimate one column from the right and one column from the left side of the image. At any step k, by solving the linear inverse problem in (6), we estimate two columns $\hat{\mathbf{x}}_{q-k}$ and $\hat{\mathbf{x}}_{k+1}$.

$$\mathbf{r}_{k+1} - \sum_{i=1}^{i=k-1}\hat{C}_{\mathbf{x}_{q-k+i}}\hat{\mathbf{x}}_{i+1} \approx \begin{bmatrix}C^l_{\mathbf{x}_1} & C_{\mathbf{x}_q}\end{bmatrix}\begin{bmatrix}\mathbf{x}_{q-k}\\ \mathbf{x}_{k+1}\end{bmatrix}. \tag{6}$$

A pseudocode of the proposed method is presented in Algorithm 1. In our experiments, K is computed based on the total number of columns in the image (q) and the number of known columns. We observed that if a small number of columns from both sides of an image are known as a reference, the conditioning of the overall system in (3) improves, which results in stable recovery. SVD or

Algorithm 1 Proposed sequential recovery method

Inputs: $\mathbf{r}_2, ..., \mathbf{r}_{K+1}, \mathbf{x}_1, \mathbf{x}_q$, and K
for k = 1,. . . K **do**

$$\begin{bmatrix} \hat{\mathbf{x}}_{q-k} \\ \hat{\mathbf{x}}_{k+1} \end{bmatrix} = \begin{bmatrix} C^l_{\mathbf{x}_1} & C_{\mathbf{x}_q} \end{bmatrix}^{-1} (\mathbf{r}_{k+1} - \sum_{i=1}^{i=k-1} \hat{C}_{\mathbf{x}_{q-k+i}} \hat{\mathbf{x}}_{i+1})$$

end for
Output: $\hat{\mathbf{x}}_2, ..., \hat{\mathbf{x}}_{q-1}$

QR decomposition of the system matrix, which is the same at each iteration, can be used to implement Algorithm 1. Overall, the main computational cost involves computing the residual in (6) and solving the resulting least squares problem using precomputed SVD or QR factors.

2.2.2. *Reference as known columns on one side*

In this scenario, we assume that the known columns are on the right side of the image. If we only know one column, $\mathbf{x}_q$, the only difference compared to the previous section is the estimation of column $\mathbf{x}_1$. We can estimate $\mathbf{x}_1$ from the first line in (3), then the remaining steps are similar to what we discussed in the previous section.

If we know a large number of columns on one side of the image, the estimation becomes even simpler. In this scenario, we estimate the first few columns by solving a linear inverse problem, then we estimate the other columns using the proposed sequential method. For example, if the known patch has h columns at the right side of the image, we can first estimate h columns from the left side using a linear approach, then we can employ the sequential method to estimate the remaining columns.

2.3. Stability and Recovery Conditions

In this section, we discuss required conditions for stable recovery.

2.3.1. *Stability of the algorithm at each step*

Stability of the recovery algorithm at each iteration depends on the conditioning of the matrix $H = \begin{bmatrix} C^l_{\mathbf{x}_1} & C_{\mathbf{x}_q} \end{bmatrix}$. For matrix H to be invertible and well-conditioned, the matrices $C^l_{\mathbf{x}_1}$ and $C_{\mathbf{x}_q}$ need to be well-conditioned and incoherent with respect to each other. This condition is a necessary condition for perfect reconstruction; however, it is not sufficient. We can either design this matrix by choosing the reference signal appropriately or verify that this condition is satisfied by the given reference. One special example is two pinhole columns. In this case, matrix H is a full-column rank matrix. If we also know some rows of the image, we can further improve the conditioning of the system.

2.3.2. *Stability of the overall system*

The stability of the overall system is a necessary condition for the sequential method to be successful. The overall system, as shown in (3), depends on the pixel values in the unknown image. In contrast, the stability of the system at each step (H) only depends on the pixel value of the first and last columns of the image. For the overall matrix to be well-conditioned, we may need all the columns to be incoherent.

2.3.3. *Stability of the proposed sequential method*

As shown in Algorithm 1, at every step, we use the estimates from all the previous steps. Presence of measurement noise and finite precision of multiplications and additions in the cross-correlation terms causes an accumulation of error, which can cause instability if the system is not well-conditioned. To avoid this, we include an ℓ_∞ regularization on the estimated pixel value. This regularization is implemented by a hard-thresholding that limits the range for the estimated pixel values. Our empirical results show that adding a small border of all-zero columns to the left and right sides of the image can improve the condition of the overall system matrix in (3) and provide stable reconstruction.

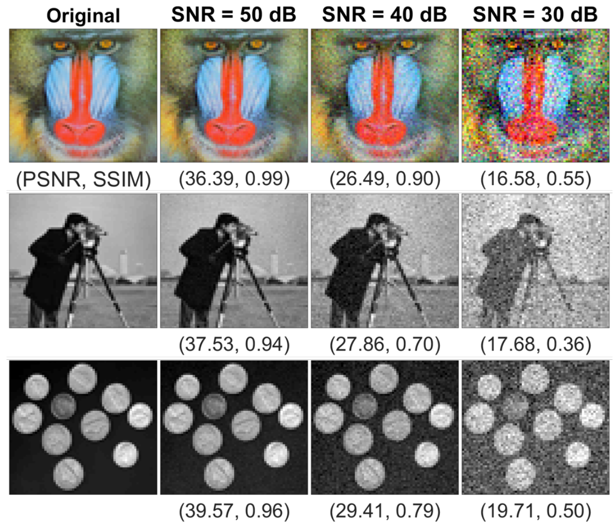

Fig. 2: Fourier phase retrieval reconstruction for sample images when we add a known reference border around the images.

3. SIMULATION RESULTS

We performed a number of simulations to evaluate the performance of our proposed method in different settings for reference and measurement noise levels. The main motivation behind our experiments comes from "looking around the corner problem," in which we estimate the reflectivity of the target objects in the scene that are hidden from the camera view. The imaging system captures correlation of the entire scene that includes objects within the direct line of sight and those hidden around the corner. In our experiments, we assume that some parts of the scene, such as the background or border around the object, are known apriori. Another motivation is related to the Fourier phase retrieval for a video sequence in which we may perfectly know parts of the scene that are static (e.g., background) and can be incorporated as side information. The reported SSIM and PSNR are calculated only based on the unknown parts of the image.

3.1. Sequential recovery with a known border

In this experiment, we evaluate the performance of our method assuming that a border around the image is known.

In the first experiment, we assume we can design such a border to make sure that the system is well-conditioned. One example of such a reference border is using two pinholes on both sides of the image. Sample reconstructions using our proposed method in this scenario are shown in Fig. 2. In Fig. 2, the border width is set to 8 pixels and the experiment is performed in the presence of different amounts of Gaussian measurement noise. Figure 3 presents PSNR and SSIM curves to show the performance of our proposed method in the presence of different amounts of measurement noise. Figure 4 shows a comparison to our proposed method with classical alternating minimization method with and without reference for a sample image.

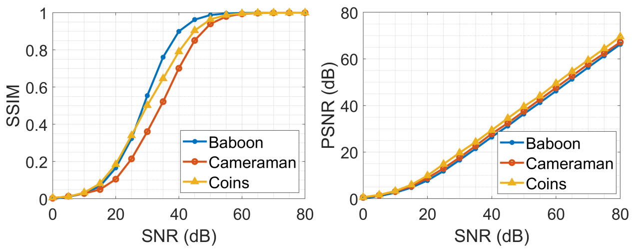

Fig. 3: Reconstruction performance of the proposed method when a known reference border is added around the image for different noise levels.

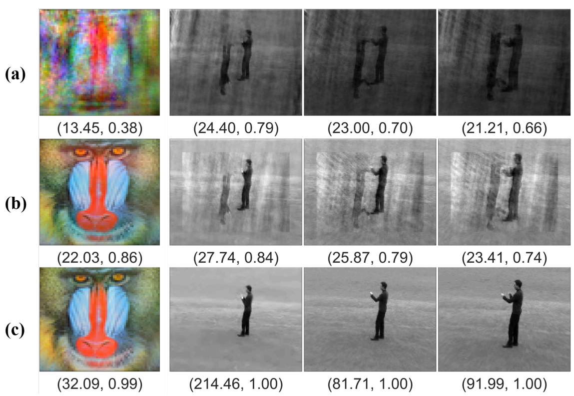

Fig. 4: Recovered images using (a) alternating minimization without side information [4], (b) nonlinear method in [28], and (c) our proposed sequential method. For each reconstructed image, the values of PSNR and SSIM are shown as (PSNR, SSIM). For a baboon image, we added a pinhole at location $h = 16$ as side information and for the KTH video sequence, we considered a border of width 15 pixels to be known.

In the second experiment, we solve Fourier phase retrieval for a video sequence where a border around the object-of-interest in the scene is known. In this case, we can use the region around the object as a reference in our proposed sequential method. As shown in Fig. 5, the reconstruction performance improves as we increase the size of reference. Known area in Fig. 5 is defined as the ratio of the number of pixels in the known border to the total number of pixels. Each frame of the videos from KTH dataset has 120×160 pixels. Figure 4 shows a comparison to our proposed method with classical alternating minimization method with and without reference for a sample video sequence.

3.2. Sequential recovery with a known patch

In this section, we evaluate the performance of our proposed sequential method when a small patch is known and the Fourier phase retrieval problem is considered nonlinear. Figure 6 shows sample reconstructions for the scenario where the known patch is concatenated to the right side of the image. The known patch contains a pinhole and multiple all zero columns. The number of columns in the known patch which also defines the separation between the pinhole and the unknown image is shown by h. Figure 7 summarizes the performance of the proposed method for this scenario. The SSIM and PSNR values for recovered images confirm that the proposed method perform well even if the separation condition is not satisfied. Figure 4 shows a comparison with the existing methods in reconstruction of the baboon image in this scenario where $h = 16$.

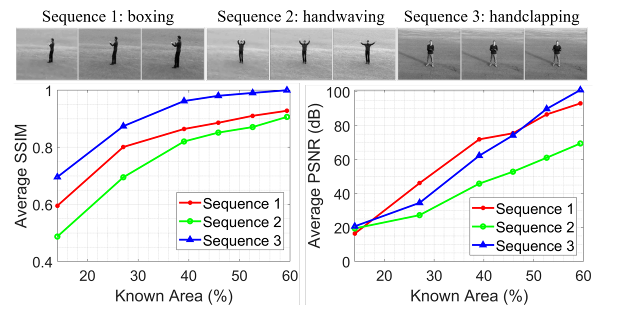

Fig. 5: Reconstruction performance of our proposed sequential method for three sequences in KTH dataset. Sequences 1 to 3 used in this experiment are person 1 boxing, person 1 handwaving, and person 10 handclapping videos respectively. PSNR and SSIM are averaged over 10 frames.

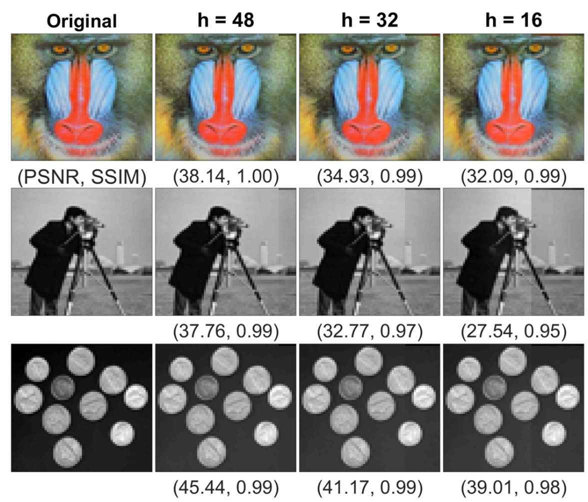

Fig. 6: Sample reconstructions using the proposed method for different separation, defined as h, between the reference and the unknown image of size 64×64.

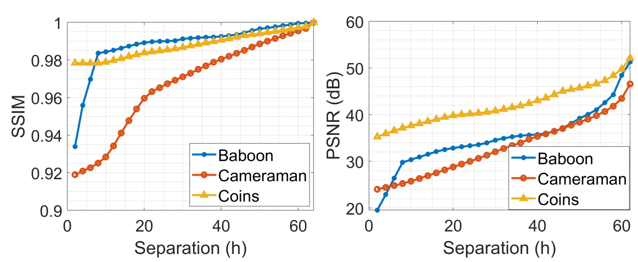

Fig. 7: Reconstruction performance of the proposed method for different positions of the pinhole reference.

4. CONCLUSION

We proposed a sequential method to solve Fourier phase retrieval problem with a known reference. Existing methods either assume a minimum separation between reference and unknown image that converts phase retrieval into a linear problem or solve a nonlinear problem to recover the entire image at once. Our method solves the (nonlinear) phase retrieval problem using a sequence of small linear deconvolution problems over small parts of the unknown image. Our simulation results demonstrate that our method can reliably recover images from Fourier amplitude measurements under different settings for reference and measurement noise levels.

5. REFERENCES

[1] Emmanuel J. Candès, Xiaodong Li, and Mahdi Soltanolkotabi, "Phase retrieval from coded diffraction patterns," *Applied and Computational Harmonic Analysis*, vol. 39, 10 2013.

[2] Yoav Shechtman, Yonina C. Eldar, Oren Cohen, Henry Nicholas Chapman, Jianwei Miao, and Mordechai Segev, "Phase retrieval with application to optical imaging: A contemporary overview," *IEEE Signal Processing Magazine*, vol. 32, no. 3, pp. 87–109, 2015.

[3] Guoan Zheng, Roarke Horstmeyer, and Changhuei Yang, "Wide-field, high-resolution fourier ptychographic microscopy," in *Nature photonics*, 2013.

[4] James R. Fienup, "Reconstruction of an object from the modulus of its fourier transform," *Opt. Lett.*, vol. 3, pp. 27–29, Jul 1978.

[5] Jianwei Miao, Richard L. Sandberg, and Changyong Song, "Coherent x-ray diffraction imaging," *IEEE Journal of Selected Topics in Quantum Electronics*, vol. 18, no. 1, pp. 399–410, 2012.

[6] Jianwei Miao, Tetsuya Ishikawa, Ian K. Robinson, and Margaret M. Murnane, "Beyond crystallography: Diffractive imaging using coherent x-ray light sources," *Science*, vol. 348, no. 6234, pp. 530–535, 2015.

[7] Dani Kogan, Yonina C. Eldar, and Dan Oron, "On the 2d phase retrieval problem," *IEEE Transactions on Signal Processing*, vol. 65, no. 4, pp. 1058–1067, 2017.

[8] R. W. Gerchberg and W. O. Saxton, "A practical algorithm for the determination of phase from image and diffraction plane pictures," *Optik*, vol. 35, pp. 237–246, 1972.

[9] Emmanuel J. Candès, Thomas Strohmer, and Vladislav Voroninski, "Phaselift: Exact and stable signal recovery from magnitude measurements via convex programming," *Communications on Pure and Applied Mathematics*, vol. 66, 2012.

[10] Emmanuel J. Candès, Yonina C. Eldar, Thomas. Strohmer, and Vladislav. Voroninski, "Phase retrieval via matrix completion," *SIAM Journal on Imaging Sciences*, vol. 6, no. 1, pp. 199–225, 2013.

[11] Irène Waldspurger, Alexandre d'Aspremont, and Stéphane Mallat, "Phase recovery, maxcut and complex semidefinite programming," *Mathematical Programming*, vol. 149, pp. 47–81, Feb 2015.

[12] Tom Goldstein and Christoph Studer, "Phasemax: Convex phase retrieval via basis pursuit," *IEEE Transactions on Information Theory*, vol. 64, no. 4, pp. 2675–2689, 2018.

[13] Sohail Bahmani and Justin Romberg, "A flexible convex relaxation for phase retrieval," *Electronic Journal of Statistics*, vol. 11, no. 2, pp. 5254–5281, 2017.

[14] Emmanuel J. Candès, Xiaodong Li, and Mahdi Soltanolkotabi, "Phase retrieval via wirtinger flow: Theory and algorithms," *IEEE Transactions on Information Theory*, vol. 61, no. 4, pp. 1985–2007, 2015.

[15] Yuxin Chen and Emmanuel J. Candès, "Solving random quadratic systems of equations is nearly as easy as solving linear systems," *Communications on Pure and Applied Mathematics*, vol. 70, 05 2015.

[16] Ziyang Yuan and Hongxia Wang, "Phase retrieval via reweighted wirtinger flow," *Appl. Opt.*, vol. 56, pp. 2418–2427, Mar 2017.

[17] Gang Wang, Georgios B. Giannakis, and Yonina C. Eldar, "Solving systems of random quadratic equations via truncated amplitude flow," *IEEE Transactions on Information Theory*, vol. 64, no. 2, pp. 773–794, 2018.

[18] Gang Wang, Georgios B. Giannakis, Yousef Saad, and Jie Chen, "Phase retrieval via reweighted amplitude flow," *IEEE Transactions on Signal Processing*, vol. 66, no. 11, pp. 2818–2833, 2018.

[19] Edouard Jean Robert Pauwels, Amir Beck, Yonina C. Eldar, and Shoham Sabach, "On fienup methods for sparse phase retrieval," *IEEE Transactions on Signal Processing*, vol. 66, no. 4, pp. 982–991, 2018.

[20] Kishore Jaganathan, Samet Oymak, and Babak Hassibi, "Phase retrieval for sparse signals using rank minimization," in *2012 IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP)*, 2012, pp. 3449–3452.

[21] Kishore Jaganathan, Samet Oymak, and Babak Hassibi, "Sparse phase retrieval: Uniqueness guarantees and recovery algorithms," *Trans. Sig. Proc.*, vol. 65, no. 9, pp. 2402–2410, May 2017.

[22] Gauri Jagatap and Chinmay Hegde, "Fast, sample-efficient algorithms for structured phase retrieval," *Advances in Neural Information Processing Systems 30*, pp. 4917–4927, 2017.

[23] Manuel Guizar-Sicairos and James R. Fienup, "Holography with extended reference by autocorrelation linear differential operation," *Opt. Express*, vol. 15, no. 26, pp. 17592–17612, Dec 2007.

[24] David A. Barmherzig, Ju Sun, Emmanuel J. Candès, T. J. Lane, and Po-Nan Li, "Holographic phase retrieval and optimal reference design," *Inverse Problems*, vol. 35, no. 9, 2019.

[25] Manuel Guizar-Sicairos and James R. Fienup, "Direct image reconstruction from a fourier intensity pattern using heraldo.," *Optics letters*, vol. 33 22, pp. 2668–70, 2008.

[26] Ziyang Yuan and Hongxia Wang, "Phase retrieval with background information," *Inverse Problems*, vol. 35, no. 5, 2019.

[27] Rakib Hyder, Chinmay Hegde, and M. Salman Asif, "Fourier phase retrieval with side information using generative prior," in *2019 53rd Asilomar Conference on Signals, Systems, and Computers*, 2019, pp. 759–763.

[28] Fahimeh Arab and M. Salman Asif, "Fourier phase retrieval with arbitrary reference signal," in *ICASSP 2020 - 2020 IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP)*, 2020, pp. 1479–1483.