Proceedings of the 2007 American Control Conference
Marriott Marquis Hotel at Times Square
New York City, USA, July 11-13, 2007

Synchronized Multiple Spacecraft Rotations: A Revisit in the Context of Consensus Building

Wei Ren

Abstract— **The problem of synchronized multiple spacecraft rotations is revisited in the context of the emergent consensus building paradigm. We propose control laws so that multiple spacecraft can maintain given (time-varying) relative attitudes and angular velocities during formation maneuvers. We also propose control laws to guarantee that multiple spacecraft can track a given (time-varying) reference attitude when the reference attitude is only available to a part of the team members. The proposed control laws for reference attitude tracking allow information to flow from any spacecraft to any other spacecraft to introduce information feedback between neighbors, which generalizes the leader-follower approach in the literature. Simulation results on reference attitude tracking are presented to validate our approach.**

I. INTRODUCTION

Multi-vehicle coordination is often tackled by means of a behavioral approach [1], where the control action for each vehicle is defined by a weighted average of the control corresponding to each desired behavior for the vehicle. In [2], several behavioral strategies are presented for formation maneuvers of groups of mobile robots, where a bidirectional ring topology is used to reduce the communication overhead for the whole system and formation patterns are also defined to achieve a sequence of maneuvers. As a decentralized scheme, the behavioral approach can achieve more flexibility, reliability, and robustness than centralized schemes. For example, one advantage of the behavioral approach is that explicit information feedback is included through the information exchange between adjacent neighbors.

Related to the behavioral approach [2] are the consensus-type problems in cooperative control of mobile autonomous agents, where each agent in a team updates its information state based on the information states of its local neighbors in such a way that the final information state of each agent converges to a common value (see [3] for a survey).

In some applications, it is desirable that multiple rigid bodies maintain relative or the same attitudes during formation maneuvers. One example is deep space interferometry, where a formation of networked spacecraft are required to perform a sequence of maneuvers while maintaining relative attitudes accurately. The attitude synchronization problem for multiple spacecraft have been studied in [4]–[8] via information exchange with adjacent neighbors, to name a few, which can be thought of as an extension of the consensus-type problem from single or double integrator dynamics to rigid body rotational dynamics.

W. Ren is with the Department of Electrical and Computer Engineering, Utah State University, Logan, UT 84322, USA wren@engineering.usu.edu

The main purpose of this paper is to extend the results in [4]–[8]. In [6]–[8] multiple spacecraft are restricted to achieve the same attitude while under certain circumstances it might be desirable that multiple spacecraft maintain (time-varying) relative attitudes and angular velocities. Also a reference attitude for the group is not taken into account in [6]–[8]. In addition, the convergence results of the control laws in [8] rely on the assumption that the scalar parts of the unit quaternions are non-negative for all time. However, no conditions are given in [8] to guarantee that this assumption is always valid. In [4], [5], a leader-follower approach is applied for attitude synchronization among a group of spacecraft, where each spacecraft tracks its leader's attitude and information only flows from leaders to followers. While the leader-follower approach is easy to understand and implement, there are limitations. For example, the team leader is a single point of failure for the whole team. Also, there is no explicit feedback from the followers to the leaders: if the follower is perturbed by some disturbances, the attitude synchronization cannot be maintained.

The contributions of the current paper are twofold. First, we propose control laws so that multiple spacecraft can maintain given (time-varying) relative attitudes and angular velocities during formation maneuvers. Second, we propose control laws to guarantee that multiple spacecraft can track a given (time-varying) reference attitude when the reference attitude is only available to a part of the team members. The proposed control laws for reference attitude tracking allow information to flow from any spacecraft to any other spacecraft to introduce information feedback between adjacent neighbors, which generalizes the leader-follower approach in the literature (e.g., [4], [5]). It is worthwhile to mention that although we study the attitude synchronization problem in the context of spacecraft formation flying, the results hereafter are valid for attitude synchronization among other rigid bodies that satisfy the rotational dynamics. Note that the extension from double integrator dynamics to rigid body attitude dynamics is nontrivial. It is also worthwhile to mention that compared to other work in spacecraft attitude control (e.g., [9]) the novelty of this paper lies in the analysis of how inter-spacecraft information exchange plays a key role in attitude synchronization from a consensus-building point of view.

II. BACKGROUND AND PRELIMINARIES

A. Matrix Theory Notations

Let $\mathbf{1}$ denote the $n \times 1$ column vector of all ones. Let I_n denote the $n \times n$ identity matrix. Given a real scalar γ,

1-4244-0989-6/07/$25.00 ©2007 IEEE.

we use $\gamma > 0$ to denote that γ is positive. Given an $n \times n$ real matrix P, we use $P > 0$ to denote that matrix P is symmetric positive definite. In the following, a lower case symbol denotes a scalar or vector while an upper case symbol denotes a matrix.

B. Spacecraft Attitude Dynamics

We use unit quaternions to represent spacecraft attitudes in this paper. A unit quaternion is defined as $q = [\hat{q}^T, \bar{q}]^T \in \mathbb{R}^4$, where $\hat{q} = a \cdot \sin(\frac{\phi}{2}) \in \mathbb{R}^3$ denotes the vector part and $\bar{q} = \cos(\frac{\phi}{2}) \in \mathbb{R}$ denotes the scalar part of the unit quaternion. In this notation, $a \in \mathbb{R}^3$ is a unit vector, known as the Euler axis, and $\phi \in \mathbb{R}$ is the rotation angle about a, called the Euler angle. Note that $q^T q = 1$ by definition. A unit quaternion is not unique since q and $-q$ represent the same attitude. However, uniqueness can be achieved by restricting ϕ to the range $0 \leq \phi \leq \pi$ so that $\bar{q} \geq 0$ [10].

The product of two unit quaternions p and q is defined by $qp = \begin{bmatrix} \bar{q}\hat{p} + \bar{p}\hat{q} + \hat{q} \times \hat{p} \\ \bar{q}\bar{p} - \hat{q}^T \hat{p} \end{bmatrix}$, which is also a unit quaternion. The conjugate of the unit quaternion q is defined by $q^* = [-\hat{q}^T, \bar{q}]^T$. The conjugate of qp is given by $(qp)^* = p^* q^*$. The multiplicative identity quaternion is denoted by $\mathbf{q}_I = [0, 0, 0, 1]^T$, where $qq^* = q^* q = \mathbf{q}_I$ and $q\mathbf{q}_I = \mathbf{q}_I q = q$ [10].

Spacecraft attitude dynamics are given by

$$\begin{aligned} \dot{\widehat{q_i}} &= -\frac{1}{2}\omega_i \times \widehat{q_i} + \frac{1}{2}\overline{q_i}\omega_i, \qquad \dot{\overline{q_i}} = -\frac{1}{2}\omega_i \cdot \widehat{q_i} \\ J_i \dot{\omega}_i &= -\omega_i \times (J_i \omega_i) + \tau_i, \quad i = 1, \cdots, n \end{aligned} \tag{1}$$

where n is the total number of spacecraft in the team, $\widehat{q_i} \in \mathbb{R}^3$ and $\overline{q_i} \in \mathbb{R}$ are vector and scalar parts of the unit quaternion of the i^{th} spacecraft, $\omega_i \in \mathbb{R}^3$ is the angular velocity, and $J_i \in \mathbb{R}^{3\times 3}$ and $\tau_i \in \mathbb{R}^3$ are inertia tensor and control torque [10].

Before moving on, we need the following lemma for our main results.

Lemma 2.1: If the unit quaternion and angular velocity pairs (q_k, ω_k) and (q_ℓ, ω_ℓ) satisfy the quaternion kinematics defined by the first two equations in Eq. (1), then the unit quaternion and angular velocity pair $(q_\ell^* q_k, \omega_k - \omega_\ell)$ also satisfies the quaternion kinematics. In addition, if $V_q = \|q_\ell^* q_k - \mathbf{q}_I\|^2$, then $\dot{V}_q = (\omega_k - \omega_\ell)^T \widehat{q_\ell^* q_k}$, where $\hat{p}$ denotes the vector part of quaternion p.
Proof: See [9]. ■

C. Graph Theory

It is natural to model information exchange between spacecraft by directed/undirected graphs. A digraph (directed graph) consists of a pair $(\mathcal{N}, \mathcal{E})$, where $\mathcal{N}$ is a finite nonempty set of nodes and $\mathcal{E} \in \mathcal{N}^2$ is a set of ordered pairs of nodes, called edges. As a comparison, the pairs of nodes in an undirected graph are unordered. If there is a directed edge from node v_i to node v_j, then v_i is defined as the parent node and v_j is defined as the child node. A directed path is a sequence of ordered edges of the form $(v_{i_1}, v_{i_2}), (v_{i_2}, v_{i_3}), \cdots$, where $v_{i_j} \in \mathcal{N}$, in a digraph. An undirected path in an undirected graph is defined accordingly. In a digraph, a cycle is a path that starts and ends at the same node. A digraph is called strongly connected if there is a directed path from every node to every other node. An undirected graph is called connected if there is a path between any distinct pair of nodes. A directed tree is a digraph, where every node has exactly one parent except for one node, called root, which has no parent, and the root has a directed path to every other node. Note that in a directed tree, each edge has a natural orientation away from the root, and no cycle exists. In the case of undirected graphs, a tree is a graph in which every pair of nodes is connected by exactly one path. A directed spanning tree of a digraph is a directed tree formed by graph edges that connect all the nodes of the graph. A graph has (or contains) a directed spanning tree if there exists a directed spanning tree being a subset of the graph. Note that the condition that a digraph has a directed spanning tree is equivalent to the case that there exists at least one node having a directed path to all the other nodes. In the case of undirected graphs, having an undirected spanning tree is equivalent to being connected. However, in the case of directed graphs, having a directed spanning tree is a weaker condition than being strongly connected.

The adjacency matrix $G = [g_{ij}] \in \mathbb{R}^{n\times n}$ of a graph is defined as $g_{ii} = 0$ and $g_{ij} = 1$ if $(j, i) \in \mathcal{E}$ where $i \neq j$. For a weighted graph, G is defined as $g_{ii} = 0$ and $g_{ij} > 0$ if $(j, i) \in \mathcal{E}$ where $i \neq j$. Note that the adjacency matrix of an undirected graph is symmetric since $(j, i) \in \mathcal{E}$ implies $(i, j) \in \mathcal{E}$. However, the adjacency matrix of a digraph does not have this property. Let matrix $L = [\ell_{ij}] \in \mathbb{R}^{n\times n}$ be defined as $\ell_{ii} = \sum_{j\neq i} g_{ij}$ and $\ell_{ij} = -g_{ij}$ where $i \neq j$. The matrix L satisfies the following conditions:

$$\ell_{ij} \leq 0, \quad i \neq j, \qquad \sum_{j=1}^{n} \ell_{ij} = 0, \quad i = 1, \cdots, n. \tag{2}$$

For an undirected graph, L is called the Laplacian matrix [11], which is symmetric positive semi-definite. However, L for a digraph does not have this property.

In the case of an undirected information-exchange graph, L has a simple zero eigenvalue with an associated eigenvector $\mathbf{1}$ and all the other eigenvalues are positive if and only if the graph is connected [11]. In the case of a directed information-exchange graph, L has a simple zero eigenvalue with an associated eigenvector $\mathbf{1}$ and all the other eigenvalues have positive real parts if and only if the digraph has a directed spanning tree [12]. Let $x = [x_1, \cdots, x_n]^T$, where $x_j \in \mathbb{R}$, $j = 1, \cdots, n$, and $y = [y_1^T, \cdots, y_n^T]^T$, where $y_j \in \mathbb{R}^m$, $j = 1, \cdots, n$. Under the conditions of both cases, $Lx = 0$ implies that $x = \alpha\mathbf{1}$ (i.e. $x_1 = \cdots = x_n$), where $\alpha \in \mathbb{R}$, and $(L \otimes I_m)y = 0$, where $\otimes$ is the Kronecker product, implies that $y = \mathbf{1} \otimes \beta$ (i.e. $y_1 = \cdots = y_n$), where $\beta \in \mathbb{R}^m$.

The digraph of an $n \times n$ real matrix $S = [s_{ij}]$, denoted by $\Gamma(S)$, is the digraph on n nodes such that there is a directed edge in $\Gamma(S)$ from v_j to v_i if and only if $s_{ij} \neq 0$ (c.f. [13]).

III. RELATIVE ATTITUDE MAINTENANCE

In this section, we consider the case that multiple spacecraft are required to maintain given relative attitudes and angular velocities during formation maneuvers. Before moving on, we need the following lemmas.

Lemma 3.1: [8] Let

$$\tau_i = -k_G \widehat{q^{d*}q_i} - D_{Gi}\omega_i - \sum_{j=1}^{n} g_{ij}[a_{ij}\widehat{q_j^* q_i} + b_{ij}(\omega_i - \omega_j)], \tag{3}$$

where $k_G \in I\!R \geq 0$, $D_{Gi} \in I\!R^{3\times 3} > 0$, $q^d \in I\!R^4$ denotes the desired constant attitude for each spacecraft, $a_{ij} = a_{ji} \in I\!R > 0$, $b_{ij} = b_{ji} \in I\!R > 0$, $g_{ii} \stackrel{\triangle}{=} 0$, and g_{ij} is 1 if spacecraft i receives information from spacecraft j and 0 otherwise. If $V = k_G \sum_{i=1}^n \left\| q^{d*}q_i - \mathbf{q}_I \right\|^2 + \frac{1}{2}\sum_{i=1}^n \sum_{j=1}^n g_{ij}a_{ij}\left\| q_j^* q_i - \mathbf{q}_I \right\|^2 + \frac{1}{2}\sum_{i=1}^n (\omega_i^T J_i \omega_i)$, then $\dot{V} = -\sum_{i=1}^n (\omega_i^T D_{Gi}\omega_i) - \frac{1}{2}\sum_{i=1}^n \sum_{j=1}^n g_{ij}b_{ij}\left\|\omega_i - \omega_j\right\|^2$.

Lemma 3.2: [8] Let

$$k_G \widehat{q^{d*}q_i} + \sum_{j=1}^n g_{ij}a_{ij}\widehat{q_j^* q_i} = 0, \quad i = 1, \cdots, n. \tag{4}$$

Then Eq. (4) can be written in matrix form as

$$(P(t) \otimes I_3)\,\widehat{q_s} = 0, \tag{5}$$

where $\otimes$ is the Kronecker product, I_3 is the 3×3 identity matrix, $\widehat{q_s} \in I\!R^{3n}$ is a column vector stack composed of $\widehat{q^{d*}q_\ell}$, $\ell = 1, \cdots, n$, and $P(t) = [p_{ij}(t)] \in I\!R^{n\times n}$ is given by $p_{ii}(t) = k_G + \sum_{j=1}^n g_{ij}a_{ij}\overline{q^{d*}q_j}$ and $p_{ij}(t) = -g_{ij}a_{ij}\overline{q^{d*}q_i}$.

Lemma 3.3: [14, Corollary 4.1] Let $x = 0$ be an equilibrium for $\dot{x} = f(x)$, where $f : D \to I\!R^n$ is a locally Lipschitz map from a domain $D \in I\!R^n$ into $I\!R^n$. Let $V : D \to I\!R$ be a continuously differentiable positive-definite function on a domain D containing the origin $x = 0$, such that $\dot{V}(x) \leq 0$ in D. Let $S = \{x \in D | \dot{V}(x) = 0\}$ and suppose that no solution can stay identically in S, other than the trivial solution $x(t) \equiv 0$. Then, the origin is asymptotically stable.

A. Fixed Relative Attitudes with Zero Final Angular Velocities

In the case that multiple spacecraft are required to maintain fixed relative attitudes with zero final angular velocities, we propose the control torque to the i^{th} spacecraft as

$$\tau_i = -k_G \widehat{q^{d*}q_i q_{\delta_i}} - D_{Gi}\omega_i - \sum_{j=1}^n g_{ij}[a_{ij} q_{\delta_j}^* \widehat{q_j^* q_i q_{\delta_i}} + b_{ij}(\omega_i - \omega_j)], \tag{6}$$

where $q_{\delta_\ell} \in I\!R^4$, $\ell = 1, \cdots, n$, are constant quaternions defining the relative attitudes between the ℓ^{th} spacecraft and the desired constant attitude q^d, and k_G, D_{Gi}, a_{ij}, b_{ij}, g_{ij} are defined as in Eq. (3).

Note that the i^{th} spacecraft defines q_{δ_i}. Also note that product $q_{\delta_j} q_{\delta_i}^*$ defines the relative attitudes between the i^{th} spacecraft and the j^{th} spacecraft. As a result, relative attitudes between the spacecraft can be achieved by appropriately choosing q_{δ_i}, $i = 1, \cdots, n$.

Theorem 3.1: Assume that the control torque is given by Eq. (6) and the undirected communication graph is connected. Let $\mathcal{E}$ denote the edge set of unordered pairs of spacecraft, where an edge $(k, \ell) \in \mathcal{E}$ implies that $g_{k\ell} = g_{\ell k} = 1$[1]. Also let $|\mathcal{E}|$ denote the cardinality of $\mathcal{E}$. If $k_G > 2\sum_{j=1}^n g_{ij}a_{ij}$, then $q_i \to q^d q_{\delta_i}^*$, $i = 1, \cdots, n$, and $\omega_i \to \omega_j \to 0$ asymptotically, $\forall i \neq j$. If $k_G = 0$ and $|\mathcal{E}| \leq n$, then $q_j^* q_i \to q_{\delta_j} q_{\delta_i}^*$ and $\omega_i \to \omega_j \to 0$ asymptotically, $\forall i \neq j$.

Proof: Consider a Lyapunov function candidate V defined in Lemma 3.1 with q_ℓ replaced by $q_\ell q_{\delta_\ell}$, $\ell = 1, \ldots, n$. The derivative of V is negative semi-definite from Lemma 3.1.

In Subcase A ($k_G > 2\sum_{j=1}^n g_{ij}a_{ij}$), let $S = \{q^{d*}q_i q_{\delta_i} - \mathbf{q}_I, \omega_i | \dot{V} = 0\}$. Note that $\dot{V} \equiv 0$ implies that $\omega_i \equiv 0$, $i = 1, \cdots, n$. Because $\omega_i \equiv 0$, Eq. (4) with q_ℓ replaced by $q_\ell q_{\delta_\ell}$ holds from Eqs. (1) and (6). From Lemma 3.2, we know that Eq. (4) with q_ℓ replaced by $q_\ell q_{\delta_\ell}$ can be written in matrix form as Eq. (5) with q_ℓ replaced by $q_\ell q_{\delta_\ell}$.

Noting that $|\overline{q^{d*}q_j q_{\delta_j}}| \leq 1$, $j = 1, \cdots, n$, and $k_G > 2\sum_{j=1}^n g_{ij}a_{ij}$, we see that $P(t)$ is strictly diagonally dominant and therefore has full rank, which in turn implies that $\widehat{q_s} \equiv 0$. Thus, we see that $\widehat{q^{d*}q_i q_{\delta_i}} \equiv 0$, $i = 1, \cdots, n$, which implies that $q^{d*}q_i q_{\delta_i} - \mathbf{q}_I \equiv 0$ if $\dot{V} \equiv 0$.

Therefore, by Lemma 3.3, it follows that $q^{d*}q_i q_{\delta_i} - \mathbf{q}_I \to 0$ and $\omega_i \to \omega_j \to 0$ asymptotically. Equivalently, we know that $q_i \to q^d q_{\delta_i}^*$, $i = 1, \cdots, n$, and $\omega_i \to \omega_j \to 0$.

In Subcase B ($k_G = 0$), let $S = \{(q_j q_{\delta_j})^* q_i q_{\delta_i} - \mathbf{q}_I, \omega_i | \dot{V} = 0\}$. Note that $\dot{V} \equiv 0$ implies that $\omega_i \equiv 0$, $i = 1, \cdots, n$. Because $\omega_i \equiv 0$, we know that

$$\sum_{j=1}^n g_{ij}a_{ij}\widehat{(q_j q_{\delta_j})^* q_i q_{\delta_i}} = 0, \quad i = 1, \cdots, n \tag{7}$$

from Eqs. (1) and (6).

Let $\widehat{(q_j q_{\delta_j})^* q_i q_{\delta_i}}$ be a variable associated with an edge $(i, j) \in \mathcal{E}$, where $i < j$. Noting that the undirected communication graph is connected and $|\mathcal{E}| \leq n$, we know that $|\mathcal{E}| = n-1$ or $|\mathcal{E}| = n$, which implies that there are $n-1$ or n variables associated with the edge set $\mathcal{E}$. Let $\widehat{q_u}$ be a column vector stack composed of all $\widehat{(q_j q_{\delta_j})^* q_i q_{\delta_i}}$, $\forall (i, j) \in \mathcal{E}$, where $i < j$. By noting that $\widehat{(q_i q_{\delta_i})^* q_j q_{\delta_j}} = -\widehat{(q_j q_{\delta_j})^* q_i q_{\delta_i}}$, Eq. (7) can be rewritten as

$$(Q \otimes I_3)\,\widehat{q_u} = 0, \tag{8}$$

where $Q \in I\!R^{n\times n-1}$ and $\widehat{q_u} \in I\!R^{3(n-1)}$ if $|\mathcal{E}| = n - 1$, and $Q \in I\!R^{n\times n}$ and $\widehat{q_u} \in I\!R^{3n}$ if $|\mathcal{E}| = n$.

Consider a system given by $Q\tilde{x} = 0$, where $\tilde{x}$ is a column vector stack composed of $x_{ij} = x_i - x_j$, $\forall (i, j) \in \mathcal{E}$, where $i < j$ and $x_k \in I\!R$, $k = 1, \cdots, n$. Note that $Q\tilde{x} = 0$ can be written as $Lx = 0$, where $x = [x_1, \cdots, x_n]^T$ and L is the graph Laplacian matrix. Noting that the undirected

[1]Note that (k, ℓ) and (ℓ, k) denote the same element in $\mathcal{E}$ in the case of undirected graphs. In the following we assume that $k < \ell$ without loss of generality.

communication graph is connected, we know that $x_1 = \cdots = x_n$, which in turn implies that $\tilde{x} = 0$. As a result, we know that Q can be transformed to a row echelon form to show that $Q\tilde{x} = 0$ implies that $x_{ij} = 0$, $\forall (i,j) \in \mathcal{E}$, where $i < j$. The same transformation procedure can be applied to Eq. (8) to show that $(Q \otimes I_3)\widehat{q_u} = 0$ implies that $\widehat{(q_j q_{\delta_j})^* q_i q_{\delta_i}} = 0$, $\forall (i,j) \in \mathcal{E}$, where $i < j$. Thus, we see that $(q_j q_{\delta_j})^* q_i q_{\delta_i} - \mathbf{q}_I \equiv 0$, $\forall i \neq j$, if $\dot{V} \equiv 0$.

Therefore, by Lemma 3.3, it follows that $(q_j q_{\delta_j})^* q_i q_{\delta_i} - \mathbf{q}_I \to 0$ and $\omega_i \to 0$ asymptotically. Equivalently, we know that $q_j^* q_i \to q_{\delta_j} q_{\delta_i}^*$ and $\omega_i \to \omega_j \to 0$ asymptotically, $\forall i \neq j$. ■

B. Time-varying Relative Attitudes and Angular Velocities

In the case that multiple spacecraft are required to maintain time-varying relative attitudes and angular velocities, we propose the control torque to the i^{th} spacecraft as

$$\tau_i = -J_i \dot{\omega}_{\delta_i} + \omega_i \times (J_i \omega_i) - J_i \sum_{j=1}^{n} g_{ij} \{ a_{ij} (\widehat{q_{\delta_j}^* q_j^* q_i q_{\delta_i}}) + b_{ij} [(\omega_i - \omega_j) + (\omega_{\delta_i} - \omega_{\delta_j})] \}, \tag{9}$$

where a_{ij}, b_{ij}, and g_{ij} are defined as in Eq. (3), and the unit quaternion and angular velocity pair $(q_{\delta_i}, \omega_{\delta_i})$ satisfies the quaternion kinematics and defines the desired relative attitudes and angular velocities between the spacecraft.

Theorem 3.2: With the control torque (9), if the undirected communication graph is connected and $|\mathcal{E}| \leq n$, where $\mathcal{E}$ is defined as in Theorem 3.1, then $q_j^* q_i \to q_{\delta_j} q_{\delta_i}^*$ and $\omega_i \to \omega_j + \omega_{\delta_j} - \omega_{\delta_i}$ asymptotically, $\forall i \neq j$.

Proof: Consider a Lyapunov function candidate: $V = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} g_{ij} a_{ij} \left\| \tilde{q}_j^* \tilde{q}_i - \mathbf{q}_I \right\|^2 + \frac{1}{2} \sum_{i=1}^{n} \tilde{\omega}_i^T \tilde{\omega}_i$, where $\tilde{q}_i = q_i q_{\delta_i}$ and $\tilde{\omega}_i = \omega_i + \omega_{\delta_i}$. Following Lemma 2.1, we obtain $\dot{V} = -\frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} g_{ij} b_{ij} \left\| \tilde{\omega}_i - \tilde{\omega}_j \right\|^2 \leq 0$.

Let $S = \{ (\tilde{q}_j^* \tilde{q}_i - \mathbf{q}_I, \tilde{\omega}_i | \dot{V} = 0 \}$. Note that $\dot{V} \equiv 0$ implies that $\tilde{\omega}_i \equiv \tilde{\omega}_j$, $\forall i \neq j$, since the undirected communication graph is connected. Following the proof to Theorem 4.2 in [8], we can show that $\sum_{j=1}^{n} g_{ij} a_{ij} \widehat{\tilde{q}_j^* \tilde{q}_i} = 0$, $i = 1, \cdots, n$. Then following the proof to Subcase B in Theorem 3.1, we know that $\tilde{q}_i \to \tilde{q}_j$ and $\tilde{\omega}_i \to \tilde{\omega}_j$ asymptotically, $\forall i \neq j$. Therefore, we see that $q_j^* q_i \to q_{\delta_j} q_{\delta_i}^*$ and $\omega_i \to \omega_j + \omega_{\delta_j} - \omega_{\delta_i}$ asymptotically, $\forall i \neq j$. ■

Note that in the case of $q_{\delta_i} = \mathbf{q}_I$ and $\omega_{\delta_i} = 0$, the control torque (9) guarantees that $q_i \to q_j$ and $\omega_i \to \omega_j$, $\forall i \neq j$. Also note that the convergence results of the control laws in this paper do not rely on the assumption that the scalar parts of the unit quaternions are non-negative for all time as in [8], where no conditions are given to guarantee that this assumption is always valid.

IV. REFERENCE ATTITUDE TRACKING

In this section, we consider the case that multiple spacecraft are required to track a (time-varying) reference attitude. Let $q^d(t) \in \mathbb{R}^4$ and $\omega^d(t) \in \mathbb{R}^3$ denote, respectively, the (time-varying) reference attitude and angular velocity, which satisfy rotational dynamics (1). The goal is to guarantee that $q_i \to q_j \to q^d(t)$ and $\omega_i \to \omega_j \to \omega^d(t)$, $\forall i \neq j$.

A. Time-varying Reference Attitude: Full Access

In the case that the reference attitude and angular velocity are available to each spacecraft, we propose the control torque to the i^{th} spacecraft as

$$\tau_i = \omega_i \times (J_i \omega_i) + J_i \dot{\omega}^d - k_G \widehat{q^{d*} q_i} - D_{Gi} (\omega_i - \omega^d) - \sum_{j=1}^{n} g_{ij} [a_{ij} \widehat{q_j^* q_i} + b_{ij} (\omega_i - \omega_j)], \tag{10}$$

where k_G, D_{Gi}, a_{ij}, b_{ij}, g_{ij} are defined as in Eq. (3).

Corollary 4.1: Assume that the control torque is given by Eq. (10) and the undirected communication graph is connected. Also assume that $q^d(t)$ and $\omega^d(t)$ satisfy the quaternion kinematics. Let $\mathcal{E}$ be defined as in Theorem 3.1. If $k_G > 2 \sum_{j=1}^{n} g_{ij} a_{ij}$, then $q_i \to q_j \to q^d(t)$ and $\omega_i \to \omega_j \to \omega^d(t)$ asymptotically, $\forall i \neq j$. If $k_G = 0$ and $|\mathcal{E}| \leq n$, then $q_i \to q_j$ and $\omega_i \to \omega_j \to \omega^d(t)$ asymptotically, $\forall i \neq j$.

Proof: If $k_G > 2 \sum_{j=1}^{n} g_{ij} a_{ij}$, we let $\tilde{q}_i = q^{d*} q_i$ and $\tilde{\omega}_i = \omega_i - \omega^d$. Note that $\tilde{q}_i$ and $\tilde{\omega}_i$ also satisfy the quaternion kinematics. Following the proof to Theorem 3.1 with q_i, ω_i, q^d, and q_{δ_i} replaced, respectively, by $\tilde{q}_i$, $\tilde{\omega}_i$, $\mathbf{q}_I$, and $\mathbf{q}_I$, we see that $\tilde{q}_i \to \tilde{q}_j \to \mathbf{q}_I$ and $\tilde{\omega}_i \to \tilde{\omega}_j \to 0$, that is, $q_i \to q_j \to q^d(t)$ and $\omega_i \to \omega_j \to \omega^d(t)$ asymptotically, $\forall i \neq j$. If $k_G = 0$ and $|\mathcal{E}| \leq n$, we see that $\tilde{q}_i \to \tilde{q}_j$ and $\tilde{\omega}_i \to \tilde{\omega}_j \to 0$, that is, $q_i \to q_j$ and $\omega_i \to \omega_j \to \omega^d(t)$ asymptotically, $\forall i \neq j$. ■

B. Time-varying Reference Attitude: Partial Access

Note that the control law (10) assumes that the reference attitude is available to each spacecraft in the team, which might be restrictive under certain circumstances. In the case that the reference attitude is only available to a part of the team members, we propose the control torque to the i^{th} spacecraft as

$$\tau_i = \omega_i \times (J_i \omega_i) + \frac{1}{|\mathcal{N}_i| + 1} J_i (\dot{\omega}^d + \sum_{j \in \mathcal{N}_i} \dot{\omega}_j) - \frac{1}{|\mathcal{N}_i| + 1} \{ k_{qi} \widehat{p_{\pi_i}} + K_{\omega i} [(\omega_i - \omega^d) + \sum_{j \in \mathcal{N}_i} (\omega_i - \omega_j)] \}, \quad i \in \mathcal{L} \tag{11}$$

$$\tau_i = \omega_i \times (J_i \omega_i) + \frac{1}{|\mathcal{N}_i|} J_i \sum_{j \in \mathcal{N}_i} \dot{\omega}_j - \frac{1}{|\mathcal{N}_i|} [k_{qi} \widehat{q_{\pi_i}} + \sum_{j \in \mathcal{N}_i} K_{\omega i} (\omega_i - \omega_j)], \quad i \notin \mathcal{L} \tag{12}$$

where $\mathcal{N}_i$ denotes the set of spacecraft whose information is available to spacecraft i, $|\mathcal{N}_i|$ denotes the cardinality of $\mathcal{N}_i$, $\mathcal{L}$ denotes the set of vehicles to which q^d, ω^d and $\dot{\omega}^d$ are available, $k_{qi} \in \mathbb{R} > 0$, $K_{\omega i} \in \mathbb{R}^{3 \times 3} > 0$, $p_{\pi_i} = [\prod_{j \in \mathcal{N}_i} (q_j^* q_i)] q^{d*} q_i$, and $q_{\pi_i} = \prod_{j \in \mathcal{N}_i} (q_j^* q_i)$. Note that $i \notin \mathcal{N}_i$. Also note that $j \in \mathcal{N}_i$ does not imply that $i \in \mathcal{N}_j$ in the case of directed information exchange.

Theorem 4.2: Let $G = [g_{ij}] \in \mathbb{R}^{(n+1) \times (n+1)}$, where g_{ij}, $i, j \in \{1, \cdots, n\}$, is 1 if information flows from spacecraft j to spacecraft i and 0 otherwise, $g_{i(n+1)}$, $i = 1, \cdots, n$, is 1 if q^d, ω^d and $\dot{\omega}^d$ are available to spacecraft i and 0 otherwise, and $g_{(n+1)j} = 0$, $j = 1, \cdots, n+1$. With the control torques (11) and (12), if the graph of G has a directed spanning tree[2], then $\widehat{p_{\pi_i}} \to 0$, $i \in \mathcal{L}$, $\widehat{q_{\pi_i}} \to 0$, $i \notin \mathcal{L}$, and $\omega_i \to \omega^d$, $i = 1, \cdots, n$, asymptotically.[3]

Proof: Let $q_{n+1} \equiv q^d$ and $\omega_{n+1} \equiv \omega^d$. Also let $\mathcal{J}_i = \mathcal{N}_i$ if $g_{i(n+1)} = 0$ and $\mathcal{J}_i = \mathcal{N}_i \cup \{n+1\}$ if $g_{i(n+1)} = 1$. Then Eqs. (11) and (12) can be rewritten as

$$\begin{aligned}\tau_i = {} & \omega_i \times (J_i \omega_i) + \frac{1}{|\mathcal{J}_i|} J_i \sum_{j \in \mathcal{J}_i} \dot{\omega}_j \\ & - \frac{1}{|\mathcal{J}_i|}[k_{qi}\widehat{s_{\pi_i}} + K_{\omega i} \sum_{j \in \mathcal{J}_i} (\omega_i - \omega_j)], \quad i = 1, \cdots, n,\end{aligned} \tag{13}$$

where $s_{\pi_i} = \prod_{j \in \mathcal{J}_i} (q_j^* q_i)$. Combining Eqs. (1) and (13), we get that

$$J_i \dot{\omega}_{\sigma_i} = -k_{qi} \widehat{s_{\pi_i}} - K_{\omega i} \omega_{\sigma_i}, \quad i = 1, \cdots, n, \tag{14}$$

where $\omega_{\sigma_i} = \sum_{j \in \mathcal{J}_i} (\omega_i - \omega_j)$. Note that the quaternion and angular velocity pair $(s_{\pi_i}, \omega_{\sigma_i})$ satisfies the quaternion kinematics. Then Eq. (14) implies that $\widehat{s_{\pi_i}} \to 0$ and $\omega_{\sigma_i} \to 0$, $i = 1, \cdots, n$ [9]. Let $L_\omega = [\ell_{ij}]$ be an $(n+1) \times (n+1)$ matrix, where $\ell_{ii} = \sum_{j=1}^{n+1} g_{ij}$ and $\ell_{ij} = -g_{ij}$, $i \neq j$. Note that L_ω satisfies the property (2) and all entries of the n+1th row of L_ω are zero. Also note that $\omega_{\sigma_i} \to 0$, $i = 1, \cdots, n$, can be written in matrix form as $(L_\omega \otimes I_3)\omega \to 0$, where $\omega = [\omega_1^T, \cdots, \omega_{n+1}^T]^T$. Noting that the graph of G has a directed spanning tree, we know that $\omega_i \to \omega_j$, $i, j \in \{1, \cdots, n+1\}$, which in turn implies that $\omega_i \to \omega^d$, since $\omega_{n+1} \equiv \omega^d$. ∎

Note that in the leader-follower approach (e.g., [4], [5]) information only flows from leaders to followers and each spacecraft except the team leader has exactly one parent (e.g., no information loops allowed). As a comparison, the control laws (11) and (12) allow information to flow from any spacecraft to any other spacecraft (e.g., followers to leaders), which introduces information feedback between neighboring spacecraft and increases redundancy and robustness to the whole system in the case of failures of certain information-exchange links.

In the information-exchange topology, if a node k has exactly one parent, node ℓ, then $\widehat{s_{\pi_k}} = \widehat{q_\ell^* q_k} \to 0$ implies that $q_k \to q_\ell$. That is, spacecraft k approaches the reference attitude q_0^d if $\ell = n+1$, or spacecraft k and ℓ approach the same attitude if $\ell \neq n+1$. As a result, edge (ℓ, k) can be deleted in the information-exchange topology, and nodes k and ℓ can be combined as one single node whose incoming and outgoing edges are the union of the incoming and outgoing edges of nodes k and ℓ. By repeating this procedure, we can simplify the information-exchange topology. If the information-exchange topology can be simplified to a graph with only one node, then $\widehat{p_{\pi_i}} \to 0$, $i \in \mathcal{L}$, and $\widehat{q_{\pi_i}} \to 0$, $i \notin \mathcal{L}$, directly imply that $q_i \to q_j \to q_0^d$. The leader-follower approach for attitude alignment (e.g., [4], [5]) can be considered a special case of the control laws (11) and (12), where each spacecraft has at most one neighbor (i.e., its leader).

As an illustrative example, Fig. 1 shows the information-exchange topologies between four spacecraft, where node q^d denotes the reference attitude and node A_i, $i = 1, \cdots, 4$, denote the i^{th} spacecraft. Note that in Fig. 1 a link from node q^d to node A_j denotes that the reference attitude is available to spacecraft j. In particular, the leader-follower approach (e.g., [4], [5]) corresponds to Subplots (a) and (b) in Fig. 1, where each spacecraft has only one parent node. As a comparison, control laws (11) and (12) correspond to Subplots (c) and (d) in Fig. 1, which are more general than Subplots (a) and (b) in the sense that information can flow between any neighboring spacecraft to introduce feedback between neighbors and the reference attitude might be available to one or more spacecraft in the team. Note that node q^d has a directed path to all the spacecraft in the team in Fig. 1.

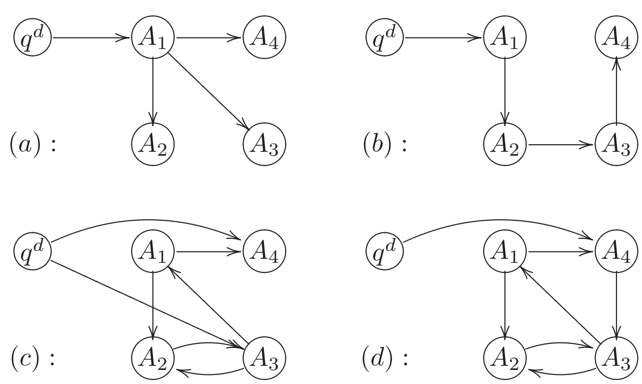

Fig. 1. Information-exchange topologies between four spacecraft where Subplots (a) and (b) correspond to the leader-follower approach and Subplots (c) and (d) correspond to control laws (11) and (12).

V. SIMULATION

In this section, we apply the control laws (11) and (12) to guarantee that four spacecraft follow a time-varying reference attitude $q^d(t)$ and angular velocity $\omega^d(t)$ that satisfy the rotational dynamics given by

$$\begin{aligned}\dot{\widehat{q}}^d &= -\frac{1}{2}\omega^d \times \widehat{q}^d + \frac{1}{2}\overline{q}^d \omega^d, \qquad \dot{\overline{q}}^d = -\frac{1}{2}\omega^d \cdot \widehat{q}^d \\ J^d \dot{\omega}^d &= -\omega^d \times (J^d \omega^d) + \tau^d.\end{aligned}$$

In the simulation, we let $\tau^d = [0, 0, 0]^T$, $J^d = \text{diag}\{1, 2, 1\}$, $q^d(0) = [0, 0, 0, 1]^T$, and $\omega^d(0) = [0.1, 0.3, 0.5]^T$. We also choose $q_i(0)$ and $\omega_i(0)$, $i = 1, \cdots, 4$, randomly. Also let $k_{qi} = 1$ and $K_{\omega i} = 2I_3$, $i = 1, \cdots, 4$, in Eqs. (11) and (12). The information flow between the four spacecraft is given by Subplot (c) in Fig. 1, where the reference attitude is available to spacecraft 3

[2]Define a virtual node $n+1$ whose states are q^d, ω^d and $\dot{\omega}^d$. With G defined in Theorem 4.2, where $g_{(n+1)j} = 0$, $j = 1, \cdots, n+1$, the condition that the graph of G has a directed spanning tree implies that node $n+1$ is the root, which is also equivalent to the condition that node $n+1$ has a directed path to all the spacecraft in the team.

[3]Note that in Theorem 4.2 we consider the general case of directed information exchange.

and 4. We also assume that the control torque of each spacecraft satisfies $|\tau_i^{(j)}| \leq 1$ Nm, where $j = 1, 2, 3$ denotes each component of the control torque. In the following, a superscript (j) denotes the j^{th} component of a quaternion or vector.

Figs. 2 shows the actual attitudes of each spacecraft and the reference attitude. Fig. 3 shows the actual angular velocities of each spacecraft and the reference angular velocity. Note that the actual attitudes and angular velocities of each spacecraft converge to their reference values. Fig. 4 shows the control torques of each spacecraft, which satisfy the saturation constraint.

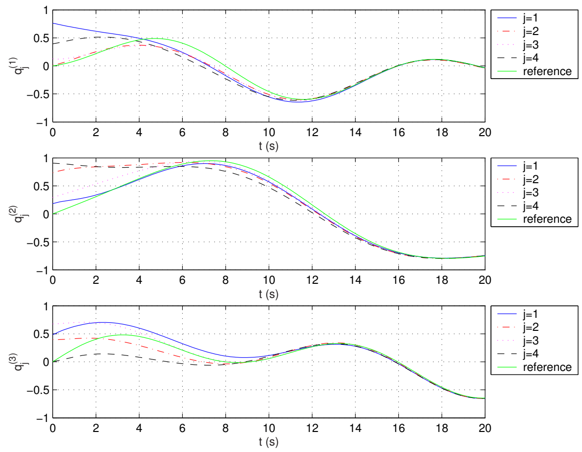

Fig. 2. Actual attitudes of each spacecraft and the reference attitude q^d.

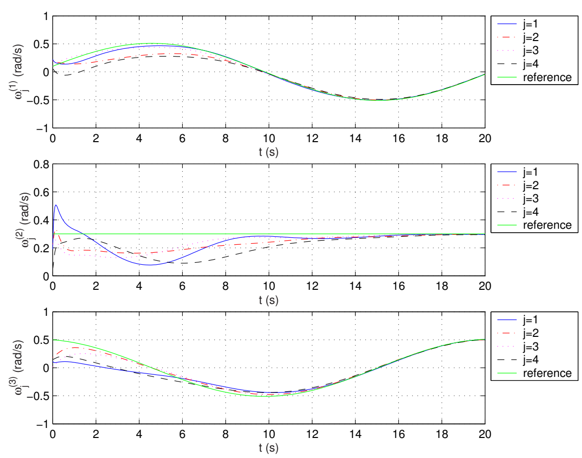

Fig. 3. Actual angular velocities of each spacecraft and the reference angular velocity ω^d.

VI. CONCLUSION AND FUTURE WORK

We have revisited the problem of synchronized multiple spacecraft rotations in the context of consensus building. Control laws have been proposed for relative attitude maintenance between the spacecraft during formation maneuvers. Reference attitude tracking in the presence of information

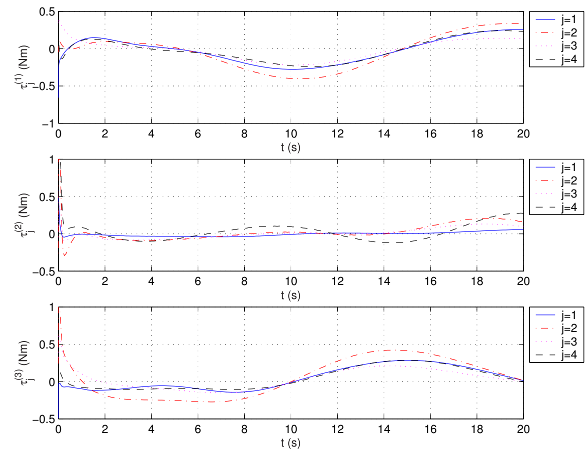

Fig. 4. Spacecraft control torques.

loops or feedback between neighbors has also been studied. Simulation results on reference attitude tracking for multiple spacecraft have demonstrated the effectiveness of our approach. Future work will address attitude alignment under switching directed information-exchange topologies. In addition, the extension of the current work to an orbital environment will also be a topic of future research.

REFERENCES

[1] T. Balch and R. C. Arkin, "Behavior-based formation control for multirobot teams," *IEEE Transactions on Robotics and Automation*, vol. 14, no. 6, pp. 926–939, December 1998.

[2] J. R. Lawton, R. W. Beard, and B. Young, "A decentralized approach to formation maneuvers," *IEEE Transactions on Robotics and Automation*, vol. 19, no. 6, pp. 933–941, December 2003.

[3] W. Ren, R. W. Beard, and E. M. Atkins, "Information consensus in multivehicle cooperative control: Collective group behavior through local interaction," *IEEE Control Systems Magazine*, April 2007.

[4] P. K. C. Wang and F. Y. Hadaegh, "Coordination and control of multiple microspacecraft moving in formation," *The Journal of the Astronautical Sciences*, vol. 44, no. 3, pp. 315–355, 1996.

[5] P. K. C. Wang, F. Y. Hadaegh, and K. Lau, "Synchronized formation rotation and attitude control of multiple free-flying spacecraft," *AIAA Journal of Guidance, Control and Dynamics*, vol. 22, no. 1, pp. 28–35, January 1999.

[6] J. R. Lawton and R. W. Beard, "Synchronized multiple spacecraft rotations," *Automatica*, vol. 38, no. 8, pp. 1359–1364, 2002.

[7] W. Ren and R. W. Beard, "Decentralized scheme for spacecraft formation flying via the virtual structure approach," *AIAA Journal of Guidance, Control, and Dynamics*, vol. 27, no. 1, pp. 73–82, January-February 2004.

[8] W. Ren, "Distributed attitude consensus among multiple networked spacecraft," in *Proceedings of the American Control Conference*, Minneapolis, MN, June 2006, pp. 1760–1765.

[9] J. T.-Y. Wen and K. Kreutz-Delgado, "The attitude control problem," *IEEE Transactions on Automatic Control*, vol. 36, no. 10, pp. 1148–1162, October 1991.

[10] P. C. Hughes, *Spacecraft Attitude Dynamics*. John Wiley & Sons, 1986.

[11] F. R. K. Chung, *Spectral Graph Theory*. American Mathematical Society, 1997.

[12] W. Ren and R. W. Beard, "Consensus seeking in multiagent systems under dynamically changing interaction topologies," *IEEE Transactions on Automatic Control*, vol. 50, no. 5, pp. 655–661, May 2005.

[13] R. A. Horn and C. R. Johnson, *Matrix Analysis*. Cambridge University Press, 1985.

[14] H. K. Khalil, *Nonlinear Systems*, 3rd ed. Upper Saddle River, NJ: Prentice Hall, 2002.