AIAA Guidance, Navigation, and Control Conference and Exhibit
15 - 18 August 2005, San Francisco, California

AIAA 2005-6238

Second-order Consensus Protocols in Multiple Vehicle Systems with Local Interactions

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In this paper, a distributed coordination scheme with local interactions is studied for multiple vehicle systems. We introduce a second-order consensus protocol and derive necessary and/or sufficient conditions under which consensus can be reached in the context of uni-directional interaction topologies. The consensus protocol is then applied to achieve altitude alignment among a team of micro air vehicles as an illustrative example.

Nomenclature

h	Altitude, m
λ_*	Autopilot parameters
κ_*	Autopilot parameters
Subscript	
i	Variable number
Superscript	
c	Command

I. Introduction

Cooperative control for multiple vehicle systems has been a topic of significant interest in recent years. For cooperative control strategies to be successful, numerous issues must be addressed, among which the study of shared information in a group of vehicles facilitates the coordination of these vehicles. As a result, a critical problem for cooperative control is to design appropriate protocols and algorithms so that the group of vehicles can converge to a consistent view of the shared information in the presence of limited and unreliable information exchange and dynamically changing interaction topologies.

Convergence to a common value is called the consensus or agreement problem in the literature. Consensus problems have a history in computer science[1] and have recently been studied in the context of cooperative control of multiple vehicle systems.[2–8] Ref. 9 provides a survey of consensus problems in multi-agent coordination. Consensus protocols have potential applications in formation control problems for mobile robots, satellites, or spacecraft and cooperative timing or search missions for multiple unmanned air vehicles. For example, information consensus for dynamically evolving information was applied in Ref. 10 to formation flying of multiple space-based interferometers.

One approach to consensus relies on algebraic graph theory, in which graph topologies are connected with the algebraic properties of the corresponding graph matrices. In Ref. 2 information exchange techniques are studied to improve stability margins and formation accuracy of vehicle formations. In Ref. 3, sufficient conditions are given for consensus of the heading angles of a group of agents under undirected switching

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interaction topologies. In Ref. 4, average consensus problems are solved for a network of integrators using directed graphs. Using directed graphs, Refs. 7 and 8 show necessary and/or sufficient conditions for consensus of information under time-invariant and switching interaction topologies respectively.

Meanwhile, some other researchers make use of nonlinear mathematical tools to study consensus problems. In Ref. 5, a set-valued Lyapunov approach is used to consider consensus problems with uni-directional time-dependent communication links. In Ref. 11, nonlinear contraction theory is used to study synchronization and schooling applications, which are related to the consensus problems.

Optimality issues related to consensus problems are also studied in the literature. For example, in Ref. 12, the fastest distributed linear averaging (FDLA) problem are addressed in the context of consensus-seeking among multiple autonomous agents.

All the previously mentioned references except Ref. 10 focus on consensus protocols that take the form of first-order dynamics. In reality, equations of motion of a broad class of vehicles require second-order dynamic models. For example, some vehicle dynamics can be feedback linearized as double integrators, e.g. mobile robot dynamic models. In the case of first-order consensus protocols, the final consensus value is a constant. In contrast to the constant final consensus value, it might be proper to derive second-order consensus protocols such that some information states converge to a consistent value (e.g. position of the formation center) while others converge to another consistent value (e.g. velocity of the formation center). However, the extension of consensus protocols from first order to second order is nontrivial. In Refs. 10,13–16, formation keeping algorithms taking the form of second-order dynamics are addressed to guarantee attitude alignment, agreement of position deviations and velocities, and/or collision avoidance in a group of vehicles. However, each algorithm mentioned above assumes an undirected interaction topology. The case of directed interaction topologies is much more challenging than that of undirected interaction topologies. In this paper, we assume a directed interaction topology to take into account the general case where information flow may be uni-directional. The main contributions of this paper are to introduce a second-order consensus protocol and derive necessary and/or sufficient conditions under which consensus can be reached in the context of uni-directional interaction topologies.

II. Background and Preliminaries

It is natural to model interaction between vehicles by directed/undirected graphs. A digraph (directed graph) consists of a pair $(\mathcal{N}, \mathcal{E})$, where $\mathcal{N}$ is a finite nonempty set of nodes and $\mathcal{E} \in \mathcal{N}^2$ is a set of ordered pairs of nodes, called edges. As a comparison, the pairs of nodes in an undirected graph are unordered. If there is a directed edge from node v_i to node v_j, then v_i is defined as the parent node and v_j is defined as the child node. A directed path is a sequence of ordered edges of the form $(v_{i_1}, v_{i_2}), (v_{i_2}, v_{i_3}), \cdots$, where $v_{i_j} \in \mathcal{N}$, in a digraph. An undirected path in an undirected graph is defined accordingly. A digraph is called strongly connected if there is a directed path from every node to every other nodes. An undirected graph is called connected if there is a path between any distinct pair of nodes. A directed tree is a digraph, where every node, except the root, has exactly one parent. A spanning tree of a digraph is a directed tree formed by graph edges that connect all the nodes of the graph. We say that a graph has (or contains) a spanning tree if there exists a spanning tree being a subset of the graph. Note that the condition that a digraph has a spanning tree is equivalent to the case that there exists a node having a directed path to all the other nodes.

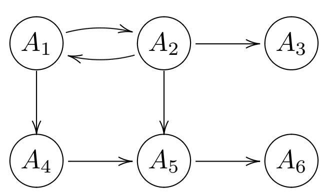

Figure 1. A digraph that has more than one possible spanning trees, but is not strongly connected.

The adjacency matrix $A = [a_{ij}]$ of a weighted digraph is defined as $a_{ii} = 0$ and $a_{ij} > 0$ if $(j, i) \in \mathcal{E}$ where $i \neq j$. The Laplacian matrix of the weighted digraph is defined as $L = [\ell_{ij}]$, where $\ell_{ii} = \sum_{j \neq i} a_{ij}$ and $\ell_{ij} = -a_{ij}$ where $i \neq j$. For an undirected graph, the Laplacian matrix is symmetric positive semi-definite.

As an example of a Laplacian matrix for a weighted digraph, the following matrix

$$L = \begin{bmatrix} 1.5 & -1.5 & 0 & 0 & 0 & 0 \\ -0.7 & 0.7 & 0 & 0 & 0 & 0 \\ 0 & -1.1 & 1.1 & 0 & 0 & 0 \\ -0.8 & 0 & 0 & 0.8 & 0 & 0 \\ 0 & -0.2 & 0 & -0.3 & 0.5 & 0 \\ 0 & 0 & 0 & 0 & -1.2 & 1.2 \end{bmatrix}$$

can be a valid Laplacian matrix corresponding to the digraph in Fig. 1.

Let $\mathcal{I} = \{1, 2, \cdots, n\}$. Let $\mathbf{1}$ and $\mathbf{0}$ denote the $n \times 1$ column vector of all ones and all zeros respectively. Let I_n denote the $n \times n$ identity matrix and $0_{m\times n}$ denote the $m \times n$ matrix with all zero entries. Let $M_n(\mathbb{R})$ represent the set of all $n \times n$ real matrices. Given a matrix $A = [a_{ij}] \in M_n(\mathbb{R})$, the digraph of A, denoted by $\Gamma(A)$, is the digraph on n nodes v_i, $i \in \mathcal{I}$, such that there is a directed edge in $\Gamma(A)$ from v_j to v_i if and only if $a_{ij} \neq 0$ (c.f. Ref. 17).

III. Consensus Protocols

A first-order consensus protocol is proposed in Refs. 3, 4, 6, 7 as

$$\dot{\xi}_i = -\sum_{j=1}^{n} g_{ij} k_{ij} (\xi_i - \xi_j), \qquad i \in \mathcal{I} \tag{1}$$

where $\xi_i \in \mathbb{R}$, $k_{ij} > 0$, $g_{ii} \triangleq 0$, and $g_{ij} = 1$ if information flows from vehicle j to vehicle i and 0 otherwise, $\forall i \neq j$. The adjacency matrix A of the interaction topology is defined accordingly as $a_{ii} = 0$ and $a_{ij} = g_{ij}k_{ij}$, $\forall i \neq j$.

Eq. (1) can be written in matrix form as

$$\dot{\xi} = -L\xi,$$

where $\xi = [\xi_1, \cdots, \xi_n]^T$, and $L = [\ell_{ij}]$ is the Laplacian matrix with $\ell_{ii} = \sum_{j\neq i} g_{ij}k_{ij}$ and $\ell_{ij} = -g_{ij}k_{ij}$, $\forall i \neq j$.

The final consensus value using Eq. (1) is given by $\xi^* = \sum_{i=1}^{n} \alpha_i \xi_i(0)$, where $\alpha_i \geq 0$ and $\sum_{i=1}^{n} \alpha_i = 1$.[7]

Taking into account second-order vehicle dynamics, we propose the following second-order consensus protocol:

$$\begin{aligned} \dot{\xi}_i &= \zeta_i \\ \dot{\zeta}_i &= -\sum_{j=1}^{n} g_{ij} [k_{ij}(\xi_i - \xi_j) + \gamma k_{ij} (\zeta_i - \zeta_j)], \qquad i \in \mathcal{I} \end{aligned} \tag{2}$$

where $\xi_i \in \mathbb{R}$, $\zeta_i \in \mathbb{R}$, $k_{ij} > 0$, $\gamma > 0$, $g_{ii} \triangleq 0$, and $g_{ij} = 1$ if information flows from vehicle j to vehicle i and 0 otherwise, $\forall i \neq j$.

Note that consensus protocols (1) and (2) are distributed in the sense that each vehicle only needs information from its (possibly time-varying) local neighbors. The goal of consensus protocol (2) is to guarantee that $|\xi_i - \xi_j| \to 0$ and $|\zeta_i - \zeta_j| \to 0$ as $t \to \infty$. In the case that ξ_i and ζ_i denote the position and velocity of the i^{th} vehicle respectively, Eq. (2) represents the motion of that vehicle.

Let $\xi = [\xi_1, \cdots, \xi_n]^T$ and $\zeta = [\zeta_1, \cdots, \zeta_n]^T$. Eq. (2) can be written in matrix form as

$$\begin{bmatrix} \dot{\xi} \\ \dot{\zeta} \end{bmatrix} = \Gamma \begin{bmatrix} \xi \\ \zeta \end{bmatrix}, \tag{3}$$

where

$$\Gamma = \begin{bmatrix} 0_{n\times n} & I_n \\ -L & -\gamma L \end{bmatrix}.$$

IV. Convergence Analysis for the Second-order Consensus Protocol

In this paper, we focus on the convergence analysis for consensus protocol (2) under a time-invariant interaction topology. The convergence analysis for consensus protocol (2) under time-varying interaction topologies will be addressed in future work.

Given a block matrix

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix},$$

it is known that $\det(M) = \det(AD - CB)$ if A and C commute, where $\det(\cdot)$ denotes the determinant of a matrix.

To find the eigenvalues of Γ, we can solve the equation $\det(\lambda I_{2n} - \Gamma) = 0$, where $\det(\lambda I_{2n} - \Gamma)$ is the characteristic polynomial of matrix Γ. Note that

$$\begin{aligned} \det(\lambda I_{2n} - \Gamma) &= \det\left(\begin{bmatrix} \lambda I_n & -I_n \\ L & \lambda I_n + \gamma L \end{bmatrix} \right) \\ &= \det(\lambda^2 I_n + (1+\gamma\lambda)L). \end{aligned} \tag{4}$$

Also note that

$$\det(\lambda I_n + L) = \prod_{i=1}^{n} (\lambda - \mu_i), \tag{5}$$

where μ_i is the i^{th} eigenvalue of $-L$.

By comparing Eqs. (4) and (5), we see that

$$\det\left(\lambda^2 I_n + (1+\gamma\lambda)L\right) = \prod_{i=1}^{n} (\lambda^2 - (1+\gamma\lambda)\mu_i),$$

which implies that the roots of Eq. (4) can be obtained by solving $\lambda^2 = (1+\gamma\lambda)\mu_i$. Therefore, it is straightforward to see that the eigenvalues of Γ are given by

$$\begin{aligned} \lambda_{i+} &= \frac{\gamma\mu_i + \sqrt{\gamma^2\mu_i^2 + 4\mu_i}}{2} \\ \lambda_{i-} &= \frac{\gamma\mu_i - \sqrt{\gamma^2\mu_i^2 + 4\mu_i}}{2}, \end{aligned} \tag{6}$$

where λ_{i+} and λ_{i-} are called eigenvalues of Γ that are associated with μ_i.

From Eq. (6), we can see that Γ has $2m$ zero eigenvalues if and only if $-L$ has m zero eigenvalues. It is straightforward to see that $-L$ has at least one zero eigenvalue since all its row sums are equal to zero. Therefore, we know that Γ has at least two zero eigenvalues. Without loss of generality, we let $\lambda_{1+} = \lambda_{1-} = 0$. In addition, we know that all non-zero eigenvalues of $-L$ have negative real parts from the Gersgorin disc theorem.[17]

We have the following lemma regarding a necessary and sufficient condition for information consensus using consensus protocol (2).

Lemma IV.1 *Consensus protocol* (2) *achieves consensus asymptotically if and only if matrix* Γ *has exactly two zero eigenvalues and all the other eigenvalues have negative real parts. Specifically,* $\xi \to \mathbf{1}p^T\xi(0) + t\mathbf{1}p^T\zeta(0)$ *and* $\zeta \to \mathbf{1}p^T\zeta(0)$, *where* p *is a nonnegative left eigenvector of* $-L$ *associated with eigenvalue* 0 *and* $p^T\mathbf{1} = 1$.

Proof: (Sufficiency.) Noting that Γ has two exactly zero eigenvalues, we can verify that eigenvalue zero has geometric multiplicity equal to one. As a result, we know that Γ can be written in Jordan canonical form as

$$\begin{aligned} \Gamma &= PJP^{-1} \\ &= [w_1, \cdots, w_{2n}] \begin{bmatrix} 0 & 1 & 0_{1\times(2n-2)} \\ 0 & 0 & 0_{1\times(2n-2)} \\ 0_{(2n-2)\times 1} & 0_{(2n-2)\times 1} & J' \end{bmatrix} \begin{bmatrix} \nu_1^T \\ \vdots \\ \nu_{2n}^T \end{bmatrix}, \end{aligned} \tag{7}$$

where w_j, $j = 1, \cdots, 2n$, can be chosen to be the right eigenvectors or generalized eigenvectors of Γ, ν_j^T, $j = 1, \cdots, 2n$, can be chosen to be the left eigenvectors or generalized eigenvectors of Γ, and J' is the Jordan upper diagonal block matrix corresponding to non-zero eigenvalues λ_{i+} and λ_{i-}, $i = 2, \cdots, n$.

Without loss of generality, we choose $w_1 = [\mathbf{1}^T, \mathbf{0}^T]^T$ and $w_2 = [\mathbf{0}^T, \mathbf{1}^T]^T$, where it can be verified that w_1 and w_2 are an eigenvector and generalized eigenvector of Γ associated with eigenvalue 0 respectively. Noting that Γ has exactly two zero eigenvalues, we know that $-L$ has a simple zero eigenvalue, which in turn implies that there exists a nonnegative vector p such that $p^T L = 0$ and $p^T \mathbf{1} = 1$ as shown in Ref. 7. It can be verified that $\nu_1 = [p^T, \mathbf{0}^T]^T$ and $\nu_2 = [\mathbf{0}^T, p^T]^T$ are a generalized left eigenvector and left eigenvector of Γ associated with eigenvalue 0 respectively, where $\nu_1^T w_1 = 1$ and $\nu_2^T w_2 = 1$. Noting that eigenvalues λ_{i+} and λ_{i-}, $i = 2, \cdots, n$, have negative real parts, we see that

$$
\begin{aligned}
\lim_{t\to\infty} e^{\Gamma t} &= \lim_{t\to\infty} P e^{Jt} P^{-1} \\
&= P \lim_{t\to\infty} \begin{bmatrix} 1 & t & 0_{1\times(2n-2)} \\ 0 & 1 & 0_{1\times(2n-2)} \\ 0_{(2n-2)\times 1} & 0_{(2n-2)\times 1} & e^{J't} \end{bmatrix} P^{-1} \\
&= \begin{bmatrix} \mathbf{1}p^T & t\mathbf{1}p^T \\ 0_{n\times n} & \mathbf{1}p^T \end{bmatrix},
\end{aligned}
$$

where we have used the fact that $\lim_{t\to\infty} e^{J't} \to 0_{(2n-2)\times(2n-2)}$.

Noting that as $t \to \infty$

$$
\begin{bmatrix} \xi(t) \\ \zeta(t) \end{bmatrix} \to \begin{bmatrix} \mathbf{1}p^T & t\mathbf{1}p^T \\ 0_{n\times n} & \mathbf{1}p^T \end{bmatrix} \begin{bmatrix} \xi(0) \\ \zeta(0) \end{bmatrix},
$$

we see that $\xi(t) \to \mathbf{1}p^T\xi(0) + t\mathbf{1}p^T\zeta(0)$ and $\zeta(t) \to \mathbf{1}p^T\zeta(0)$ as $t \to \infty$. As a result, we know that $|\xi_i(t) - \xi_j(t)| \to 0$ and $|\zeta_i(t) - \zeta_j(t)| \to 0$ as $t \to \infty$. That is, consensus is achieved for the group of vehicles.

(Necessity.) Suppose that the sufficient condition that Γ has exactly two zero eigenvalues and all the other eigenvalues have negative real parts does not hold. Noting that Γ has at least two zero eigenvalues, the fact that the sufficient condition does not hold implies that Γ has either more than two zero eigenvalues or it has two zero eigenvalues but has at least another eigenvalue having positive real part. In either case, it can be verified that $\lim_{t\to\infty} e^{Jt}$ has a rank larger than two, which implies that $\lim_{t\to\infty} e^{\Gamma t}$ has a rank larger than two. Note that consensus is reached asymptotically if and only if $\lim_{t\to\infty} e^{\Gamma t} \to \begin{bmatrix} \mathbf{1}p^T \\ \mathbf{1}q^T \end{bmatrix}$, where p and q are $n \times 1$ vectors. As a result, the rank of $\lim_{t\to\infty} e^{\Gamma t}$ cannot exceed two. This results in a contradiction. ■

If all non-zero eigenvalues of $-L$ are real and therefore negative, it is straightforward to verify that all non-zero eigenvalues of Γ have negative real parts following Eq. (6). In the general case, some non-zero eigenvalues of Γ may have positive real parts even if all non-zero eigenvalues of $-L$ have negative real parts as shown in the following examples.

We consider several cases as follows.

Case 1: Interaction Topology Having Separated Subgroups

In the case that the interaction topology has separated subgroups as shown in Fig. 2, consensus cannot be achieved for the team of vehicles since the information states from different separated groups do not affect one another. In fact, we also know that $-L$ has at least two zero eigenvalues in this case,[7] which in turn implies that Γ has at least four zero eigenvalues.

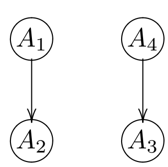

Figure 2. A digraph that has separated subgroups.

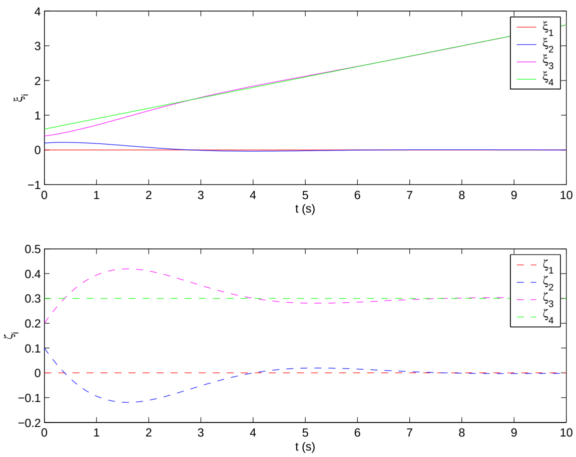

Figure 3. Evolution of the information states under the interaction topology given by Fig. 2.

Hereafter we assume that $k_{ij} = 1$ and $\gamma = 1$ in Eq. (2) unless explicitly mentioned. In addition, we let $\xi_i(0) = 0.2(i-1)$ and $\zeta_i(0) = 0.1(i-1)$, $i = 1, \cdots, 4$. Fig. 3 shows the evolution of the information states ξ_i and ζ_i, $i = 1, \cdots, 4$, using consensus protocol (2) under the interaction topology given by Fig. 2. Note that A_1 and A_2 reach consensus, and A_3 and A_4 also reach consensus although the whole group cannot reach consensus.

Case 2: Interaction Topology Having Multiple Leaders

In the case that the interaction topology has multiple leaders as shown in Fig. 4, where both A_1 and A_4 are leaders, consensus cannot be achieved for the team of vehicles since each leader's information state is not affected by any other vehicle's information state in the team. Noting that $-L$ has at least two rows with all zero entries in this case, we know that $-L$ has at least two zero eigenvalues, which in turn implies that Γ has at least four zero eigenvalues.

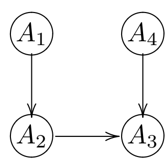

Figure 4. A digraph that has multiple leaders.

Fig. 5 shows the evolution of the information states ξ_i and ζ_i, $i = 1, \cdots, 4$, using the consensus protocol (2) under the interaction topology given by Fig. 4. Note that only A_1 and A_2 reach consensus.

Case 3: Connected Undirected Interaction Topology

If the interaction topology is undirected as shown in Fig. 6, we know that the graph Laplacian L is symmetric positive semi-definite, which implies that all eigenvalues of L are real. Therefore, all non-zero eigenvalues of Γ have negative real parts.

In the case of undirected graphs, graph Laplacian L has a simple zero eigenvalue if and only if the graph is connected. Therefore, we know that consensus is achieved asymptotically if and only if the undirected graph is connected.

Fig. 7 shows the evolution of the information states ξ_i and ζ_i, $i = 1, \cdots, 4$, using the consensus protocol (2) under the interaction topology given by Fig. 6.

Case 4: leader-follower Interaction Topology

In the case that the interaction topology is a leader-follower one as shown in Fig. 8, it is straightforward to see that L can be written as an upper diagonal matrix by permutation transformations. As a result, we know that zero is a simple eigenvalue of L and all non-zero eigenvalues are real. Therefore, we know that consensus is achieved asymptotically in the case of leader-follower interaction topologies.

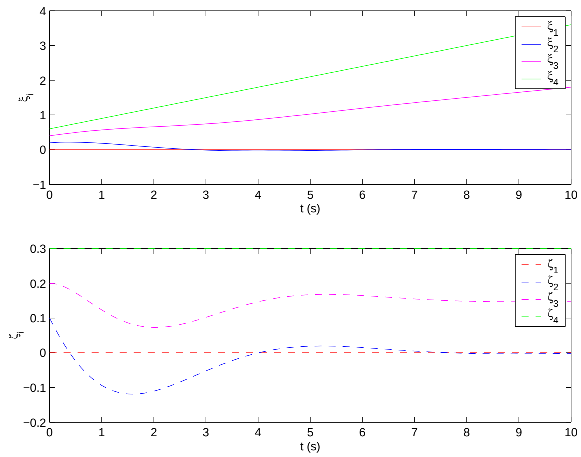

Figure 5. Evolution of the information states under the interaction topology given by Fig. 4.

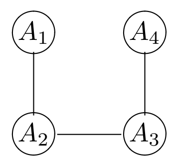

Figure 6. A connected undirected graph.

Fig. 9 shows the evolution of the information states ξ_i and ζ_i, $i = 1, \cdots, 4$, using the consensus protocol (2) under the interaction topology given by Fig. 8.

Case 5: Interaction Topology Having a Spanning Tree

Note that the connected undirected topology and the leader following topology can be thought of as special cases of an interaction topology having a spanning tree.

In the case that the interaction topology has a spanning tree as shown in Fig. 10, consensus may not be achieved as in the case where the consensus protocol is given by Eq. (1). However, having a spanning tree is a necessary condition for information consensus as will be shown below.

Fig. 11 and 12 show the evolution of the information states ξ_i and ζ_i, $i = 1, \cdots, 4$, using the consensus protocol (2) under the interaction topology given by Fig. 10 with $\gamma = 1$ and $\gamma = 0.4$ respectively. Note that consensus cannot be reached in the case that $\gamma = 0.4$. Unlike the previous cases where convergence of the consensus protocol does not depend upon γ, consensus may not be reached in the general case where the interaction topology has a spanning tree other than Cases 3 and 4 if γ is too small.

By comparing Figs. 8 and 10, we see that more interactions are involved in Fig. 10 than in Fig. 8 in the sense that A_3 sends information to A_1 in Fig. 10. However, while the consensus protocol converges under the interaction topology given by Fig. 8 for any $\gamma > 0$, the consensus protocol cannot converge under the interaction topology given by Fig. 10 if γ is too small. This is somewhat contradictory to our intuition in the sense that more interactions may lead to instability for the whole group.

In the special case that all eigenvalues of L are real, we have the following lemma.

Lemma IV.2 *If $-L$ has a simple zero eigenvalue and all the other eigenvalues are real, consensus protocol* (2) *achieves consensus for any $\gamma > 0$.*

To show that having a spanning tree is a necessary condition for information consensus, we need the following lemma.

Lemma IV.3 [7] *The graph Laplacian of a directed weighted graph has a simple zero eigenvalue if and only if the graph has a spanning tree.*

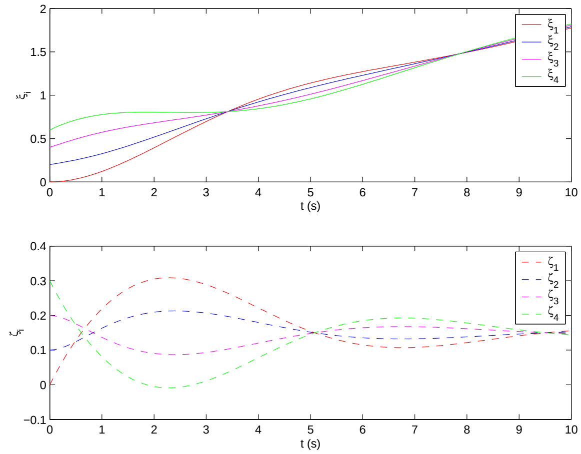

Figure 7. Evolution of the information states under the interaction topology given by Fig. 6.

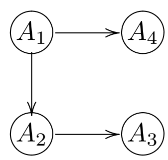

Figure 8. A digraph that has a leader following topology.

We have the following necessary condition for information consensus.

Theorem IV.1 *Consensus protocol* (2) *achieves consensus asymptotically only if the interaction topology has a spanning tree.* [a]

Proof: If consensus protocol (2) achieves consensus asymptotically, we know that Γ has exactly two zero eigenvalues following Lemma IV.1. Therefore, we see that matrix L has a simple zero eigenvalue, which in turn implies that the interaction topology has a spanning tree following Lemma IV.3. ■

Next, we show a sufficient condition for information consensus.

Theorem IV.2 *Consensus protocol* (2) *achieves consensus asymptotically if the interaction topology has a spanning tree and*

$$\gamma > \max_{i=2,\cdots,n} \sqrt{\frac{2}{|\mu_i| \cos(\frac{\pi}{2} - \tan^{-1} \frac{-\mathrm{Re}(\mu_i)}{\mathrm{Im}(\mu_i)})}}, \tag{8}$$

where μ_i, $i = 2, \cdots, n$, *are the non-zero eigenvalues of* $-L$, *and* $\mathrm{Re}(\cdot)$ *and* $\mathrm{Im}(\cdot)$ *represent the real and imaginary parts of a number respectively.*

Proof: If the interaction topology has a spanning tree, we know that $-L$ has one zero eigenvalue and all the other eigenvalues have negative real parts. Therefore, we know that Γ has two zero eigenvalues. It is left to show that non-zero eigenvalues of Γ have negative real parts. If inequality (8) is true, we know that λ_{i+} and λ_{i-}, $i = 2, \cdots, n$, have negative real parts following the proof of Theorem 6 in Ref. 18, where λ_{i+} and λ_{i-} are eigenvalues of Γ associated with μ_i. As a result, we see that consensus can be achieved asymptotically from Lemma IV.1. ■

We also have the following lemma regarding the final consensus value.

[a] As a comparison, the first-order consensus protocol (1) achieves consensus asymptotically if and only if the interaction topology has a spanning tree (see Ref. 7).

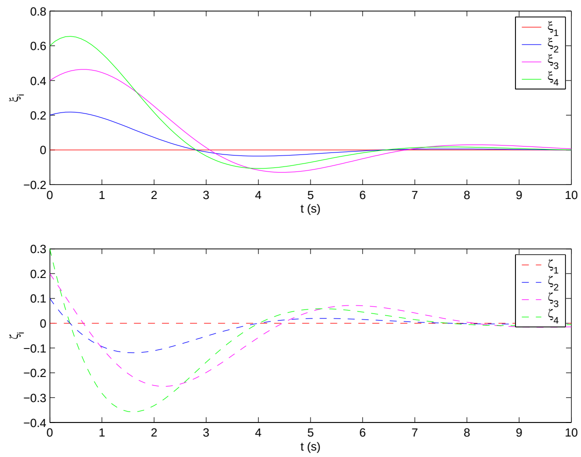

Figure 9. Evolution of the information states under the interaction topology given by Fig. 8.

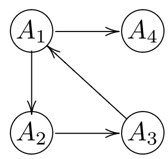

Figure 10. A digraph that has a spanning tree.

Lemma IV.4 *Suppose that* Γ *has two zero eigenvalues and all the other eigenvalues have negative real parts. If* $\zeta_i(0) = 0$, $i \in \mathcal{I}$, *then as* $t \to \infty$, $\xi_i(t) \to \sum_{i=1}^n p_i \xi_i(0)$ *and* $\zeta_i(t) \to 0$, *where* $i \in \mathcal{I}$ *and* $p = [p_1, \cdots, p_n]^T$ *is a nonnegative left eigenvector of* $-L$ *associated with eigenvalue* 0 *satisfying* $\sum_{i=1}^n p_i = 1$. *In addition, if* $\zeta_i(0) = 0$, $i \in \mathcal{I}_L$, *where* $\mathcal{I}_L$ *denotes the set of vehicles that have a directed path to all the other vehicles in the interaction topology, then* $\xi_i(t) \to \sum_{i \in \mathcal{I}_L} p_i \xi_i(0)$ *and* $\zeta_i(t) \to 0$, $i \in \mathcal{I}$, *as* $t \to \infty$.

Proof: The first part of the lemma follows directly from the fact that $\xi(t) \to \mathbf{1}p^T\xi(0) + t\mathbf{1}p^T\zeta(0)$ and $\zeta(t) \to \mathbf{1}p^T\zeta(0)$ as $t \to \infty$.

For the second part of the lemma, we note that $p_i > 0$ if vehicle i has a directed path to all the other vehicles in the interaction topology and $p_i = 0$ otherwise.[7] As a result, we know that $\xi_i(t) \to \sum_{i \in \mathcal{I}_L} p_i \xi_i(0) + t \sum_{i \in \mathcal{I}_L} p_i \zeta_i(0)$ and $\zeta_i(t) \to \sum_{i \in \mathcal{I}_L} p_i \zeta_i(0)$ and the second part of the lemma is proved. ■

Note that $\xi \to \mathbf{1}p^T\xi(0) + t\mathbf{1}p^T\zeta(0)$ and $\zeta \to \mathbf{1}p^T\zeta(0)$ with consensus protocol (2). Under some circumstances, it might be desirable that $\xi \to \mathbf{1}q^T$ and $\zeta \to \mathbf{0}$, where q is an $n \times 1$ vector. For example, in formation stabilization applications, we want each vehicle to agree on their *a priori* unknown fixed formation center, which has a constant position and zero velocity. In this case, we propose the following second-order consensus protocol:

$$\begin{aligned} \dot{\xi}_i &= \zeta_i \\ \dot{\zeta}_i &= -\alpha\zeta_i - \sum_{j=1}^n g_{ij}[k_{ij}(\xi_i - \xi_j) + \gamma k_{ij}(\zeta_i - \zeta_j)], \end{aligned} \tag{9}$$

where $\alpha > 0$.

The analysis for consensus protocol (9) is similar to that for consensus protocol (2) and is omitted for simplicity.

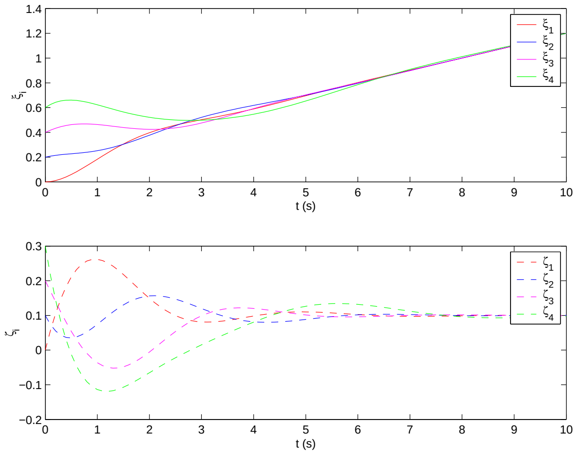

Figure 11. Evolution of the information states under the interaction topology given by Fig. 10 with $\gamma = 1$.

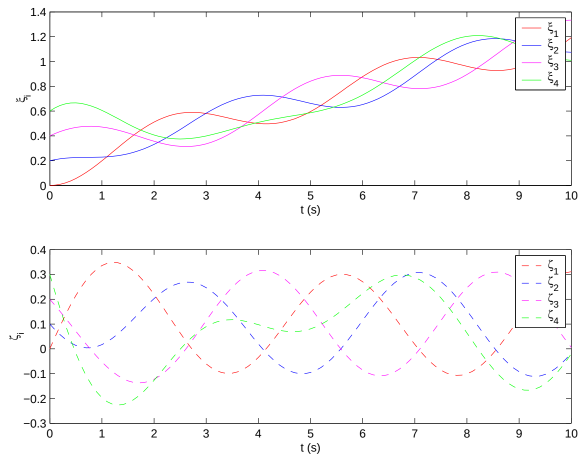

Figure 12. Evolution of the information states under the interaction topology given by Fig. 10 with $\gamma = 0.4$.

V. Illustrative Example

In this section, we apply the second-order consensus protocol to achieve altitude alignment among multiple micro unmanned air vehicles.

Let h_i denote the altitude of the i^{th} unmanned air vehicle (UAV). For UAVs equipped with efficient low-level autopilots, the resulting UAV/autopilot models are assumed to be second order for altitude hold.[19,20]

For a fixed-wing micro air vehicle, the simplified equation of motion for altitude is given by

$$\ddot{h}_i = -\lambda_{\dot{h}i}\dot{h}_i + \lambda_{hi}(h_i^c - h_i), \tag{10}$$

where h_i^c is the altitude command to the low-level controllers, and λ_* are positive constants.[19]

For a rotary-wing micro air vehicle, the simplified equation of motion for altitude is given by

$$\ddot{h}_i = \kappa_{\dot{h}i}(\dot{h}_i^c - \dot{h}_i) \tag{11}$$

where $\dot{h}_i^c$ is the vertical velocity command to the low-level controllers, and $\kappa_{\dot{h}i}$ is a positive constant.[20]

Let ν_i be defined as

$$\nu_i = -\alpha\dot{h}_i - \sum_{j=1}^{n} g_{ij}k_{ij}(h_i - h_j) - \sum_{j=1}^{n} g_{ij}\gamma k_{ij}(\dot{h}_i - \dot{h}_j),$$

where $\alpha \geq 0$, $k_{ij} > 0$, $\gamma > 0$, and g_{ij} is defined the same as in Eq. (2).

For Eq. (10), we propose the altitude command as

$$h_i^c = h_i + \frac{\lambda_{\dot{h}i}}{\lambda_{hi}}\dot{h}_i + \frac{1}{\lambda_{hi}}\nu_i. \tag{12}$$

For Eq. (11), we propose the vertical velocity command as

$$\dot{h}_i^c = \dot{h}_i + \frac{1}{\kappa_{\dot{h}i}}\nu_i. \tag{13}$$

In the following, we only consider altitude alignment for multiple rotary-wing micro air vehicles with control law (13). Results for multiple fixed-wing micro air vehicles with control law (12) are similar. The interaction topology for the micro air vehicles is given by Fig. 13, where a directed edge from the jth vehicle to the ith vehicle means that the ith vehicle can obtain h_j and $\dot{h}_j$ from the jth vehicle through a uni-directional communication link. Note that Fig. 13 has a spanning tree.

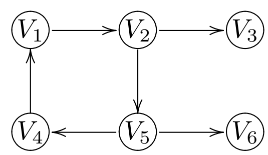

Figure 13. The interaction topology between the six micro air vehicles.

We assume that $\kappa_{\dot{h}i} = 1$, and the vertical velocity command is saturated and satisfies $|\dot{h}_i^c| \leq 0.5$ m/s.

In the first case, we let $\alpha = 0$, $k_{ij} = 1$, and $\gamma = 1$, which guarantees that Γ has two zero eigenvalues and all the other eigenvalues have negative real parts. Fig. 14 shows the altitudes and vertical velocities of each vehicle. Note that altitude is aligned between those vehicles. Fig. 15 shows the vertical velocity commands of each vehicle.

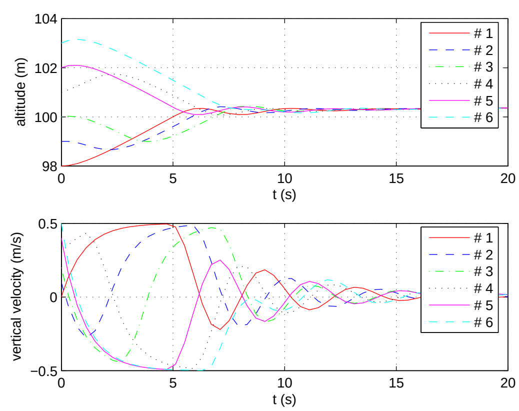

Figure 14. Altitudes and vertical velocities of each vehicle with $\gamma = 1$.

As a comparison, we let $\alpha = 0$, $k_{ij} = 1$, and $\gamma = 0.1$ in the second case. It can be shown that two eigenvalues of Γ have positive real parts, which implies that consensus cannot be achieved. Fig. 16 shows the altitudes and vertical velocities of each vehicle. Note that altitude cannot be aligned between those vehicles in this case. Fig. 17 shows the vertical velocity commands of each vehicle.

VI. Conclusion

We have proposed a second-order protocol for information consensus among multiple vehicles. We have also shown necessary and/or sufficient conditions under which consensus can be achieved in the context of uni-directional interaction topologies. The second-order consensus protocol has been applied to align the altitudes of multiple rotary-wing micro air vehicles in a distributed manner as a proof of concept.

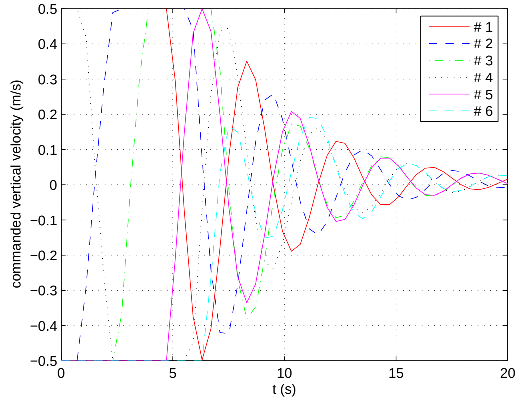

Figure 15. Commanded vertical velocities of each vehicle with $\gamma = 1$.

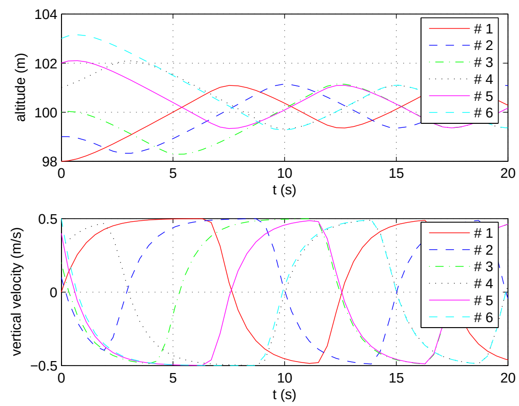

Figure 16. Altitudes and vertical velocities of each vehicle with $\gamma = 0.1$.

Acknowledgments

This research is supported by the Army Research Office through the MAV MURI Program (Grant No. ARMY-W911NF0410176) with Technical Monitor as Dr. Gary Anderson.

References

[1] Lynch, N. A., *Distributed Algorithms*, Morgan Kaufmann Publishers, Inc., San Francisco, California, 1996.

[2] Fax, J. A. and Murray, R. M., "Information flow and cooperative control of vehicle formations," *IEEE Transactions on Automatic Control*, Vol. 49, No. 9, September 2004, pp. 1465–1476.

[3] Jadbabaie, A., Lin, J., and Morse, A. S., "Coordination of Groups of Mobile Autonomous Agents Using Nearest Neighbor Rules," *IEEE Transactions on Automatic Control*, Vol. 48, No. 6, June 2003, pp. 988–1001.

[4] Olfati-Saber, R. and Murray, R. M., "Consensus problems in networks of agents with switching topology and time-delays," *IEEE Transactions on Automatic Control*, Vol. 49, No. 9, September 2004, pp. 1520–1533.

[5] Moreau, L., "Stability of Multi-agent Systems with Time-dependent Communication Links," *IEEE Transactions on Automatic Control*, Vol. 50, No. 2, February 2005, pp. 169–182.

[6] Lin, Z., Broucke, M., and Francis, B., "Local Control Strategies for Groups of Mobile Autonomous Agents," *IEEE Transactions on Automatic Control*, Vol. 49, No. 4, 2004, pp. 622–629.

[7] Ren, W., Beard, R. W., and McLain, T. W., "Coordination Variables and Consensus Building in Multiple Vehicle Systems," *Cooperative Control: A Post-Workshop Volume 2003 Block Island Workshop on Cooperative Control*, edited by V. Kumar, N. E. Leonard, and A. S. Morse, Vol. 309, Springer-Verlag Series: Lecture Notes in Control and Information Sciences, 2004, pp. 171–188.

[8] Ren, W. and Beard, R. W., "Consensus Seeking in Multiagent Systems Under Dynamically Changing Interaction Topologies," *IEEE Transactions on Automatic Control*, Vol. 50, No. 5, May 2005, pp. 655–661.

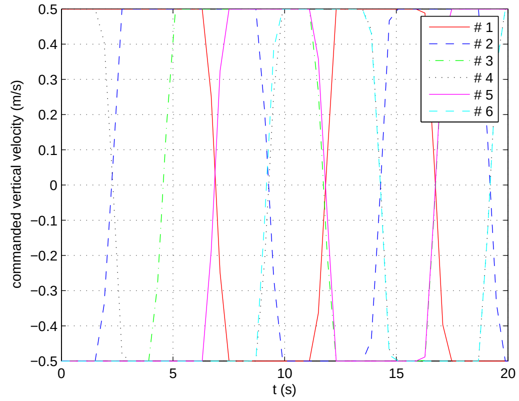

Figure 17. Commanded vertical velocities of each vehicle with $\gamma = 0.1$.

[9]Ren, W., Beard, R. W., and Atkins, E. M., "A Survey of Consensus Problems in Multi-agent Coordination," *Proceedings of the American Control Conference*, Portland, OR, June 2005.

[10]Ren, W. and Beard, R. W., "Decentralized Scheme for Spacecraft Formation Flying via the Virtual Structure Approach," *AIAA Journal of Guidance, Control, and Dynamics*, Vol. 27, No. 1, January-February 2004, pp. 73–82.

[11]Slotine, J.-J. E. and Wang, W., "A Study of Synchronization and Group Cooperation Using Partial Contraction Theory," *Cooperative Control: A Post-Workshop Volume 2003 Block Island Workshop on Cooperative Control*, edited by V. Kumar, N. E. Leonard, and A. S. Morse, Vol. 309, Springer-Verlag Series: Lecture Notes in Control and Information Sciences, 2004, pp. 207–228.

[12]Xiao, L. and Boyd, S., "Fast Linear Iterations for Distributed Averaging," *Systems and Control Letters*, Vol. 53, 2004, pp. 65–78.

[13]Lawton, J. R. and Beard, R. W., "Synchronized Multiple Spacecraft Rotations," *Automatica*, Vol. 38, No. 8, 2002, pp. 1359–1364.

[14]Lawton, J. R., Beard, R. W., and Young, B., "A Decentralized Approach To Formation Maneuvers," *IEEE Transactions on Robotics and Automation*, Vol. 19, No. 6, December 2003, pp. 933–941.

[15]Tanner, H. G., Jadbabaie, A., and Pappas, G. J., "Flocking in Fixed and Switching Networks," *Automatica*, 2003, (submitted).

[16]Olfati-Saber, R., "Flocking for Multi-Agent Dynamic Systems: Algorithms and Theory," *IEEE Transactions on Automatic Control*, 2004, (submitted).

[17]Horn, R. A. and Johnson, C. R., *Matrix Analysis*, Cambridge University Press, 1985.

[18]Roy, S., Saberi, A., and Herlugson, K., *Formation and alignment of distributed sensing agents with double-integrator dynamics and actuator saturation*, Sensor Network Applications, IEEE Press, 2004, (accepted).

[19]Proud, A. W., Pachter, M., and D'Azzo, J. J., "Close Formation Flight Control," *Proceedings of the AIAA Guidance, Navigation, and Control Conference*, Portland, OR, August 1999, pp. 1231–1246, Paper No. AIAA-99-4207.

[20]Gavrilets, V., Mettler, B., and Feron, E., "Human-Inspired Control Logic for Automated Maneuvering of Miniature Helicopter," *AIAA Journal of Guidance, Control, and Dynamics*, Vol. 27, No. 5, September–October 2004, pp. 752–759.