High-Order Consensus Algorithms in Cooperative Vehicle Systems

Wei Ren, *Member, IEEE*; Kevin Moore, *Senior Member, IEEE*; YangQuan Chen, *Senior Member, IEEE*

***Abstract*— In this paper we study ℓ^{th} order ($\ell >= 3$) consensus algorithms, which generalize the existing first-order and second-order consensus algorithms in the literature. We will show sufficient conditions under which each information variable and their higher-order derivatives converge to common values. We will present the idea of higher-order consensus with a leader and introduce the concept of an ℓ^{th} order model-reference consensus problem, where each information variable and their high-order derivatives not only reach consensus but also converge to the solution of a prescribed dynamic model. The effectiveness of these algorithms are demonstrated through simulations and a multi-vehicle cooperative control application which mimics flocking behavior in birds.**

I. Introduction

In plain language, when several entities or agents agree on a common value of a variable of interest they are said to have come to "*consensus*." In a group of networked mobile agents with a common mission or task, the consensus problem can play a pivotal role, particularly when the communication capability for each agent is limited and/or purposely constrained. For example, when the dynamic environment changes, the agents in the team must be in agreement as to what changes have taken place, even when every agent cannot talk directly to every other agent. To achieve consensus, there must be a shared variable of interest (called the coordination data or variable) as well as appropriate algorithmic methods for negotiating to consensus about the value of that variable (called a consensus algorithm or protocol).

Cooperative control for multi-agent systems have primarily been applied to formation control problems with applications to mobile robots, unmanned air vehicles (UAVs), autonomous underwater vehicles (AUVs), satellites, aircrafts, spacecrafts, and automated highway systems [1], but also include non-formation cooperative control problems such as task assignment, payload transport, role assignment, air traffic control, timing, and search [2]. As pointed out in [1], for cooperative control strategies to be successful, numerous issues must be addressed, including the definition and management of shared information among a group of agents to facilitate the coordination of these agents. Information necessary for cooperation may be shared in a variety of ways. However, the question "*consensus to what?*" has not been explicitly answered in the literature, although a similar question "*formation to what form or shape?*" was asked in [3] within the context of mobile actuator and sensor networks [4].

W. Ren and Y.-Q. Chen are with the Department of Electrical and Computer Engineering, Utah State University, Logan, UT 84322-4120, USA email: `wren@engineering.usu.edu, yqchen@ece.usu.edu`).

K.L. Moore is with Division of Engineering, Colorado School of Mines, 1610 Illinois Street, Golden, CO 80401, USA (email: `kmoore@mines.edu`).

1-4244-0065-1/06/$20.00 ©2006 IEEE

In this paper we generalize the first-order and second-order consensus algorithms in the literature (see [5] and the references therein). We show *sufficient* conditions under which the consensus information variable and its higher-order derivatives converge to common values. One motivation for studying higher-order consensus comes from observing the behavior of flocks of birds. It is often noted that such flocks fly in somewhat of a formation, maintaining a nominal separation from each other, but each traveling with the same velocity vector. In [6] it was shown how second-order consensus can produce the behavior of a separation and common velocity under directed information exchange. However, sometimes a bird flock abruptly changes direction, perhaps when one of them suddenly perceives a source of danger or food. Clearly the birds in this setting need to build consensus on not only their relative position and their velocity, but also on acceleration. This motivates the idea of higher-order consensus. Higher order consensus makes obvious sense for cooperative control of a team of UAVs when confronting another team of "*hostile intelligent gaming (HIG) UAVs*". With the question "*consensus to what*" in mind, we are also motivated to consider an ℓ^{th} order model-reference consensus problem, where each information variable and their high-order derivatives not only reach consensus but also converge to the solution of a prescribed dynamic model. We introduce this model-reference consensus problem and establish sufficient conditions for consensus convergence.

The remainder of the paper is organized as follows. Section II presents some background materials and some mathematical preliminaries for our later development. Higher-order consensus algorithms are given in Sec. III, including the standard, unforced case and the cases of setpoint tracking, which lead to the idea of consensus with a leader, and model-reference consensus. The effectiveness of the proposed algorithms is illustrated throughout by simulations, including an example of flocking behavior in Section IV. Section V concludes the paper.

II. Background and Preliminaries

It is natural to model information exchange between vehicles by directed/undirected graphs. A digraph (directed graph) consists of a pair $(\mathcal{N}, \mathcal{E})$, where $\mathcal{N}$ is a finite nonempty set of nodes and $\mathcal{E} \in \mathcal{N}^2$ is a set of ordered pairs of nodes, called edges. As a comparison, the pairs of nodes in an undirected graph are unordered. If there is a directed edge from node v_i to node v_j, then v_i is defined as the parent node and v_j is defined as the child node. A directed path is a sequence of ordered edges of the form $(v_{i_1}, v_{i_2}), (v_{i_2}, v_{i_3}), \cdots$, where $v_{i_j} \in \mathcal{N}$, in a digraph. An undirected path in an undirected graph is defined accordingly.

A digraph is called strongly connected if there is a directed path from every node to every other node. An undirected graph is called connected if there is a path between any distinct pair of nodes. A directed tree is a digraph, where every node, except the root, has exactly one parent. A (directed) spanning tree of a digraph is a directed tree formed by graph edges that connect all the nodes of the graph. We say that a graph has (or contains) a (directed) spanning tree if there exists a (directed) spanning tree being a subset of the graph. Note that the condition that a digraph has a (directed) spanning tree is equivalent to the case that there exists a node having a directed path to all the other nodes. In the case of undirected graphs, having an undirected spanning tree is equivalent to being connected. However, in the case of directed graphs, having a directed spanning is not equivalent to being strongly connected. The union of a group of digraphs is a digraph with nodes given by the union of the node sets and edges given by the union of the edge sets of those digraphs.

The adjacency matrix $A = [a_{ij}]$ of a weighted digraph is defined as $a_{ii} = 0$ and $a_{ij} > 0$ if $(j, i) \in \mathcal{E}$ where $i \neq j$. The Laplacian matrix of the weighted digraph is defined as $L = [\ell_{ij}]$, where $\ell_{ii} = \sum_{j \neq i} a_{ij}$ and $\ell_{ij} = -a_{ij}$ where $i \neq j$. For an undirected graph, the Laplacian matrix is symmetric positive semi-definite.

Let $\mathbf{1}$ and $\mathbf{0}$ denote the $n \times 1$ column vector of all ones and all zeros respectively. Let I_n denote the $n \times n$ identity matrix and 0_n denote the $n \times n$ zero matrix. Let $M_n(I\!R)$ represent the set of all $n \times n$ real matrices. Given a matrix $A = [a_{ij}] \in M_n(I\!R)$, the digraph of A, denoted by $\Gamma(A)$, is the digraph on n nodes v_i, $i \in \mathcal{I}$, such that there is a directed edge in $\Gamma(A)$ from v_j to v_i if and only if $a_{ij} \neq 0$ (c.f. [7]).

III. Higher-order Consensus Algorithms

We begin by presenting the general ℓ^{th}-order extension to the standard consensus protocol algorithm, followed by two extensions: (1) setpoint tracking and consensus with a leader, and (2) model-reference consensus.

A. ℓ^{th}-order consensus

Consider information variables with ℓ^{th}-order dynamics given by

$$\begin{aligned} \dot{\xi}_i^{(0)} &= \xi_i^{(1)} \\ &\vdots \\ \dot{\xi}_i^{(\ell-2)} &= \xi_i^{(\ell-1)} \\ \dot{\xi}_i^{(\ell-1)} &= u_i \end{aligned} \tag{1}$$

where $\xi_i^{(k)} \in I\!R^m$, $k = 0, 1, \cdots, \ell - 1$, are the states, $u_i \in I\!R^m$ is the control input, and $\xi_i^{(k)}$ denotes the k^{th} derivative of ξ_i with $\xi_i^{(0)} = \xi_i$.

We propose the following consensus algorithm:

$$u_i = -\sum_{j=1}^{n} g_{ij} k_{ij} [\sum_{k=0}^{\ell-1} \gamma_k (\xi_i^{(k)} - \xi_j^{(k)})], \quad i \in \{1, \cdots, n\} \tag{2}$$

where $k_{ij} > 0$, $\gamma_k > 0$, $g_{ii} \stackrel{\triangle}{=} 0$, and g_{ij} is 1 if information flows from vehicle j to vehicle i and 0 otherwise. We say that consensus is reached among the n vehicles if $\xi_i^{(k)} \to \xi_j^{(k)}$, $k = 0, 1, \cdots, \ell - 1$, $\forall i \neq j$. Note that in addition to the references above the authors' work, a number of other researchers have considered consensus problems in the context of multi-agent systems. See, for example [8], [9], [10], [11]. However, the linear consensus strategies reported in the literature are special cases of (2) when $l = 1$ or $l = 2$.

Let $\xi = [\xi_1^T, \cdots, \xi_n^T]^T$. By applying consensus algorithm (2), Eq. (1) can be written in matrix form as

$$\begin{bmatrix} \dot{\xi}^{(0)} \\ \dot{\xi}^{(1)} \\ \vdots \\ \dot{\xi}^{(\ell-1)} \end{bmatrix} = (\Gamma \otimes I_m) \begin{bmatrix} \xi^{(0)} \\ \xi^{(1)} \\ \vdots \\ \xi^{(\ell-1)} \end{bmatrix}, \tag{3}$$

where

$$\Gamma = \begin{bmatrix} 0_n & I_n & 0_n & \cdots & 0_n \\ 0_n & 0_n & I_n & \cdots & 0_n \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0_n & 0_n & 0_n & \cdots & I_n \\ -\gamma_0 L & -\gamma_1 L & -\gamma_2 L & \cdots & -\gamma_{\ell-1} L \end{bmatrix},$$

where $L = [\ell_{ij}]$ with $\ell_{ii} = \sum_{j \neq i} g_{ij} k_{ij}$ and $\ell_{ij} = -g_{ij} k_{ij}$, $\forall i \neq j$.

In the following, we assume $m = 1$ for simplicity. However, all the results hereafter remain valid for $m > 1$. In addition, we only consider the case when $\ell = 3$. Similar analyses are applicable to the case when $\ell > 3$.

Before stating our main results, we need the following lemma.

Lemma 3.1: In the case of $\ell = 3$, Γ has at least three zero eigenvalues. It has exactly three zero eigenvalues if and only if $-L$ has a simple zero eigenvalue. Moreover, if $-L$ has a simple zero eigenvalue, the zero eigenvalue of Γ has geometricity equal to one.

Proof: Let μ and p be an eigenvalue and eigenvector of $-L$. Consider a vector of the form $q = [p^T, \alpha p^T, \beta p^T]^T$, where $\alpha, \beta \in \mathbb{C}$. We see that

$$\begin{aligned} \Gamma q &= \begin{bmatrix} 0_n & I_n & 0_n \\ 0_n & 0_n & I_n \\ -\gamma_0 L & -\gamma_1 L & -\gamma_2 L \end{bmatrix} \begin{bmatrix} p \\ \alpha p \\ \beta p \end{bmatrix} \\ &= \begin{bmatrix} \alpha p \\ \beta p \\ (\gamma_0 + \alpha\gamma_1 + \beta\gamma_2)\mu p \end{bmatrix}, \end{aligned}$$

where we have used the fact that $-Lp = \mu p$. Note that q is an eigenvector of Γ with eigenvalue λ if and only if $\alpha = \lambda$, $\beta = \lambda\alpha$, and $(\gamma_0 + \alpha\gamma_1 + \beta\gamma_2)\mu = \lambda\beta$. After some computation, we know that

$$\lambda^3 - \gamma_2 \mu \lambda^2 - \gamma_1 \mu \lambda - \gamma_0 \mu = 0, \tag{4}$$

which implies that three roots exist for each μ. That is, each eigenvalue of $-L$ corresponds to three eigenvalues of Γ.

Let μ_i, $i = 1, \cdots, n$, be the i^{th} eigenvalue of $-L$. Also let λ_{3i-2}, λ_{3i-1}, and λ_{3i}, $i = 1, \cdots, n$, be the eigenvalues of Γ

corresponding to μ_i. From Eq. (4), we can see that $\mu_j = 0$ implies that $\lambda_{3j-2} = \lambda_{3j-1} = \lambda_{3j} = 0$. It is straightforward to see that $-L$ has at least one zero eigenvalue with an associated eigenvector $\mathbf{1}$ since all its row sums are equal to zero. Therefore, we know that Γ has at least three zero eigenvalues.

From Eq. (4) we can also see that $-L$ has a simple zero eigenvalue if and only if Γ has exactly three zero eigenvalues. In addition, if $-L$ has a simple zero eigenvalue, denoted as $\mu_1 = 0$ without loss of generality, then there is only one linearly independent eigenvector p for $-L$ associated with eigenvalue zero. Note that $\mu_1 = 0$ implies that $\lambda_1 = \lambda_2 = \lambda_3 = 0$, which in turn implies that $\alpha = \beta = 0$. Therefore, there is only one linearly independent eigenvector $q = [p^T, \mathbf{0}^T, \mathbf{0}^T]^T$ for Γ associated with eigenalue zero. That is, the zero eigenvalue of Γ has geometricity equal to one. ■

Using this lemma we can prove the following main result.

Theorem 3.1: In the case of $\ell = 3$, consensus algorithm (2) achieves consensus asymptotically if and only if matrix Γ has exactly three zero eigenvalues and all the other eigenvalues have negative real parts.

Proof: (Sufficiency.) Noting that Γ has exactly three zero eigenvalues, we know that eigenvalue zero has geometric multiplicity equal to one from Lemma 3.1. As a result, we know that Γ can be written in Jordan canonical form as

$$\begin{aligned}\Gamma &= PJP^{-1}\\ &= [w_1, \cdots, w_{2n}]\cdot\\ &\begin{bmatrix} 0 & 1 & 0 & 0_{1\times(3n-3)}\\ 0 & 0 & 1 & 0_{1\times(3n-3)}\\ 0 & 0 & 0 & 0_{1\times(3n-3)}\\ 0_{(3n-3)\times 1} & 0_{(3n-3)\times 1} & 0_{(3n-3)\times 1} & J' \end{bmatrix} \begin{bmatrix} \nu_1^T\\ \vdots\\ \nu_{2n}^T \end{bmatrix},\end{aligned} \tag{5}$$

where $w_j \in I\!R^{3n}$, $j = 1, \cdots, 3n$, can be chosen to be the right eigenvectors or generalized eigenvectors of Γ, $\nu_j \in I\!R^{3n}$, $j = 1, \cdots, 3n$, can be chosen to be the left eigenvectors or generalized eigenvectors of Γ, and J' is the Jordan upper diagonal block matrix corresponding to $3n-3$ non-zero eigenvalues of Γ.

Without loss of generality, we choose $w_1 = [\mathbf{1}^T, \mathbf{0}^T, \mathbf{0}^T]^T$, $w_2 = [\mathbf{0}^T, \mathbf{1}^T, \mathbf{0}^T]^T$, and $w_3 = [\mathbf{0}^T, \mathbf{0}^T, \mathbf{1}^T]^T$, where it can be verified that w_1, w_2, and w_3 are an eigenvector and two generalized eigenvectors of Γ associated with eigenvalue 0 respectively. Noting that Γ has exactly three zero eigenvalues, denoted as $\lambda_1 = \lambda_2 = \lambda_3 = 0$ without loss of generality, we know that $-L$ has a simple zero eigenvalue, which in turn implies that there exists a nonnegative vector p such that $p^T L = 0$ and $p^T \mathbf{1} = 1$ as shown in [12]. It can be verified that $\nu_1 = [p^T, \mathbf{0}^T, \mathbf{0}^T]^T$, $\nu_2 = [\mathbf{0}^T, p^T, \mathbf{0}^T]^T$, and $\nu_3 = [\mathbf{0}^T, \mathbf{0}^T, p^T]^T$ are two generalized left eigenvectors and a left eigenvector of Γ associated with eigenvalue 0 respectively, where $\nu_j^T w_j = 1$, $j = 1, 2, 3$. Noting that eigenvalues λ_{3i-2}, λ_{3i-1} and λ_{3i}, $i = 2, \cdots, n$, have negative real parts, we see that

$$\begin{aligned}&\lim_{t\to\infty} e^{\Gamma t}\\ &= \lim_{t\to\infty} P e^{Jt} P^{-1}\\ &= P \lim_{t\to\infty} \begin{bmatrix} 1 & t & \frac{1}{2}t^2 & 0_{1\times(3n-3)}\\ 0 & 1 & t & 0_{1\times(3n-3)}\\ 0 & 0 & 1 & 0_{1\times(3n-3)}\\ 0_{(3n-3)\times 1} & 0_{(3n-3)\times 1} & 0_{(3n-3)\times 1} & e^{J't} \end{bmatrix} P^{-1}\\ &= \begin{bmatrix} \mathbf{1}p^T & t\mathbf{1}p^T & \frac{1}{2}t^2 p^T\\ 0_n & \mathbf{1}p^T & t\mathbf{1}p^T\\ 0_n & 0_n & \mathbf{1}p^T \end{bmatrix},\end{aligned}$$

where we have used the fact that $\lim_{t\to\infty} e^{J't} \to 0_{3n-3}$. Noting that $\xi(t) \to \Gamma\xi(0)$, we know that $\xi_i^{(k)} \to \xi_j^{(k)}$, $\forall i \neq j$, $k = 0, 1, 2$.

(Necessity.) Suppose that the sufficient condition that Γ has exactly three zero eigenvalues and all the other eigenvalues have negative real parts does not hold. Noting that Γ has at least three zero eigenvalues, the fact that the sufficient condition does not hold implies that Γ has either more than three zero eigenvalues or it has three zero eigenvalues but has at least another eigenvalue having positive real part. In either case, it can be verified that $\lim_{t\to\infty} e^{Jt}$ has a rank larger than three, which implies that $\lim_{t\to\infty} e^{\Gamma t}$ has a rank larger than three. Note that consensus is reached asymptotically if and only if $\lim_{t\to\infty} e^{\Gamma t} \to \begin{bmatrix} \mathbf{1}q^T\\ \mathbf{1}s^T\\ \mathbf{1}t^T \end{bmatrix}$, where q, s, and t are $n \times 1$ vectors. As a result, the rank of $\lim_{t\to\infty} e^{\Gamma t}$ cannot exceed three. This results in a contradiction. ■

Note that in the case of $\ell = 1$, having a (directed) spanning tree is a necessary and sufficient condition for consensus seeking. However, in the case of $\ell = 3$, having a (directed) spanning tree is only a necessary condition for consensus seeking, which is similar to the case of $l = 2$. Both the information exchange topology and values of γ_* will effect the convergence of the ℓ^{th} order consensus algorithm. In fact, in the case of $\ell = 3$, if consensus algorithm (2) achieves consensus asymptotically, we know that Γ has exactly three zero eigenvalues following Theorem 3.1. Therefore, we see that matrix $-L$ has a simple zero eigenvalue, which in turn implies that the information exchange topology has a (directed) spanning tree following Corollary 1 in [12].

From Eq. (4) we can see that γ_k, $k = 0, 1, 2$, plays an important role in the eigenvalues of Γ. Although in the case of $\ell = 2$, we know that if $-L$ has a simple zero eigenvalue and all the other eigenvalues are real and therefore negative (e.g., undirected connected information exchange topology), then consensus protocol (2) achieves consensus for arbitrary $\gamma_k > 0$, $k = 0, 1$ [5], this argument is no longer valid for the case of $\ell = 3$. However, for each given $-L$ whose graph has a (directed) spanning tree, by appropriately choosing γ_k, $k = 0, 1, 2$, we can guarantee that the conditions of Theorem 3.1 are satisfied.

To illustrate these points, consider the following simulation example. In Case 1, we let $\gamma_0 = 2$, $\gamma_1 = 1$, and $\gamma_2 = 2$.

Suppose that $\mu_j = -1$, where μ_j is an eigenvalue of $-L$. Then from Eq. (4) we see that $\lambda_{3j-2} = -2$, $\lambda_{3j-1} = i$, and $\lambda_{3j} = -i$, where λ_* is the eigenvalue of Γ corresponds to μ_j. As a result, consensus cannot be achieved. However, if for Case 2 we choose $\gamma_0 = 1$, $\gamma_1 = 2$, and $\gamma_2 = 3$, then consensus can be reached. Figure 1 shows the plots of $\xi_i^{(2)}$, $i = 1, \cdots, 4$, for Cases 1 and 2 with different γ_k, $k = 0, 1, 2$, values, where matrix L is given by

$$L_0 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}.$$

Note that the graph of L has a (directed) spanning tree. [1] But, clearly in Case 1 the consensus system is not stable, whereas in Case 2 it is stable. Thus, the gains γ_k must be chosen properly to ensure consensus is achieved.

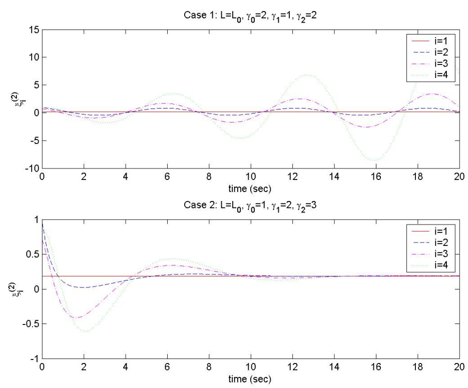

Fig. 1. Plots of $\xi_i^{(k)}$, $k = 2$, for Cases 1 and 2 with different γ_* values.

B. Setpoint tracking and consensus with a leader

In [13] the idea of a leader-node is introduced, whereby a single node is chosen that ignores all the other nodes, but continues to broadcast, and the controllability properties of the resulting graph are explored. In [14] it was shown how to modify the first-order consensus protocol to introduce setpoint tracking, but with a less stringent requirement than full state controllability (and not requiring that the leader ignore all the other nodes, though this is effectively what happens). The algorithm in [14] caused all the nodes to converge to the leader's setpoint. This is called *consensus with a leader*.

Though [14] considered only first-order consensus, in the same way for higher-order consensus we can modify Eq. (2) as follows:

$$u_i = -\sum_{j=1}^{n} g_{ij} k_{ij} [\sum_{k=0}^{\ell-1} \gamma_k (\xi_i^{(k)} - \xi_j^{(k)})] - \alpha_i (\xi_i^{(\ell-1)} - \xi_i^{(\ell-1)^*}), \tag{6}$$

for $i \in \{1, \cdots, n\}$, where $\xi_i^{(\ell-1)^*}$ is the local setpoint on node i. Using Eq. (6) we claim that if $\alpha_i = 0$ for all but node k, with $\alpha_k = 1$, (i.e., consensus with a leader) then all the nodes will converge to $\xi_i^{(l-1)} \to \xi_k^{(\ell-1)^*}$. Note that this assertion requires $\xi_k^{(\ell-1)^*}$ to be piecewise constant. If $\xi_k^{(\ell-1)^*}$ is time-varying it would be necessary to modify the term $\alpha_i(\xi_k^{(\ell-1)^*} - \xi_i^{(l-1)})$ in Eq. (6) using an internal model controller. The topics in this subsection are subjects of on-going research. However, in the next subsection we generalize these ideas further by extending them to include setpoints for all the derivatives, where the setpoints come from a reference model.

[1] In fact, the graph of L is itself a (directed) spanning tree in this case.

C. Model-reference consensus

Consider a prescribed reference dynamic model given by

$$\begin{aligned} \dot{\xi}_r^{(0)} &= \xi_r^{(1)} \\ &\vdots \\ \dot{\xi}_r^{(\ell-2)} &= \xi_r^{(\ell-1)} \\ \dot{\xi}_r^{(\ell-1)} &= u_r \end{aligned}$$

where $\xi_r^{(k)} \in I\!R^m$, $k = 0, 1, \cdots, \ell - 1$, are the reference states, and $u_r \in I\!R^m$ is the reference control input.

We say that a model reference consensus problem is solved if $\xi_i^{(k)} \to \xi_r^{(k)}$, $k = 0, \cdots, \ell - 1$, asymptotically and $\xi_i^{(k)} \to \xi_j^{(k)}$, $\forall i \neq j$, during the transition.

We propose a model-reference consensus algorithm:

$$\begin{aligned} u_i = &- \sum_{j=1}^{n} g_{ij} k_{ij} [\sum_{k=0}^{\ell-1} \gamma_k (\xi_i^{(k)} - \xi_j^{(k)})] \\ &- \eta \sum_{k=0}^{\ell-1} \gamma_k (\xi_i^{(k)} - \xi_r^{(k)}) + u_r \quad i \in \{1, \cdots, n\} \end{aligned} \tag{7}$$

where $\eta > 0$.

Let $\tilde{\xi}^{(k)} = \xi^{(k)} - \xi_r^{(k)}$, $k = 0, \cdots, \ell$. By applying consensus algorithm (7), Eq. (1) can be written in matrix form as

$$\begin{bmatrix} \dot{\tilde{\xi}}^{(0)} \\ \dot{\tilde{\xi}}^{(1)} \\ \vdots \\ \dot{\tilde{\xi}}^{(\ell-1)} \end{bmatrix} = (\Sigma \otimes I_m) \begin{bmatrix} \tilde{\xi}^{(0)} \\ \tilde{\xi}^{(1)} \\ \vdots \\ \tilde{\xi}^{(\ell-1)} \end{bmatrix}, \tag{8}$$

where

$$\Sigma = \begin{bmatrix} 0_n & I_n & 0_n & \cdots & 0_n \\ 0_n & 0_n & I_n & \cdots & 0_n \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0_n & 0_n & 0_n & \cdots & I_n \\ -\gamma_0 M & -\gamma_1 M & -\gamma_2 M & \cdots & -\gamma_{\ell-1} M \end{bmatrix},$$

with $M = L + \eta I_n$.

Letting μ_i and ρ_i be the i^{th} eigenvalue of $-L$ and $-M$ respectively, then we see that $\rho_i = \mu_i - \eta$. Following the argument of Theorem 3.1, we know that each eigenvalue of $-M$ corresponds to three eigenvalues of Σ. Letting ς_{3i-j},

$j = 1, 2, 3$, be the eigenvalue of Σ corresponding to ρ_i, then they are related by the following equation:

$$\varsigma^3 - \gamma_2 \rho \varsigma^2 - \gamma_1 \rho \varsigma - \gamma_0 \rho = 0. \tag{9}$$

Note that if $\mathrm{Re}(\varsigma_i) < 0$, $i = 1, \cdots, 3n$, that is, Σ is a stable matrix, then $\tilde{\xi}_i^{(k)} \to 0$ asymptotically, $k = 0, \cdots, \ell - 1$, which in turn implies that $\xi_i^{(k)} \to \xi_r^{(k)}$ asymptotically.

It is straightforward to see that $\mathrm{Re}(\rho_i) < 0$, $i = 1, \cdots, 3n$, due to the fact that $\mathrm{Re}(\mu_i) \leq 0$ and $\eta > 0$. Similar to Eq. (4), γ_k, $k = 0, 1, 2$, plays an important role in the eigenvalues of Σ in Eq. (9). Note that even if the information exchange topology does not have a (directed) spanning tree (e.g., the worse case of no information exchange between vehicles, that is, $L = 0_n$), it is still possible to choose γ_k, $k = 0, 1, 2$, such that all eigenvalues of Σ have negative real parts. However, having a (directed) spanning tree guarantees that $\xi_i^{(k)} \to \xi_j^{(k)}$, $k = 0, \cdots, \ell - 1$, $\forall i \neq j$ during the transition when $\xi_i^{(k)} \to \xi_r^{(k)}$.

To give a simulation example of model-reference consensus, let $u_r = \sin(t)$ and $\xi_r^{(k)}(0) = 0$, $k = 0, 1, 2$. Also let $\eta = 0.3$. Figure 2 shows the plots of $\xi_i^{(2)}$, $i = 1, \cdots, 4$, for Case 1, where $L = L_0$, and Case 2, where $L = 0_n$. Note that although $\xi_i^{(2)}$ approaches $\xi_r^{(2)}$ asymptotically in both Cases 1 and 2, we can see that $\xi_i^{(2)}$ stays close to $\xi_j^{(2)}$, $\forall i \neq j$, during the transition in Case 1 but not in Case 2.

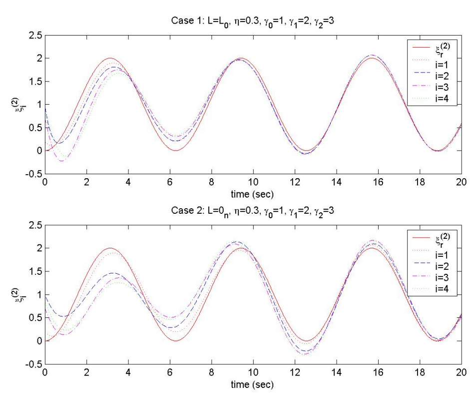

Fig. 2. Plots of $\xi_i^{(k)}$, $k = 2$, for Cases 1 and 2 with different information exchange topologies.

Also note that the value of η also has an effect on convergence. Let u_r and $\xi_r^{(k)}(0) = 0$ be defined the same as above. Also let $L = L_0$. Figure 3 shows the plots of $\xi_i^{(2)}$, $i = 1, \cdots, 4$, for Case 1, where $\eta = 0.3$, and Case 2, where $\eta = 2$. Note that $\xi_i^{(2)}$ does not approach $\xi_r^{(2)}$ in Case 1 due to small η. However, when we increase η to 2, convergence to $\xi_r^{(r)}$ is guaranteed in Case 2.

In the case of $\ell = 2$, if $L = 0_n$, then $\mathrm{Re}(\varsigma_i) < 0$ for any $\gamma_k > 0$, $k = 0, 1$, due to the fact that $\rho_i = -\eta$ is real. That is, the model reference consensus problem is solved for arbitrary $\gamma_k > 0$, $k = 0, 1$. However, in the case of $\ell = 3$, this argument is no longer valid and the gains γ_k must be chosen properly to ensure consensus is achieved.

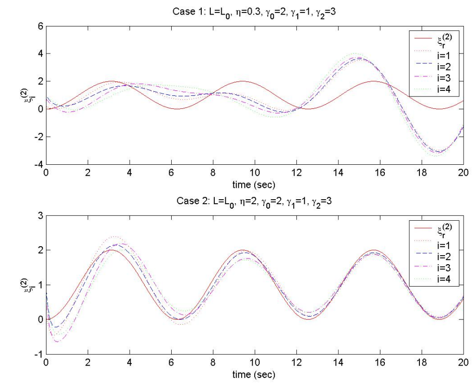

Fig. 3. Plots of $\xi_i^{(k)}$, $k = 2$, for Cases 1 and 2 with different η values.

IV. Multi-vehicle Coordination Example

Here we illustrate via simulation how the higher-order consensus ideas presented above can be used in a formation control scenario for multiple vehicles, whereby a desired separation is maintained between each vehicle, each vehicle travels at a common velocity, and a leader vehicle responds to setpoint commands in acceleration, which are communicated to the other vehicles through the consensus protocol. We assume there are five vehicles, communicating through a topology defined by the matrix

$$-L = \begin{bmatrix} -2 & 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 1 & 0 & -2 & 0 & 1 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & -1 \end{bmatrix}.$$

The equation running on each vehicle i is given by:

$$\begin{aligned} \dot{x}_i &= v_i \\ \dot{v}_i &= a_i \\ \dot{a}_i &= -\sum_{j=1}^{n} g_{ij} k_{ij} \{\gamma_0 [(x_i - \delta_i) - (x_j - \delta_j)] \\ &\quad + \gamma_1 (v_i - v_j) + \gamma_2 (a_i - a_j)\} - \alpha_i (a_i - a_i^*) \end{aligned}$$

where x_i denotes the position of vehicle i in two dimensions ($x_i \in R^2$), v_i denotes the velocity of vehicle i in two dimensions, and a_i denotes acceleration of vehicle i in two dimensions. The terms δ_i denote the desired formation separations (so that $(\delta_i - \delta_j)$ is the desired separation between vehicle i and vehicle j [6]), again in two dimensions (we used the same separation distances in both the x-axis and the y-axis). We used $\gamma_0 = 1, \gamma_1 = \gamma_2 = 3$, which, together with the fact that L defines a communication topology that contains a (directed) spanning tree, result in a convergent consensus process (actually, in this example every node is a (directed) spanning node, but it was not fully connected). Further, we let vehicle 1 be the "leader." That is, $\alpha_1 = 1$ and $\alpha_i = 0$ for $i \neq 1$. In the simulation initially all vehicles have a different starting position, starting velocity, and starting acceleration, in both x and y. The starting acceleration setpoint is $a_1^* = 0$.

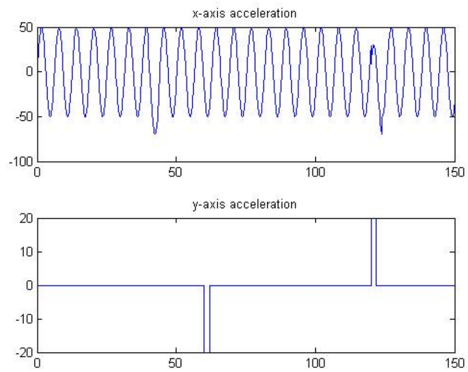

Fig. 4. The acceleration profiles used in the example.

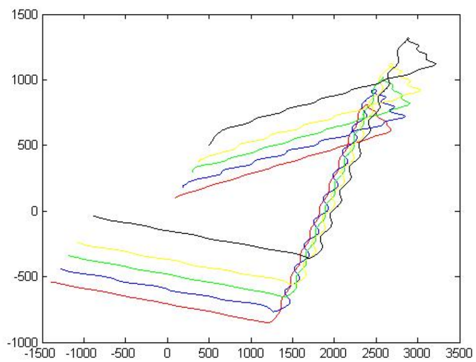

Fig. 5. Resulting $x - y$ motion.

Figure 4 shows the acceleration profile presented to vehicle 1 as a function of time for each axis. We see that the system is presented with a sinusoidal-varying acceleration in the x-axis with a negative "bumps" at 40 and 120 seconds. Along the y-axis the system is "bumped" at 60 and 120 seconds. Figure 5 shows the resulting motion in the $x - y$ plane. This makes it clear how the system maintains the desired separation between the vehicles (notice, however, that the scales are different on the two axes, which hides the fact that we specified the same vehicle-to-vehicle separations in each axis). We see that the higher-order consensus algorithms allow all the vehicles to respond to the acceleration setpoints received by vehicle 1, while maintaining their formation. This behavior can be seen to be similar to that of a flock of birds moving as a group in formation, but periodically having large changes in direction. It should be noted, however, that for the sinusoidal acceleration setpoint for the x-axis, the tracking is not error-free. Though all the vehicles track follow the same sinusoid, they can experience amplitude and phase delays. This can be corrected by incorporating an internal model in the tracking of the acceleration reference signal. As noted above, this is a topic for further research.

V. Conclusion

In this paper we have defined a class of ℓ^{th} order ($\ell >= 3$) consensus algorithms and have shown sufficient conditions under which each information variable and their higher-order derivatives converge to common values. We also introduced the idea of higher-order consensus with a leader and the concept of an ℓ^{th} order model-reference consensus problem, where each information variable and their high-order derivatives not only reach consensus but also converge to the solution of a prescribed dynamic model. Future research will focus on experimental application of these ideas, including demonstration of formation control via consensus on the MASNET testbed [3]. It also remains to study the implementation of consensus algorithms. Though convergence is a function of the eigenvalues of $\Gamma \otimes I_m$ and will not be impacted as the number of agents increases, the effects of delays in the communication between agents as well as the effect of time-varying communication topologies must be addressed.

References

[1] W. Ren, R. W. Beard, and E. Atkins, "A survey of consensus problems in multi-agent coordination," in *American Control Conference*. Portland, OR: AACC, June 2005, pp. 1859–1864.

[2] K. L. Moore and D. Lucarelli, "Consensus variable approach to decentralized adaptive scheduling," in *5th International Conference on Cooperative Control and Optimization*, Gainsville, FL, January 2005.

[3] Y. Chen and Z. Wang, "Formation control: A review and a new consideration," in *Proc. of the IEEE/RSJ Int. Conf. on Intelligent Robots and Systems*, Edmonton, Alberta, Canada, August 2005, pp. 3664–3669.

[4] K. L. Moore and Y. Chen, "Model-based approach to characterization of diffusion processes via distributed control of actuated sensor networks," in *The 1st IFAC Symposium on Telematics Applications in Automation and Robotics*. Helsinki University of Technology Espoo, Finland: IFAC, June 2004.

[5] W. Ren and E. Atkins, "Second-order consensus protocols in multiple vehicle systems with local interactions," in *Proc. of AIAA Guidance, Navigation, and Contr.*, San Francisco, CA, 2005, paper no. AIAA-2005-6238.

[6] W. Ren, "Consensus based formation control strategies for multi-vehicle systems," in *American Control Conference*, 2006, (submitted).

[7] R. A. Horn and C. R. Johnson, *Matrix Analysis*. Cambridge University Press, 1985.

[8] A. Jadbabaie, J. Lin, and A. S. Morse, "Coordination of groups of mobile autonomous agents using nearest neighbor rules," *IEEE Trans. Automat. Contr.*, vol. 48, no. 6, pp. 988–1001, June 2003.

[9] R. Olfati-Saber and R. M. Murray, "Consensus problems in networks of agents with switching topology and time delays," *IEEE Trans. Automat. Contr.*, vol. 49, no. 9, pp. 1520–1533, September 2004.

[10] ——, "Flocking with obstacle avoidance: Cooperation with limited communication in mobile networks," in *Proc. of Conf. on Decision Contr.*, Maui, Hawaii, December 2003, pp. 2022–2028.

[11] H. G. Tanner, A. Jadbabaie, and G. J. Pappas, "Stable flocking of mobile agents, part i: Fixed topology," in *Proc. of Conf. on Decision Contr.*, Maui, Hawaii, December 2003, pp. 2010–2015.

[12] W. Ren, R. W. Beard, and T. W. McLain, "Coordination variables and consensus building in multiple vehicle systems," in *Cooperative Control*, V. Kumar, N. E. Leonard, and A. S. Morse, Eds., vol. 309. Springer-Verlag Series: Lecture Notes in Control and Information Sciences, 2004, pp. 171–188.

[13] H. G. Tanner, "On the controllability of nearest neighbor interconnections," in *Proc. of Conf. on Decision Contr.*, Nassau, Bahamas, 2004.

[14] K. L. Moore and D. Lucarelli, "Force and constrained consensus among cooperating agents," in *2005 IEEE International Conference on Networking, Sensing, and Control*, Tuscon, AZ, March 2005.