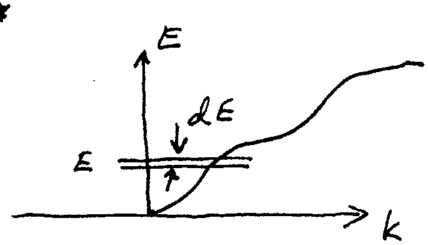
1. This product arises when calculating electron flux.



At some energy E we want the flux of electrons moving to the right. To keep this simple I assume a monotonic positive slope for $k > 0$ and a symmetric $E-k$ relation. Then if the states are fully occupied at energy E, the flux is (for 1 spin)

$$J(E) = N_{1D}(E)\, v(E)$$

$$N_{1D}(E) = \int_0^{\infty} \frac{dk}{2\pi}\, \delta(E - \epsilon(k)) = \frac{1}{2\pi} \frac{1}{dE/dk}$$

$$v(E) = \frac{1}{\hbar} \frac{dE}{dk}$$

$$J(E) = \frac{1}{h} \quad \text{per spin}$$

2. (a)

$$N_{1D}^{1spin}(E) = \frac{1}{L}\sum_k \delta(E-\epsilon(k))$$

$$= \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, \delta(E-\epsilon(k))$$

$$= 2\int_0^{\infty} \frac{dk}{2\pi}\, \delta(E-\epsilon(k)) \quad \text{symmetric } \epsilon(k)$$

$$= \frac{1}{\pi}\int_0^{\infty} d\epsilon \left(\frac{d\epsilon}{dk}\right)^{-1} \delta(E-\epsilon)$$

$$= \frac{1}{\pi}\left(\frac{dE}{dk}\right)^{-1} \qquad (E \geq 0)$$

$$\epsilon = \frac{\hbar^2 k^2}{2m}; \quad d\epsilon = \frac{\hbar^2 k}{m}\, dk = \frac{\hbar}{m}\sqrt{2m\epsilon}\, dk$$

$$\left(\frac{d\epsilon}{dk}\right)^{-1} = \frac{1}{\hbar}\sqrt{\frac{m}{2\epsilon}}$$

$$N_{1D}^{1spin}(E) = \begin{cases} \frac{1}{h}\sqrt{\frac{2m}{E}} & (E \geq 0) \\ 0 & (E < 0) \end{cases}$$

(b)

$$N_{2D}^{1spin}(E) = \frac{1}{A}\sum_{\vec{k}} \delta(E-\epsilon(\vec{k}))$$

$$= \int \frac{d^2k}{4\pi^2}\, \delta(E-\epsilon(k))$$

$$= \frac{1}{2\pi}\int_0^{\infty} dk\, k\, \delta(E-\epsilon(k))$$

$$= \frac{1}{2\pi}\int_0^{\infty} d\epsilon \left(\frac{d\epsilon}{dk}\right)^{-1} k(\epsilon)\, \delta(E-\epsilon)$$

$$= \frac{1}{2\pi}\left(\frac{dE}{dk}\right)^{-1} k(E) \qquad (E \geq 0)$$

$$k(E) = \frac{\sqrt{2mE}}{\hbar}$$

$$\frac{1}{2\pi}\left(\frac{dE}{dk}\right)^{-1} k(E) = \frac{m}{2\pi\hbar^2} = N_{2D}^{1\,spin}(E) \quad (E \geq 0)$$

(c) $$N_{3D}^{1\,spin}(E) = \int \frac{d^3k}{8\pi^3}\, \delta(E - \epsilon(k))$$

$$= \frac{1}{2\pi^2}\int_0^\infty dk\, k^2\, \delta(E - \epsilon(k))$$

$$= \frac{1}{2\pi^2}\left(\frac{dE}{dk}\right)^{-1} k^2(E)$$

$$= \frac{1}{\cancel{2}\pi^2}\,\frac{1}{\hbar}\sqrt{\frac{m}{2E}}\,\frac{\cancel{2}mE}{\hbar^2}$$

$$N_{3D}^{1\,spin}(E) = \begin{cases} \dfrac{m}{2\pi^2\hbar^3}\sqrt{2mE} & (E \geq 0) \\ 0 & (E < 0) \end{cases}$$

3. $N_{1D}^{1spin} = \frac{1}{L} \sum_{k_x} \delta(E - \epsilon(k_x))$

(a)

$$N_{1D}^{1s} \otimes N_{1D}^{1s} = \frac{1}{L^2} \sum_{k_x, k_y} \int dE' \, \delta(E - E' - \epsilon(k_x)) \, \delta(E' - \epsilon(k_y))$$

$$= \frac{1}{L^2} \sum_{k_x, k_y} \delta(E - \epsilon(k_x) - \epsilon(k_y))$$

$= N_{2D}^{1s}$ for seperable $E-k$ relation

i.e. $E(\vec{k}) = \epsilon(k_x) + \epsilon(k_y)$

eg. $E(k) = \frac{\hbar^2}{2m^*} (k_x^2 + k_y^2)$

(b) $N_{1D} \otimes N_{1D} \otimes N_{1D} = (N_{1D} \otimes N_{1D}) \otimes N_{1D}$

$$= \frac{1}{L^3} \sum_{k_x, k_y, k_z} \int dE' \, \delta(E' - \epsilon(k_x) - \epsilon(k_y)) \, \delta(E - E' - \epsilon(k_z))$$

$$= \frac{1}{L^3} \sum_{k_x, k_y, k_z} \delta(E - \epsilon(k_x) - \epsilon(k_y) - \epsilon(k_z))$$

$= N_{3D}^{1s}$ for seperable $E-k$ relation.

4. $E = E_c + \frac{\hbar^2}{ma^2}\left[1-\cos(ka)\right]$

a) $v_g = \frac{1}{\hbar}\frac{dE}{dk} = \frac{\hbar}{ma}\sin(ka)$

$$\boxed{v_g\left(k=\tfrac{\pi}{a}\right) = 0}$$

b) $\frac{F}{\hbar} = \frac{dk}{dt}$

$$\vec{k} = \vec{k}_0 + \frac{\vec{F}}{\hbar}t = \frac{-e\vec{\mathcal{E}}}{\hbar}t + \frac{-\pi}{a}$$

$$\boxed{k = \frac{-\pi}{a} + \frac{q\mathcal{E}}{\hbar}t}$$

where $\mathcal{E} = |\vec{\mathcal{E}}|$
and $q = +1.602e-19\ C$

c) $v_g(t) = \frac{\hbar}{ma}\sin\left[\left(\frac{-\pi}{a} + \frac{q\mathcal{E}}{\hbar}t\right)a\right]$

$$\boxed{v_g(t) = \frac{\hbar}{ma}\sin\left[\frac{q\mathcal{E}a}{\hbar}t - \pi\right]}$$

d) $x(t) - x_0 = \int_{t_0}^{t} dt'\, v_g(t')$

$$x(t) = \int_0^t dt'\,\frac{\hbar}{ma}\sin\left[\frac{q\mathcal{E}a}{\hbar}t' - \pi\right]$$

$$= \frac{-\hbar}{ma}\frac{\hbar}{q\mathcal{E}a}\cos\left[\frac{q\mathcal{E}a}{\hbar}t' - \pi\right]\Big|_0^t$$

$$= \frac{-\hbar^2}{q\mathcal{E}ma^2}\left[\cos\left(\frac{q\mathcal{E}at}{\hbar} - \pi\right) + 1\right]$$

$$= \frac{-\hbar^2}{q\mathcal{E}ma^2}\left[-\cos\left(\frac{q\mathcal{E}at}{\hbar}\right) + 1\right]$$

$$\boxed{x(t) = \frac{\hbar^2}{q\mathcal{E}ma^2}\left[\cos\left(\frac{q\mathcal{E}at}{\hbar}\right) - 1\right]}$$

e) $x_{max} = 0.$

$$x_{min} = \frac{\hbar^2}{q\mathcal{E}ma^2}(-2)$$

$$= \frac{(6.5855e{-}16\ eVs)^2 c^2}{10^4 \frac{eV}{cm}(0.2)(m_o c^2)(5e{-}7cm)^2}$$

$$= \frac{[1.9743e3\ (eV\AA)(10^{-8}\frac{cm}{\AA})]^2}{10^4 \frac{eV}{cm}(0.2)(0.51106e6\ eV)(5e{-}7cm)^2}(-2)$$

$$= -3.05e{-}6\ cm\ (10^7 \frac{nm}{cm})$$

$$\boxed{x_{min} = -30.5\ nm}$$

f) $\frac{q\mathcal{E}a}{\hbar}\tau = 2\pi$

$$\tau = \frac{2\pi\hbar}{q\mathcal{E}a} = \frac{2\pi(6.5855e{-}16\ eVs)}{10^4\frac{eV}{cm}(5\ nm)(10^{-7}\frac{cm}{nm})}$$

$$\tau = 8.2755e{-}13\ s$$

$$\boxed{\tau = 0.8276\ ps}$$

g) $\frac{1}{m^*} = \frac{1}{\hbar^2}\frac{\partial^2 E}{\partial k^2}$

$$\frac{\partial E}{\partial k} = \frac{\hbar^2}{ma^*}\sin ka$$

$$\frac{1}{\hbar^2}\frac{\partial^2 E}{\partial k^2} = \frac{\cancel{\hbar^2}}{m}\cos ka$$

$$\boxed{m^* = \frac{m}{\cos ka}}$$