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Distributed Attitude Synchronization for Multiple Rigid Bodies with Euler-Lagrange Equations of Motion

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Abstract— **In this paper, distributed attitude synchronization problems are considered for multiple rigid bodies with attitude dynamics represented by Euler-Lagrange equations of motion. Three distributed control laws for attitude synchronization are proposed and analyzed. The first control law introduces bounded functions to reduce the required control torque. The second control law applies a passivity approach to remove the requirement for relative angular velocity measurement between neighboring rigid bodies. The third control law incorporates a time-varying reference attitude, where the reference attitude is allowed to be available to only a subset of the group members. It is shown that the first two control laws guarantee distributed attitude synchronization under any undirected connected communication topology. The third control law guarantees that all rigid bodies track the time-varying reference attitude as long as a virtual node whose state is the time-varying reference attitude has a directed path to all of the rigid bodies in the group. Simulation results are presented to demonstrate the effectiveness of the three control laws.**

I. INTRODUCTION

Attitude control of a rigid body has been studied extensively in the literature (see e.g., [1]–[5] and references therein). Motivated by the benefits gained by having multiple inexpensive, simple rigid bodies working together, cooperative attitude synchronization of multiple rigid bodies have received recent attention in [6]–[12], to name a few. A similar problem is addressed in [13] in the context of mutual synchronization of robotic manipulators. Related are also consensus-type problems in cooperative control of multi-vehicle systems, where it is often assumed that vehicles are modeled by single- or double-integrator dynamics (see [14] and references therein).

In [6], a leader-follower strategy is applied for attitude synchronization of multiple spacecraft, where information only flows from leaders to followers. In [7], [8], a behavioral approach is used for attitude synchronization, where the control law for each spacecraft is a function of the states of its two adjacent neighbors. In particular, a passivity approach is used in [7] to derive a control law without angular velocity measurement. However, all of the results in [7], [8] require a restrictive bidirectional ring communication topology. Distributed control laws based on graph theoretic approaches are studied for attitude synchronization in [9], [10] by use of Euler parameters for attitude representation while in [12] by use of Modified Rodriguez Parameters (MRPs) for attitude representation. However, the control laws in [9], [10], [12] require relative angular velocity measurement between neighboring spacecraft. Furthermore, no group reference attitude exists in the team in [9], [10], [12]. In [11] a cooperative attitude tracking problem is addressed under directed information exchange, where a group reference attitude is available to only a subset of the group members. However, the analysis for attitude tracking in [11] is restricted to the case where the directed graph can be simplified to a graph with only one node. It is not clear whether the result still applies to a general directed graph.

In this paper, we use MRPs for attitude representation and consider attitude dynamics represented by Euler-Lagrange equations of motion [5], [15]. We propose three distributed control laws for the Euler-Lagrange equations of motion for attitude synchronization. In the first control law, we introduce bounded functions to reduce the required control torque. In the second control law, we apply a passivity approach motivated by [2], [4], [7] to remove the requirement for relative angular velocity measurement between neighboring rigid bodies. The first two control laws guarantee distributed attitude synchronization under any undirected connected communication topology, which extends the results in [7]–[10], [12]. In the third control law, we extend a single rigid body attitude tracking law to the case of multiple rigid bodies such that all of the group members can track a time-varying reference attitude even when the reference attitude is available to only a subset of the group members under a general directed information-exchange topology. The third control law guarantees that all rigid bodies track the time-varying reference attitude as long as a virtual node whose state is the time-varying reference attitude has a directed path to all of the rigid bodies in the group, which extends the results in [11]. All of the results in this paper are applicable to robotic manipulators with dynamics represented by similar Euler-Lagrange equations of motion.

II. BACKGROUND AND PRELIMINARIES

A. Graph Theory Notions

A weighted graph consists of a node set $\mathcal{V} = \{1, \ldots, p\}$, an edge set $\mathcal{E} \in \mathcal{V} \times \mathcal{V}$, and a weighted adjacency matrix $A = [a_{ij}] \in \mathbb{R}^{p \times p}$. An edge (i, j) in a weighted directed graph denotes that vehicle j can obtain information from vehicle i, but not necessarily vice versa. In contrast, the pairs of nodes in a weighted undirected graph are unordered, where an edge (i, j) denotes that vehicles i and j can obtain information from one another. The weighted adjacency matrix A of a weighted directed graph is defined such that a_{ij} is a positive weight if $(j, i) \in \mathcal{E}$, while $a_{ij} = 0$ if $(j, i) \notin \mathcal{E}$. The weighted adjacency matrix A of a weighted undirected graph is defined

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analogously except that $a_{ij} = a_{ji}$, $\forall i \neq j$, since $(j,i) \in \mathcal{E}$ implies $(i,j) \in \mathcal{E}$. If the weights are not relevant, then a_{ij} is set equal to 1 for all $(j,i) \in \mathcal{E}$. In this paper, self edges are not allowed, i.e. $a_{ii} = 0$.

For an edge (i,j) in a directed graph, i is the parent node and j is the child node. A directed path is a sequence of edges in a directed graph of the form $(i_1, i_2), (i_2, i_3), \ldots$, where $i_j \in \mathcal{V}$. A directed tree is a directed graph, where every node has exactly one parent except for one node, called the root, which has no parent, and the root has a directed path to every other node. A directed spanning tree of a directed graph is a directed tree that contains all nodes of the directed graph. A directed graph has or contains a directed spanning tree if there exists a directed spanning tree as a subset of the directed graph, that is, there exists at least one node having a directed path to all of the other nodes.

Let the matrix $L = [\ell_{ij}] \in \mathbb{R}^{p \times p}$ be defined as

$$\ell_{ii} = \sum_{j=1, j\neq i}^{p} a_{ij}, \qquad \ell_{ij} = -a_{ij}, \quad i \neq j. \tag{1}$$

The matrix L satisfies the conditions

$$\ell_{ij} \leq 0, \quad i \neq j, \qquad \sum_{j=1}^{p} \ell_{ij} = 0, \quad i = 1, \ldots, p. \tag{2}$$

For an undirected graph, the Laplacian matrix L is symmetric positive semi-definite. However, L for a directed graph does not have this property. In both the undirected and directed cases, 0 is an eigenvalue of L with the associated eigenvector $\mathbf{1}_p$, where $\mathbf{1}_p$ is a $p \times 1$ column vector of all ones. In the case of undirected graphs, 0 is a simple eigenvalue of L and all of the other eigenvalues are positive if and only if the undirected graph is connected [16]. In the case of directed graphs, 0 is a simple eigenvalue of L and all of the other eigenvalues have positive real parts if and only if the directed graph contains a directed spanning tree [17].

Given a matrix $S = [s_{ij}] \in \mathbb{R}^{p \times p}$, the directed graph of S, denoted by $\Gamma(S)$, is the directed graph on p nodes i, $i \in \{1, 2, \ldots, p\}$, such that there is an edge in $\Gamma(S)$ from node j to node i if and only if $s_{ij} \neq 0$ (cf. [18]). Again, we assume that there is no self edge (i,i).

B. Rigid Body Attitude Dynamics

Modified Rodrigues Parameters (MRPs) [4] are used to represent the attitude of a rigid body with respect to an inertial frame. Let $\sigma_i = \hat{e}_i \tan(\frac{\phi_i}{4}) \in \mathbb{R}^3$ represent the MRPs for the i^{th} rigid body, where $\hat{e}_i$ is the Euler axis and ϕ_i is the Euler angle. Given a vector $v = [v_1, v_2, v_3]^T$, the cross-product operator is denoted by $v^\times = \begin{bmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{bmatrix}$, which represents the fact that $v \times w = v^\times w$, where $w = [w_1, w_2, w_3]^T$.

Consider rigid bodies with attitude dynamics given by

$$\dot{\sigma}_i = F(\sigma_i)\omega_i, \tag{3}$$

$$J_i \dot{\omega}_i = -\omega_i \times J_i \omega_i + \tau_i \quad i = 1, \ldots, n, \tag{4}$$

where ω_i denotes the angular velocity, J_i is the inertia, τ_i is the control torque, and $F(\sigma_i) \triangleq \frac{1}{2}[(\frac{1-\sigma_i^T\sigma_i}{2})I_3 + \sigma_i^\times + \sigma_i\sigma_i^T]$.

Following [5], [15], (3) and (4) can be written as

$$H_i^*(\sigma_i)\ddot{\sigma}_i + C_i^*(\sigma_i, \dot{\sigma}_i)\dot{\sigma}_i = F^{-T}(\sigma_i)\tau_i, \tag{5}$$

where $H_i^*(\sigma_i) \triangleq F^{-T}(\sigma_i) J_i F^{-1}(\sigma_i)$, and $C_i^*(\sigma_i, \dot{\sigma}_i) \triangleq -F^{-T}(\sigma_i) J_i F^{-1}(\sigma_i)\dot{F}(\sigma_i)F^{-1}(\sigma_i) - F^{-T}(\sigma_i)(J_i F^{-1}(\sigma_i)\dot{\sigma}_i)^\times F^{-1}(\sigma_i)$. Note that $H_i^*(\sigma_i)$ is a symmetric positive-definite matrix and $\dot{H}_i^*(\sigma_i) - 2C_i^*(\sigma_i, \dot{\sigma}_i)$ is a skew-symmetric matrix [15].

III. ATTITUDE SYNCHRONIZATION WITH ZERO FINAL ANGULAR VELOCITIES

In this section, we consider a control law that guarantees multiple rigid bodies to synchronize their final attitudes with zero final angular velocities. Hereafter we assume that all of the vectors in each control law have been appropriately transformed and represented in the same coordinate frame. We propose a control torque as

$$\tau_i = F^T(\sigma_i) u_i, \tag{6}$$

where

$$\begin{aligned} u_i = &- \sum_{j=1}^{n} a_{ij} \tanh K_\sigma(\sigma_i - \sigma_j) \\ &- \sum_{j=1}^{n} b_{ij} \tanh K_{\dot{\sigma}}(\dot{\sigma}_i - \dot{\sigma}_j) - \tanh(K_{di}\dot{\sigma}_i), \end{aligned} \tag{7}$$

where $i = 1, \ldots, n$, a_{ij} is the (i,j)th entry of the weighted adjacency matrix $A \in \mathbb{R}^{n \times n}$ associated with the communication graph for σ_i, b_{ij} is the (i,j)th entry of the weighted adjacency matrix $B \in \mathbb{R}^{n \times n}$ associated with the communication graph for $\dot{\sigma}_i$, K_σ, $K_{\dot{\sigma}}$, and K_{di} are positive-definite diagonal matrices, and $\tanh(\cdot)$ is defined component-wise for a vector. Note that $\dot{\sigma}_i \triangleq F(\sigma_i)\omega_i$ in (7). Also note that in contrast to the control law in [12], bounded functions have been introduced in (7) to reduce the required control torque.

Theorem 3.1: With the control torque given by (6), $\sigma_i(t) \to \sigma_j(t)$ and $\dot{\sigma}_i(t) \to 0$ asymptotically as $t \to \infty$ if the graph of A is undirected connected and the graph of B is undirected.

Proof: Consider the nonnegative function

$$\begin{aligned} V = &\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} \mathbf{1}_3^T K_\sigma^{-1} \log(\cosh[K_\sigma(\sigma_i - \sigma_j)]) \\ &+ \frac{1}{2}\sum_{i=1}^{n} \dot{\sigma}_i^T H_i^*(\sigma_i)\dot{\sigma}_i. \end{aligned}$$

Note that the set $\{\sigma_i - \sigma_j, \dot{\sigma}_i | V \leq c\}$, where $c > 0$, is compact with respect to $\sigma_i - \sigma_j$ and $\dot{\sigma}_i$ if the graph of A is undirected connected.

Differentiating V, gives

$$\begin{aligned}
\dot{V} =& \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}(\dot{\sigma}_i - \dot{\sigma}_j)^T \tanh[K_\sigma(\sigma_i - \sigma_j)] \\
&+ \frac{1}{2}\sum_{i=1}^{n}(\ddot{\sigma}_i^T H_i^*(\sigma_i)\dot{\sigma}_i + \dot{\sigma}_i^T \dot{H}_i^*(\sigma_i)\dot{\sigma}_i + \dot{\sigma}_i^T H_i^*(\sigma_i)\ddot{\sigma}_i) \\
=& \frac{1}{2}\sum_{i=1}^{n}\dot{\sigma}_i^T\left(\sum_{j=1}^{n} a_{ij}\tanh[K_\sigma(\sigma_i - \sigma_j)]\right) \\
&- \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}\dot{\sigma}_j^T \tanh[K_\sigma(\sigma_i - \sigma_j)] \\
&+ \sum_{i=1}^{n}\dot{\sigma}_i^T H_i^*(\sigma_i)\ddot{\sigma}_i + \frac{1}{2}\sum_{i=1}^{n}\dot{\sigma}_i^T \dot{H}_i^*(\sigma_i)\dot{\sigma}_i
\end{aligned}$$

$$\begin{aligned}
=& \frac{1}{2}\sum_{i=1}^{n}\dot{\sigma}_i^T\left(\sum_{j=1}^{n} a_{ij}\tanh[K_\sigma(\sigma_i - \sigma_j)]\right) \\
&+ \frac{1}{2}\sum_{j=1}^{n}\sum_{i=1}^{n} a_{ji}\dot{\sigma}_j^T \tanh[K_\sigma(\sigma_j - \sigma_i)] \\
&- \sum_{i=1}^{n}\dot{\sigma}_i^T[C_i^*(\sigma_i,\dot{\sigma}_i)\dot{\sigma}_i + \sum_{j=1}^{n} a_{ij}\tanh K_\sigma(\sigma_i - \sigma_j) \\
&+ \sum_{j=1}^{n} b_{ij}\tanh K_{\dot{\sigma}}(\dot{\sigma}_i - \dot{\sigma}_j) + \tanh(K_{di}\dot{\sigma}_i)] \\
&+ \frac{1}{2}\sum_{i=1}^{n}\dot{\sigma}_i^T \dot{H}_i^*(\sigma_i)\dot{\sigma}_i
\end{aligned}$$

$$\begin{aligned}
=& \frac{1}{2}\sum_{i=1}^{n}\dot{\sigma}_i^T\left(\sum_{j=1}^{n} a_{ij}\tanh[K_\sigma(\sigma_i - \sigma_j)]\right) \\
&+ \frac{1}{2}\sum_{j=1}^{n}\dot{\sigma}_j^T\left(\sum_{i=1}^{n} a_{ji}\tanh[K_\sigma(\sigma_j - \sigma_i)]\right) \\
&- \sum_{i=1}^{n}\dot{\sigma}_i^T[\sum_{j=1}^{n} a_{ij}\tanh K_\sigma(\sigma_i - \sigma_j) \\
&+ \sum_{j=1}^{n} b_{ij}\tanh K_{\dot{\sigma}}(\dot{\sigma}_i - \dot{\sigma}_j) + \tanh(K_{di}\dot{\sigma}_i)]
\end{aligned}$$

$$\begin{aligned}
=& \sum_{i=1}^{n}\dot{\sigma}_i^T\left(\sum_{j=1}^{n} a_{ij}\tanh[K_\sigma(\sigma_i - \sigma_j)]\right) \\
&- \sum_{i=1}^{n}\dot{\sigma}_i^T[\sum_{j=1}^{n} a_{ij}\tanh K_\sigma(\sigma_i - \sigma_j) \\
&+ \sum_{j=1}^{n} b_{ij}\tanh K_{\dot{\sigma}}(\dot{\sigma}_i - \dot{\sigma}_j) + \tanh(K_{di}\dot{\sigma}_i)]
\end{aligned}$$

$$\begin{aligned}
&= -\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} b_{ij}(\dot{\sigma}_i - \dot{\sigma}_j)^T \tanh K_{\dot{\sigma}}(\dot{\sigma}_i - \dot{\sigma}_j) \\
&\quad - \sum_{i=1}^{n}\dot{\sigma}_i^T \tanh(K_{di}\dot{\sigma}_i) \le 0,
\end{aligned}$$

where we have used the fact that $a_{ij} = a_{ji}$ and $\tanh[K_\sigma(\sigma_j - \sigma_i)] = -\tanh[K_\sigma(\sigma_i - \sigma_j)]$ and have switched the order of the summation signs to obtain the third equality, and have used the fact that $b_{ij} = b_{ji}$ to obtain the last equality, and have used the fact that x and $\tanh(Kx)$ have the same sign when x is a vector and K is a positive-definite diagonal matrix to obtain the last inequality.

Let $S = \{\sigma_i - \sigma_j, \dot{\sigma}_i | \dot{V} = 0\}$. Note that $\dot{V} \equiv 0$ implies that $\dot{\sigma}_i \equiv 0$, which in turn implies that $\ddot{\sigma}_i \equiv 0$. From (5), (6), and (7), it follows that $\sum_{j=1}^{n} a_{ij}\tanh K_\sigma(\sigma_i - \sigma_j) \equiv 0$. Thus it follows that $\sum_{i=1}^{n}[\sigma_i^T \sum_{j=1}^{n} a_{ij}\tanh K_\sigma(\sigma_i - \sigma_j)] \equiv 0$, which implies that $\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}(\sigma_i - \sigma_j)^T \tanh K_\sigma(\sigma_i - \sigma_j) \equiv 0$, where we have used the fact that $\sum_{i=1}^{n}[\sigma_i^T \sum_{j=1}^{n} a_{ij}\tanh K_\sigma(\sigma_i - \sigma_j)] = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}(\sigma_i - \sigma_j)^T \tanh K_\sigma(\sigma_i - \sigma_j)$ by noting that $a_{ij} = a_{ji}$. Therefore, it follows that $\sigma_i \equiv \sigma_j$, $\forall i \neq j$, if the graph of A is undirected connected by noting that $\sigma_i - \sigma_j$ and $\tanh K_\sigma(\sigma_i - \sigma_j)$ have the same signs for each component. By LaSalle's invariance principle, it follows that $\sigma_i(t) \to \sigma_j(t)$ and $\dot{\sigma}_i(t) \to 0$ asymptotically as $t \to \infty$. ■

Note that Theorem 3.1 is still valid even if the undirected graph of B is not connected as long as the undirected graph of A is connected. As a result, even $b_{ij} \equiv 0$ or equivalently without the term $-\sum_{j=1}^{n} b_{ij}\tanh K_{\dot{\sigma}}(\dot{\sigma}_i - \dot{\sigma}_j)$ in (7), Theorem 3.1 is still valid as long as the undirected graph of A is connected. However, the term $-\sum_{j=1}^{n} b_{ij}\tanh K_{\dot{\sigma}}(\dot{\sigma}_i - \dot{\sigma}_j)$ in (7) introduces relative damping between neighboring rigid bodies.

IV. PASSIVITY APPROACH

Note that the control torque (6) requires $\dot{\sigma}_i$ and $\dot{\sigma}_i - \dot{\sigma}_j$, where $b_{ij} \neq 0$, if the relative damping term is included in (7). In this section, we generalize the passivity approach in [2], [4], [7] to propose a control toque for attitude synchronization, where $\dot{\sigma}_i$ and $\dot{\sigma}_i - \dot{\sigma}_j$ are not required. Before moving on, we need the following lemma.

Lemma 4.1: Suppose that $U \in \mathbb{R}^{p\times p}$, $V \in \mathbb{R}^{q\times q}$, $X \in \mathbb{R}^{p\times p}$, and $Y \in \mathbb{R}^{q\times q}$. The following arguments are valid: (i) $(U \otimes V)(X \otimes Y) = UX \otimes VY$; (ii) Suppose that U and V are invertible. Then $(U \otimes V)^{-1} = U^{-1} \otimes V^{-1}$; (iii) If U and V are symmetric, so is $U \otimes V$; and (iv) If U and V are symmetric positive definite, so is $U \otimes V$.

Proof: See [19]. ■

We propose a control torque without relative angular velocity measurement as

$$\dot{\hat{x}}_i = \Gamma\hat{x}_i + \sum_{j=1}^{n} b_{ij}(\sigma_i - \sigma_j) + \kappa\sigma_i \tag{8}$$

$$y_i = P\Gamma\hat{x}_i + P\sum_{j=1}^{n} b_{ij}(\sigma_i - \sigma_j) + \kappa P\sigma_i \tag{9}$$

$$\tau_i = -F^T(\sigma_i)\left(\sum_{j=1}^{n} a_{ij}\tanh[K_\sigma(\sigma_i - \sigma_j)] + y_i\right) \tag{10}$$

where $i = 1,\ldots,n$, $\Gamma \in \mathbb{R}^{m\times m}$ is Hurwitz, κ is a positive scalar, a_{ij} is the (i,j)th entry of the weighted adjacency matrix $A \in \mathbb{R}^{n\times n}$ associated with the communication graph for σ_i in (10), b_{ij} is the (i,j)th entry of the weighted adjacency matrix $B \in \mathbb{R}^{n\times n}$ associated with the communication graph for σ_i in (8), and $P = P^T \in \mathbb{R}^{m\times m} > 0$ is the solution to the Lyapunov equation $\Gamma^T P + P\Gamma = -Q$ with $Q = Q^T \in \mathbb{R}^{m\times m} > 0$. Note that A and B can be chosen differently. Also note that rather than requiring a bidirectional ring graph as in [7], (10) allows any undirected connected communication graph.

Theorem 4.1: With (8)-(10), $\sigma_i(t) \to \sigma_j(t)$ and $\dot{\sigma}_i(t) \to 0$ asymptotically as $t \to \infty$ if the graph of A is undirected connected and the graph of B is undirected.

Proof: Consider the function

$$V = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}\mathbf{1}_3^T K_\sigma^{-1}\log(\cosh[K_\sigma(\sigma_i - \sigma_j)]) + \frac{1}{2}\sum_{i=1}^{n}\dot{\sigma}_i^T H_i^*(\sigma_i)\dot{\sigma}_i + \frac{1}{2}\dot{\hat{x}}^T(M\otimes I_3)^{-1}(I_n\otimes P)\dot{\hat{x}},$$

where $\hat{x} = [\hat{x}_1^T,\ldots,\hat{x}_n^T]^T$, $M = L + \kappa I_n$ with $L = [\ell_{ij}] \in \mathbb{R}^{m\times m}$ defined as $\ell_{ij} = -b_{ij}$ and $\ell_{ii} = \sum_{j\neq i} b_{ij}$. Note that L satisfies the property (2) and is therefore symmetric positive semi-definite because the graph of B is undirected. It thus follows that M is symmetric positive definite, so is M^{-1}. From Lemma 4.1, note that $(M\otimes I_3)^{-1} = (M^{-1}\otimes I_3)$. Also from Lemma 4.1 note that $(M^{-1}\otimes I_3)(I_n\otimes P) = M^{-1}I_n \otimes I_3 P = I_n M^{-1}\otimes P I_3 = (I_n\otimes P)(M^{-1}\otimes I_3)$. That is, $(M\otimes I_3)^{-1}$ and $I_n\otimes P$ commute. Similarly, it is straightforward to show that $(M\otimes I_3)^{-1}$ and $I_n\otimes\Gamma^T$ also commute. Note that $M^{-1}I_n\otimes I_3 P$ is symmetric positive definite, so is $(M^{-1}\otimes I_3)(I_n\otimes P)$. Therefore, V is nonnegative. Note that the set $\{\sigma_i - \sigma_j, \dot{\sigma}_i, \dot{\hat{x}}_i | V \le c\}$, where $c > 0$, is compact with respect to $\sigma_i - \sigma_j$, $\dot{\sigma}_i$, and $\dot{\hat{x}}_i$ if the graph of A is undirected connected.

Following the proof of Theorem 3.1, we derive the derivative of V as

$$\begin{aligned}
\dot{V} = &\sum_{i=1}^{n}\dot{\sigma}_i^T\left(\sum_{j=1}^{n} a_{ij}\tanh[K_\sigma(\sigma_i - \sigma_j)]\right)\\
&- \sum_{i=1}^{n}\dot{\sigma}_i^T[\sum_{j=1}^{n} a_{ij}\tanh K_\sigma(\sigma_i - \sigma_j) + y_i]\\
&+ \frac{1}{2}\dot{\hat{x}}^T(I_n\otimes\Gamma^T)(M\otimes I_3)^{-1}(I_n\otimes P)\dot{\hat{x}}\\
&+ \frac{1}{2}\dot{\sigma}^T(M\otimes I_3)^T(M\otimes I_3)^{-1}(I_n\otimes P)\dot{\hat{x}}\\
&+ \frac{1}{2}\dot{\hat{x}}^T(M\otimes I_3)^{-1}(I_n\otimes P)(I_n\otimes\Gamma)\dot{\hat{x}}\\
&+ \frac{1}{2}\dot{\hat{x}}^T(M\otimes I_3)^{-1}(I_n\otimes P)(M\otimes I_3)\dot{\sigma}\\
= &-\sum_{i=1}^{n}\dot{\sigma}_i^T y_i + \frac{1}{2}\dot{\hat{x}}^T(M\otimes I_3)^{-1}[I_n\otimes(\Gamma^T P + P\Gamma)]\dot{\hat{x}}\\
&+ \dot{\sigma}(I_n\otimes P)\dot{\hat{x}}\\
= &-\frac{1}{2}\dot{\hat{x}}^T(M\otimes I_3)^{-1}(I_n\otimes Q)\dot{\hat{x}} \le 0,
\end{aligned}$$

where we have used the fact that

$$\ddot{\hat{x}} = (I_n\otimes\Gamma)\dot{\hat{x}} + (M\otimes I_3)\dot{\sigma} \tag{11}$$

with $\dot{\sigma} = [\dot{\sigma}_1^T,\ldots,\dot{\sigma}_n^T]^T$, $(M\otimes I_3)^{-1}$ and $I_n\otimes\Gamma^T$ commute, $(M\otimes I_3)^{-1}$ and $I_n\otimes P$ commute, $M\otimes I_3 = (M\otimes I_3)^T$, $y = (I_n\otimes P)\dot{\hat{x}}$ with $y = [y_1^T,\ldots,y_n^T]^T$, and $(M\otimes I_3)^{-1}(I_n\otimes Q) = M^{-1}I_n\otimes QI_3$ is symmetric positive definite.

Let $S = \{\sigma_i - \sigma_j, \dot{\sigma}_i, \dot{\hat{x}}_i | \dot{V} = 0\}$. Note that $\dot{V} \equiv 0$ implies that $\dot{\hat{x}} \equiv 0$, which in turn implies that $(M\otimes I_3)\dot{\sigma} \equiv 0$ according to (11) and $y_i \equiv 0$ by noting that $y_i = P\dot{\hat{x}}_i$ according to (9). Because $M\otimes I_3$ is symmetric positive definite, it follows that $\dot{\sigma}_i \equiv 0$. From (5) and (10), it follows that $\sum_{j=1}^{n} a_{ij}\tanh K_{\sigma i}(\sigma_i - \sigma_j) \equiv 0$. Therefore, following the proof to Theorem 3.1, it follows that $\sigma_i \equiv \sigma_j$, $\forall i \neq j$, if the graph of A is undirected connected. By LaSalle's invariance principle, it follows that $\sigma_i(t) \to \sigma_j(t)$ and $\dot{\sigma}_i(t) \to 0$, $\forall i \neq j$, asymptotically as $t \to \infty$. ■

Note that Theorem 4.1 is still valid even without the term $\sum_{j=1}^{n} b_{ij}(\sigma_i - \sigma_j)$ in (8) as long as the graph of A is undirected connected. However, the term introduces relative damping between neighboring rigid bodies.

V. REFERENCE ATTITUDE

In this section, we consider the case where a group reference attitude exists for the team. Let σ^r be the reference attitude for the team. Suppose that σ^r, $\dot{\sigma}^r$, and $\ddot{\sigma}^r$ are bounded. We propose a control torque as

$$\begin{aligned}
\tau_i = &F^T(\sigma_i)[H_i^*(\sigma_i)(\ddot{\sigma}_i^d - \Lambda_i\dot{\tilde{\sigma}}_i) + C_i^*(\sigma_i,\dot{\sigma}_i)(\dot{\sigma}_i^d - \Lambda_i\tilde{\sigma}_i)\\
&- K_i(\dot{\tilde{\sigma}}_i + \Lambda_i\tilde{\sigma}_i)]
\end{aligned} \tag{12}$$

where $i = 1,\ldots,n$,

$$\sigma_i^d \triangleq \frac{\sum_{j=1}^{n} a_{ij}\sigma_j + a_{i(n+1)}\sigma^r}{\sum_{j=1}^{n} a_{ij} + a_{i(n+1)}}, \tag{13}$$

a_{ij}, $i,j = 1,\ldots,n$, is the (i,j)th entry of the weighed adjacency matrix A, $a_{i(n+1)} > 0$, $i = 1,\ldots,n$, if rigid body i has access to σ^r, $\dot{\sigma}^r$ and $\ddot{\sigma}^r$, $\tilde{\sigma}_i \stackrel{\triangle}{=} \sigma_i - \sigma_i^d$, and Λ_i and K_i are symmetric positive-definite matrices. Note that in contrast to traditional trajectory tracking control laws for a single robotic manipulator (e.g. [20]), σ_i^d defined by (13) is not an external desired state but depends on each rigid body's local neighbors' attitudes as well as the group reference attitude if the rigid body has access to the group reference attitude.

Theorem 5.1: Let $A = [a_{ij}] \in \mathbb{R}^{(n+1)\times(n+1)}$, where a_{ij}, $i = 1,\ldots,n$, $j = 1,\ldots,n+1$, are defined in (12) and $a_{(n+1)j} = 0$, $j = 1,\ldots,n+1$. With (12), if the directed graph of A has a directed spanning tree, then $\sigma_i(t) \to \sigma^r(t)$ and $\dot{\sigma}_i(t) \to \dot{\sigma}^r(t)$, $i = 1,\ldots,n$, asymptotically as $t \to \infty$.

Proof: Note that σ_i^d defined by (13) has a denominator $\sum_{j=1}^n a_{ij} + a_{i(n+1)}$. We first show that the denominator is nonzero if the directed graph of A has a directed spanning tree. Note that all entries of the $(n+1)$th row of A are zero. Then the condition that the directed graph of A has a directed spanning tree implies that there exists at least one nonzero entry for each row i, $i = 1,\ldots,n$. It thus follows that $\sum_{j=1}^n a_{ij} + a_{i(n+1)} \neq 0$, $i = 1,\ldots,n$, if the directed graph of A has a directed spanning tree.

Also note that with (12), $\ddot{\sigma}_i$ exists on both sides of (5).[1] We next show that under the assumption of the theorem, there is a unique solution for $\ddot{\sigma}_i$. With (12), (5) can be written as

$$\begin{aligned} &H_i^*(\sigma_i)(\ddot{\tilde{\sigma}}_i + \Lambda_i\dot{\tilde{\sigma}}_i) + C_i^*(\sigma_i,\dot{\sigma}_i)(\dot{\tilde{\sigma}}_i + \Lambda_i\tilde{\sigma}_i) \\ &\quad + K_i(\dot{\tilde{\sigma}}_i + \Lambda_i\tilde{\sigma}_i) = 0, \quad i = 1,\ldots,n. \end{aligned} \tag{14}$$

Note that $\tilde{\sigma}_i = \sigma_i - \sigma_i^d = \sum_{i=1}^n a_{ij}(\sigma_i - \sigma_j) + a_{i(n+1)}(\sigma_i - \sigma^r)$. Letting $\tilde{\sigma} = [\tilde{\sigma}_1^T,\ldots,\tilde{\sigma}_n^T]^T$, then $\tilde{\sigma} = (W \otimes I_3)\sigma + (b \otimes I_3)\sigma^r$, where $\sigma = [\sigma_1^T,\ldots,\sigma_n^T]^T$, $W = [w_{ij}] \in \mathbb{R}^{n\times n}$ with $w_{ij} = -a_{ij}$ and $w_{ii} = \sum_{j=1,j\neq i}^{n+1} a_{ij}$, and $b = [-a_{1(n+1)},\ldots,-a_{n(n+1)}]^T \in \mathbb{R}^n$. Eq. (14) can be written in matrix form as $H^*(\sigma)(\ddot{\tilde{\sigma}} + \Lambda\dot{\tilde{\sigma}}) + C^*(\sigma,\dot{\sigma})(\dot{\tilde{\sigma}} + \Lambda\tilde{\sigma}) + K(\dot{\tilde{\sigma}} + \Lambda\tilde{\sigma}) = 0$, where $H^*(\sigma) = \mathrm{diag}\{H_1^*(\sigma_1),\ldots,H_n^*(\sigma_n)\}$, $C^*(\sigma,\dot{\sigma}) = \mathrm{diag}\{C_1^*(\sigma_1,\dot{\sigma}_1),\ldots,C_n^*(\sigma_n,\dot{\sigma}_n)\}$, $\Lambda = \mathrm{diag}\{\Lambda_1,\ldots,\Lambda_n\}$, and $K = \mathrm{diag}\{K_1,\ldots,K_n\}$. Thus there is a unique solution for $\ddot{\sigma}$ if $H^*(\sigma)$ and $W \otimes I_3$ are invertible. Note that $H^*(\sigma)$ is invertible because it is symmetric positive definite. It is left to show that W is invertible. Let $L_{n+1} = \begin{bmatrix} W & b \\ 0_{1\times n} & 0 \end{bmatrix}$. It follows that $\mathrm{Rank}(L_{n+1}) = \mathrm{Rank}([W|b])$. Also noting that $W\mathbf{1}_n + b = \mathbf{0}_n$, where $\mathbf{0}_n$ is an $n\times 1$ column vector of all zeros, it follows that $\mathrm{Rank}([W|b]) = \mathrm{Rank}(W)$. If the directed graph of A has a directed spanning tree, L_{n+1} has a simple zero eigenvalue and all of the other eigenvalues have positive real parts [17], which in turn implies that $\mathrm{Rank}(L_{n+1}) = n$. Therefore, we conclude that $\mathrm{Rank}(W) = n$, that is, W is invertible under the assumption of the theorem.

Consider a positive-definite function $V = \frac{1}{2}\sum_{i=1}^n s_i^T H_i^*(\sigma_i)s_i$, where $s_i \stackrel{\triangle}{=} \dot{\tilde{\sigma}}_i + \Lambda_i\tilde{\sigma}_i$.

Differentiating V, gives $\dot{V} = -\sum_{i=1}^n s_i^T K_i s_i$, where we have used the fact that (14) can be written as $H_i^*(\sigma_i)\dot{s}_i + C_i^*(\sigma_i,\dot{\sigma}_i)s_i + K_i s_i = 0$ and $H_i^*(\sigma_i) - 2C_i^*(\sigma_i,\dot{\sigma}_i)$ is skew symmetric. Then it follows from [20, Chapter 9] that $\tilde{\sigma}_i(t) \to 0$ and $\dot{\tilde{\sigma}}_i(t) \to 0$ asymptotically as $t \to \infty$ by use of Barbalat's lemma, which in turn implies that $(W \otimes I_3)\sigma(t) + (b \otimes I_3)\sigma^r(t) \to 0$ and $(W \otimes I_3)\dot{\sigma}(t) + (b \otimes I_3)\dot{\sigma}^r(t) \to 0$ asymptotically as $t \to \infty$. Therefore, it follows that $\sigma(t) \to \mathbf{1}_n \otimes \sigma^r(t)$ and $\dot{\sigma}(t) \to \mathbf{1}_n \otimes \dot{\sigma}^r(t)$, that is, $\sigma_i(t) \to \sigma^r(t)$ and $\dot{\sigma}_i(t) \to \dot{\sigma}_i(t)$ asymptotically as $t \to \infty$ because $-W^{-1}b = \mathbf{1}_n$ under the assumption of the theorem. ∎

Note that while the results in [11] require that the directed graph be simplified to a graph with one node for convergence analysis, Theorem 5.1 gives an explicit condition for attitude synchronization under a general directed information-exchange topology. In addition, (12) can be applied to robotic manipulators for position synchronization.

VI. SIMULATION

In this section, we apply the control torques (6), (10), and (12) in simulation to achieve attitude synchronization among six rigid bodies. The rigid body specifications are chosen to be the same as in Table I in [9]. For (6), we choose $K_\sigma = I_3$, $K_{\dot{\sigma}} = 4I_3$, $K_{di} = 4I_3$, $a_{ij} = b_{ij} = 2$ if $(j,i) \in \mathcal{E}$ and 0 otherwise. Here for simplicity we let $A = B$ although they can be chosen differently. For (10), we choose $\Gamma = -I_3$, $\kappa = 1$, $P = 2I_3$, $K_\sigma = I_3$, and $a_{ij} = b_{ij} = 2$ if $(j,i) \in \mathcal{E}$ and 0 otherwise. For (12), we choose $\Lambda_i = I_3$, $K_i = 2I_3$, $a_{ij} = 1$, $i,j = 1,\ldots,6$, if $(j,i) \in \mathcal{E}$, and $a_{i7} = 1$ if rigid body i has access to σ^r, $\dot{\sigma}^r$, and $\ddot{\sigma}^r$. Suppose that the reference attitude σ^r, reference angular velocity $\omega^r = F^{-1}(\sigma^r)\dot{\sigma}^r$, and reference torque τ^r satisfy (3) and (4) with $\tau^r = [0,0,0]^T$, $J^r = \mathrm{diag}\{1,2,1\}$, $\sigma^r(0) = [0,0,0]^T$, and $\omega^r(0) = [0.1,0.3,0.5]^T$. In the following, $\sigma_i(0)$ and $\omega_i(0)$, $i = 1,\ldots,6$, are chosen randomly. A superscript (j) denotes the j^{th} component of a vector.

The undirected graph of the weighted adjacency matrices A and B used in (6) and (10) is shown in Fig. 1, where an edge between i and j denotes that rigid bodies i and j can communicate with each other. Note that the undirected graph of A and B is connected.

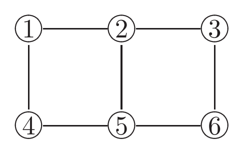

Fig. 1. Undirected graph of A and B in (6) and (10).

Figs. 2 and 3 show, respectively, the attitudes and angular velocities of rigid bodies 1, 3, and 5 with (6). Note that the attitudes are synchronized with zero final angular velocities with the control torque (6).

Figs. 4 and 5 show, respectively, the attitudes and angular velocities of rigid bodies 1, 3, and 5 with (10). Note that the attitudes are synchronized with zero final angular velocities with the control torque (10) even without relative angular velocity measurement.

[1]When (12) is implemented in practice, $\ddot{\sigma}_j$, where $a_{ij} \neq 0$, can be approximated by numerical differentiation.

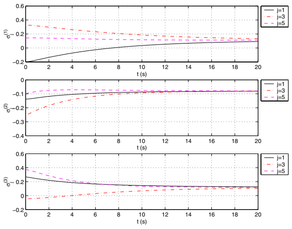

Fig. 2. Rigid body attitudes with the control torque (6).

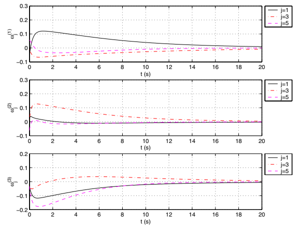

Fig. 3. Rigid body angular velocities with the control torque (6).

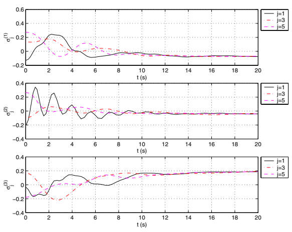

Fig. 4. Attitudes with the control torque (10).

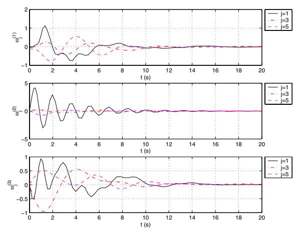

Fig. 5. Angular velocities with the control torque (10).

The directed graph of A used in (12) is shown in Fig. 6, where an edge from i to j denotes that rigid body j can receive information from rigid body i and an edge from σ^r to i denotes that rigid body i has access to σ^r, $\dot{\sigma}^r$, and $\ddot{\sigma}^r$ in (12). Note that σ^r has a directed path to all of the rigid bodies.

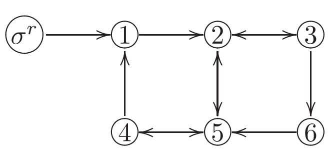

Fig. 6. Directed graph of A in (12).

Figs. 7 and 8 show, respectively, the actual attitudes and angular velocities of rigid bodies 1, 3, and 5 with (10) as well as their references. Note that the attitudes and angular velocities track, respectively, the reference attitude and angular velocity for the group with the control torque (12) even if the reference attitude and angular velocity is available to only rigid body 1.

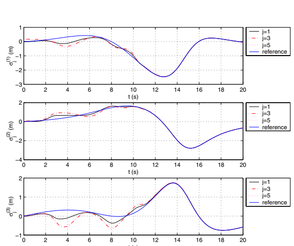

Fig. 7. Attitudes with the control torque (12).

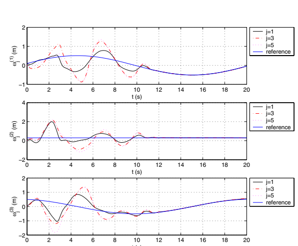

Fig. 8. Angular velocities with the control torque (12).

VII. CONCLUSION AND FUTURE WORK

We have studied distributed attitude synchronization problems with attitudes represented by MRPs and attitude dynamics represented by Euler-Lagrange equations of motion. Three distributed control laws have been proposed and their convergence conditions have also been given. Those control laws extend some existing results in the literature. Future work will involve in addressing the robustness of the proposed control laws in the presence of model uncertainties, noise, and time delays.

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