

On Consensus Algorithms for Double-integrator Dynamics

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Abstract—This paper extends some existing results in consensus algorithms for double-integrator dynamics. We propose consensus algorithms for double-integrator dynamics in four cases: (i) with a bounded control input, (ii) without relative velocity measurement, (iii) without relative velocity measurement in the presence of a group reference velocity, and (iv) with a bounded control input and with partial access to a group reference state. We show that consensus is reached asymptotically for the first two cases if the undirected interaction graph is connected. We further show that consensus is reached asymptotically for the third case if the directed interaction graph has a directed spanning tree and the gain for velocity matching with the group reference velocity is above a certain bound. We also show that consensus is reached asymptotically for the fourth case if and only if the group reference state flows directly or indirectly to all of the vehicles in the team.

I. INTRODUCTION

Consensus means that multiple vehicles reach an agreement on a common value. Consensus algorithms have a historical perspective in [1]–[4], to name a few, and have recently been studied extensively in the context of cooperative control of multiple autonomous vehicles (see [5] and references therein). Some results in consensus algorithms can be understood in the context of connective stability [6].

Consensus algorithms are primarily studied for vehicles with single-integrator kinematics in the literature. For vehicles with double-integrator dynamics, consensus related problems have been studied in [7]–[17], to name a few. In particular, formation keeping strategies are addressed in [7], [9] for multi-vehicle formation maneuvers under the assumption of a bidirectional ring interaction graph. Refs. [12], [13] study flocking algorithms that guarantee velocity matching, flock centering, and collision avoidance for a group of vehicles under undirected information exchange. In [11] the problem of decentralized stabilization of vehicle formations is studied under directed information exchange. In [14], second-order consensus algorithms are proposed and analyzed under directed information exchange, where it is shown that both the interaction graph and the coupling strength of relative velocities between neighboring vehicles affect the convergence result in the general case of directed information exchange. Refs. [16], [17] further extend [14] to incorporate a group reference velocity. A second-order consensus algorithm is also considered in [15] under undirected information exchange.

The main purpose of the current paper is to extend some existing results in consensus algorithms for double-integrator dynamics in four aspects. First, we propose and analyze a

consensus algorithm for double-integrator dynamics with a bounded control input under an undirected interaction graph. Second, we propose and analyze a consensus algorithm for double-integrator dynamics without relative velocity measurement under an undirected interaction graph. Third, we propose and analyze a consensus algorithm for double-integrator dynamics without relative velocity measurement, where a group reference velocity is available to each team member under a directed interaction graph. Finally, we propose and analyze a consensus algorithm for double-integrator dynamics with a bounded control input that allows a group reference velocity to be available to only a subgroup of the team under a directed interaction graph.

II. BACKGROUND AND PRELIMINARIES

A. Graph Theory Notions

A weighted graph consists of a node set $\mathcal{V} = \{1, \dots, p\}$, an edge set $\mathcal{E} \in \mathcal{V} \times \mathcal{V}$, and a weighted adjacency matrix $A = [a_{ij}] \in \mathbb{R}^{p \times p}$. An edge (i, j) in a weighted directed graph denotes that vehicle j can obtain information from vehicle i , but not necessarily vice versa. In contrast, the pairs of nodes in a weighted undirected graph are unordered, where an edge (i, j) denotes that vehicles i and j can obtain information from one another. The weighted adjacency matrix A of a weighted directed graph is defined such that a_{ij} is a positive weight if $(j, i) \in \mathcal{E}$, while $a_{ij} = 0$ if $(j, i) \notin \mathcal{E}$. The weighted adjacency matrix A of a weighted undirected graph is defined analogously except that $a_{ij} = a_{ji}, \forall i \neq j$, since $(j, i) \in \mathcal{E}$ implies $(i, j) \in \mathcal{E}$. If the weights are not relevant, then a_{ij} is set equal to 1 for all $(j, i) \in \mathcal{E}$. In this paper, self edges are not allowed, i.e. $a_{ii} = 0$.

For an edge (i, j) in a directed graph, i is the parent node and j is the child node. A directed path is a sequence of edges in a directed graph of the form $(i_1, i_2), (i_2, i_3), \dots$, where $i_j \in \mathcal{V}$. A directed tree is a directed graph, where every node has exactly one parent except for one node, called the root, which has no parent, and the root has a directed path to every other node. A directed spanning tree of a directed graph is a directed tree that contains all nodes of the directed graph. A directed graph has or contains a directed spanning tree if there exists a directed spanning tree as a subset of the directed graph, that is, there exists at least one node having a directed path to all of the other nodes.

Let the matrix $L = [\ell_{ij}] \in \mathbb{R}^{p \times p}$ be defined as

$$\ell_{ii} = \sum_{j=1, j \neq i}^p a_{ij}, \quad \ell_{ij} = -a_{ij}, \quad i \neq j. \quad (1)$$

The matrix L satisfies the conditions

$$\ell_{ij} \leq 0, \quad i \neq j, \quad \sum_{j=1}^p \ell_{ij} = 0, \quad i = 1, \dots, p. \quad (2)$$

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For an undirected graph, the Laplacian matrix L is symmetric positive semi-definite. However, L for a directed graph does not have this property. In both the undirected and directed cases, 0 is an eigenvalue of L with the associated eigenvector $\mathbf{1}_p$, where $\mathbf{1}_p$ is a $p \times 1$ column vector of all ones. In the case of undirected graphs, 0 is a simple eigenvalue of L and all of the other eigenvalues are positive if and only if the undirected graph is connected [18]. In the case of directed graphs, 0 is a simple eigenvalue of L and all of the other eigenvalues have positive real parts if and only if the directed graph contains a directed spanning tree [19].

Given a matrix $S = [s_{ij}] \in \mathbb{R}^{p \times p}$, the directed graph of S , denoted by $\Gamma(S)$, is the directed graph on p nodes i , $i \in \{1, 2, \dots, p\}$, such that there is an edge in $\Gamma(S)$ from node j to node i if and only if $s_{ij} \neq 0$ (cf. [20]). Again, we assume that there is no self edge (i, i) .

B. Existing Consensus Algorithms for Double-integrator Dynamics

Consider vehicles with double-integrator dynamics given by

$$\dot{r}_i = v_i, \quad \dot{v}_i = u_i, \quad i \in \mathcal{I}_n, \quad (3)$$

where $r_i \in \mathbb{R}^m$ and $v_i \in \mathbb{R}^m$ are, respectively, the position and velocity of the i^{th} vehicle, $u_i \in \mathbb{R}^m$ is the control input, and $\mathcal{I}_n \triangleq \{1, \dots, n\}$.

A consensus algorithm for (3) is proposed in [14] as

$$u_i = - \sum_{j=1}^n a_{ij} [(r_i - r_j) + \gamma(v_i - v_j)], \quad i \in \mathcal{I}_n, \quad (4)$$

where a_{ij} is the (i, j) th entry of the weighted adjacency matrix A characterizing the interaction graph for r_i and v_i , and γ is a positive gain.

In the presence of a group reference velocity $v^d \in \mathbb{R}^m$, a consensus algorithm for (3) is proposed in [16] as

$$u_i = \dot{v}^d - \alpha(v_i - v^d) - \sum_{j=1}^n a_{ij} [(r_i - r_j) + \gamma(v_i - v_j)], \quad i \in \mathcal{I}_n, \quad (5)$$

where a_{ij} is defined as in (4), and α and γ are positive gains.

Consensus is reached for (4) if for all $r_i(0)$ and $v_i(0)$, $r_i(t) \rightarrow r_j(t)$ and $v_i(t) \rightarrow v_j(t)$ asymptotically as $t \rightarrow \infty$. Consensus is reached for (5) if for all $r_i(0)$ and $v_i(0)$, $r_i(t) \rightarrow r_j(t)$ and $v_i(t) \rightarrow v^d(t)$ asymptotically as $t \rightarrow \infty$.

III. CONSENSUS WITH A BOUNDED CONTROL INPUT

Note that (4) does not explicitly take into account actuator saturation. We propose a consensus algorithm for (3) with a bounded control input as

$$u_i = - \sum_{j=1}^n (a_{ij} \tanh[K_r(r_i - r_j)] + b_{ij} \tanh[K_v(v_i - v_j)]), \quad i \in \mathcal{I}_n, \quad (6)$$

where $K_r \in \mathbb{R}^{m \times m}$ and $K_v \in \mathbb{R}^{m \times m}$ are positive-definite diagonal matrices, a_{ij} and b_{ij} are, respectively, the (i, j) th entry of the weighted adjacency matrix A and B

characterizing, respectively, the interaction graphs for r_i and v_i , and $\tanh(\cdot)$ is defined component-wise. That is, $\tanh([x_1, \dots, x_m]^T) = [\tanh(x_1), \dots, \tanh(x_m)]^T$, where $x_i \in \mathbb{R}$. Note that A and B can be chosen differently. Also note that with (6) u_i is bounded because $\tanh(\cdot)$ is bounded.

Theorem 3.1: With (6), $r_i(t) \rightarrow r_j(t)$ and $v_i(t) \rightarrow v_j(t)$ asymptotically as $t \rightarrow \infty$ if the graphs of A and B are both undirected connected.

Proof: Note that with (6), (3) can be written as

$$\begin{aligned} \dot{r}_{ij} &= v_i - v_j \\ \dot{v}_i &= - \sum_{j=1}^n (a_{ij} \tanh(K_r r_{ij}) + b_{ij} \tanh[K_v(v_i - v_j)]), \end{aligned} \quad (7)$$

where $r_{ij} \triangleq r_i - r_j$. Consider a Lyapunov function candidate for (7) as

$$V = \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n a_{ij} \mathbf{1}_m^T K_r^{-1} \log(\cosh(K_r r_{ij})) + \frac{1}{2} \sum_{i=1}^n v_i^T v_i, \quad (8)$$

where $\cosh(\cdot)$ and $\log(\cdot)$ are defined component-wise. Note that V is positive definite and radially unbounded with respect to r_{ij} , where $(j, i) \in \mathcal{E}$, and v_i if the graph of A is undirected connected. Differentiating V , gives

$$\begin{aligned} \dot{V} &= \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n a_{ij} (v_i - v_j)^T \tanh[K_r(r_i - r_j)] + \sum_{i=1}^n v_i^T u_i \\ &= \frac{1}{2} \sum_{i=1}^n v_i^T \left(\sum_{j=1}^n a_{ij} \tanh[K_r(r_i - r_j)] \right) \\ &\quad - \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n a_{ij} v_j^T \tanh[K_r(r_i - r_j)] + \sum_{i=1}^n v_i^T u_i \\ &= \frac{1}{2} \sum_{i=1}^n v_i^T \left(\sum_{j=1}^n a_{ij} \tanh[K_r(r_i - r_j)] \right) \\ &\quad + \frac{1}{2} \sum_{j=1}^n \sum_{i=1}^n a_{ji} v_j^T \tanh[K_r(r_j - r_i)] + \sum_{i=1}^n v_i^T u_i \\ &= \frac{1}{2} \sum_{i=1}^n v_i^T \left(\sum_{j=1}^n a_{ij} \tanh[K_r(r_i - r_j)] \right) \\ &\quad + \frac{1}{2} \sum_{j=1}^n v_j^T \left(\sum_{i=1}^n a_{ji} \tanh[K_r(r_j - r_i)] \right) + \sum_{i=1}^n v_i^T u_i \\ &= \sum_{i=1}^n v_i^T \left(\sum_{j=1}^n a_{ij} \tanh[K_r(r_i - r_j)] \right) + \sum_{i=1}^n v_i^T u_i \\ &= - \sum_{i=1}^n v_i^T \left(\sum_{j=1}^n b_{ij} \tanh[K_v(v_i - v_j)] \right) \\ &= - \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n b_{ij} (v_i - v_j)^T \tanh[K_v(v_i - v_j)] \leq 0, \end{aligned}$$

where we have used the fact that $\frac{d \log(\cosh(x))}{dt} = \dot{x} \tanh(x)$ with $x \in \mathbb{R}$ to obtain the first equality, have used the fact that

$a_{ij} = a_{ji}$ and $\tanh[K_r(r_j - r_i)] = -\tanh[K_r(r_i - r_j)]$ and have switched the order of the summation signs to obtain the third equality, have replaced u_i by (6) to obtain the sixth equality, have used the fact that $b_{ij} = b_{ji}$ to obtain the last equality, and have used the fact that x and $\tanh(Kx)$ have the same sign when x is a vector and K is a positive-definite diagonal matrix to obtain the last inequality.

Let $S = \{r_{ij}, v_i | \dot{V} = 0\}$. Note that $\dot{V} \equiv 0$ implies that $v_i \equiv v_j$ when the graph of B is undirected connected, which in turn implies that $\dot{v}_i \equiv \dot{v}_j$. Therefore, it follows that $\dot{v} \in \text{span}(\mathbf{1}_n \otimes \eta)$, where $v = [v_1^T, \dots, v_n^T]^T$ and $\eta \in \mathbb{R}^m$, when the graph of B is undirected connected. Because $v_i \equiv v_j$, it follows from (3) and (6) that

$$\dot{v}_i = - \sum_{j=1}^n a_{ij} \tanh[K_r(r_i - r_j)], \quad i \in \mathcal{I}_n. \quad (9)$$

Note that $(\mathbf{1}_n \otimes \eta)^T \dot{v} \equiv 0$ from (9), where we have used the fact that $a_{ij} = a_{ji}$ and $\tanh[K_r(r_i - r_j)] = -\tanh[K_r(r_j - r_i)]$. Thus it follows that \dot{v} is orthogonal to $\mathbf{1}_n \otimes \eta$. Therefore, we conclude that $\dot{v} \equiv 0$, which in turn implies that $-\sum_{j=1}^n a_{ij} \tanh[K_r(r_i - r_j)] \equiv 0$ from (9). Let $r_i = [r_{i(1)}, \dots, r_{i(m)}]^T$ and $K_r = \text{diag}\{k_{r(1)}, \dots, k_{r(m)}\}$. Noting that $\tanh(x+y) = \frac{\tanh(x)+\tanh(y)}{1+\tanh(x)\tanh(y)}$, where $x, y \in \mathbb{R}$, it follows that $-\sum_{j=1}^n a_{ij} \tanh[K_r(r_i - r_j)] \equiv 0$ is equivalent to $-\sum_{j=1}^n a_{ij} \frac{\tanh(k_{r(\kappa)} r_{i(\kappa)}) - \tanh(k_{r(\kappa)} r_{j(\kappa)})}{1 - \tanh(k_{r(\kappa)} r_{i(\kappa)}) \tanh(k_{r(\kappa)} r_{j(\kappa)})} \equiv 0$, $i = 1, \dots, n$, $\kappa = 1, \dots, m$, which can be written as $-L_{(\kappa)} \tanh(\hat{r}_{(\kappa)}) \equiv 0$, $\kappa = 1, \dots, m$, where $\hat{r}_{(\kappa)} = [k_{r(\kappa)} r_{1(\kappa)}, \dots, k_{r(\kappa)} r_{n(\kappa)}]^T$ and $L_{(\kappa)} = [\ell_{ij(\kappa)}]$ is defined as $\ell_{ij(\kappa)} = -\frac{a_{ij}}{1 - \tanh(k_{r(\kappa)} r_{i(\kappa)}) \tanh(k_{r(\kappa)} r_{j(\kappa)})}$ and $\ell_{ii(\kappa)} = \sum_{j \neq i} \frac{a_{ij}}{1 - \tanh(k_{r(\kappa)} r_{i(\kappa)}) \tanh(k_{r(\kappa)} r_{j(\kappa)})}$. Noting that $1 - \tanh(k_{r(\kappa)} r_{i(\kappa)}) \tanh(k_{r(\kappa)} r_{j(\kappa)}) > 0$, it follows that $L_{(\kappa)}$ satisfies the property (2) with $p = n$. When the graph of A is undirected connected, it follows that $\tanh(k_{r(\kappa)} r_{i(\kappa)}) \equiv \tanh(k_{r(\kappa)} r_{j(\kappa)})$, which implies that $r_i \equiv r_j$ (i.e. $r_{ij} \equiv 0$), $\forall i \neq j$. By LaSalle's invariance principle, it follows that $r_i(t) \rightarrow r_j(t)$ and $v_i(t) \rightarrow v_j(t)$, $\forall i \neq j$, asymptotically as $t \rightarrow \infty$. ■

Note that (6) guarantees that $r_i(t) \rightarrow r_j(t)$ and $v_i(t) \rightarrow v_j(t)$ asymptotically as $t \rightarrow \infty$. When it is desirable that $r_i(t) \rightarrow r_j(t)$ and $v_i(t) \rightarrow 0$ asymptotically as $t \rightarrow \infty$, we propose a consensus algorithm for (3) with a bounded control input as

$$u_i = - \sum_{j=1}^n a_{ij} \tanh[K_r(r_i - r_j)] - \tanh(K_{vi} v_i), \quad (10)$$

$$i \in \mathcal{I}_n,$$

where $K_r \in \mathbb{R}^{m \times m}$ and $K_{vi} \in \mathbb{R}^{m \times m}$ are positive-definite diagonal matrices.

Theorem 3.2: With (10), $r_i(t) \rightarrow r_j(t)$ and $v_i(t) \rightarrow 0$ asymptotically as $t \rightarrow \infty$ if the graph of A is undirected connected.

Proof: Following the proof to Theorem 3.1, consider a Lyapunov function candidate given by (8). Differentiating V , gives

$$\dot{V} = - \sum_{i=1}^n v_i^T \tanh(K_{vi} v_i) \leq 0,$$

Let $S = \{r_{ij}, v_i | \dot{V} = 0\}$. Note that $\dot{V} \equiv 0$ implies that $v_i \equiv 0$, which in turn implies that $\dot{v}_i \equiv 0$. It thus follows that $-\sum_{j=1}^n a_{ij} \tanh[K_r(r_i - r_j)] \equiv 0$. Therefore, a similar argument to that in the proof of Theorem 3.1 shows that $r_i(t) \rightarrow r_j(t)$ and $v_i(t) \rightarrow 0$, $\forall i \neq j$, asymptotically as $t \rightarrow \infty$ if the graph of A is undirected connected. ■

Note that the results in [7] are restricted to a bidirectional ring graph for convergence analysis. The algorithms (6) and (10) guarantee consensus convergence under any undirected connected interaction graph.

IV. CONSENSUS WITHOUT RELATIVE VELOCITY MEASUREMENT

Note that (4) requires measurement of relative velocities between neighboring vehicles. Motivated by [7], [21], we propose a consensus algorithm without relative velocity measurement based on a passivity approach as

$$\begin{aligned} \dot{\hat{x}}_i &= \Gamma \hat{x}_i + \sum_{j=1}^n a_{ij} (r_i - r_j) \\ y_i &= P \Gamma \hat{x}_i + P \sum_{j=1}^n a_{ij} (r_i - r_j) \\ u_i &= - \sum_{j=1}^n a_{ij} (r_i - r_j) - y_i, \quad i \in \mathcal{I}_n, \end{aligned} \quad (11)$$

where $\Gamma \in \mathbb{R}^{m \times m}$ is Hurwitz, a_{ij} is defined as in (6), $P \in \mathbb{R}^{m \times m}$ is a symmetric positive-definite matrix and is the solution to the Lyapunov equation $\Gamma^T P + P \Gamma = -Q$ with $Q \in \mathbb{R}^{m \times m}$ being a symmetric positive-definite matrix. The algorithm (11) extends the results in [7] to consensus convergence under any undirected connected interaction graph.

Theorem 4.1: With (11), $r_i(t) \rightarrow r_j(t)$ and $v_i(t) \rightarrow v_j(t)$ asymptotically as $t \rightarrow \infty$ if the graph of A is undirected connected.

Proof: Let $r = [r_1^T, \dots, r_n^T]^T$, $v = [v_1^T, \dots, v_n^T]^T$, $y = [y_1^T, \dots, y_n^T]^T$, $\hat{x} = [\hat{x}_1^T, \dots, \hat{x}_n^T]^T$, and $u = [u_1^T, \dots, u_n^T]^T$. With (11), (3) can be written as

$$\dot{\hat{x}} = (I_n \otimes \Gamma) \hat{x} + (L \otimes I_m) r \quad (12)$$

$$y = (I_n \otimes P) \hat{x} \quad (13)$$

$$u = -(L \otimes I_m) r - y, \quad (14)$$

where \otimes denotes the Kronecker product, I_n is the $n \times n$ identity matrix and L is defined in (1) with $p = n$.

Note that with (11), (3) can be written as a system with states r_{ij} , v_{ij} , and \hat{x}_i , where $r_{ij} \triangleq r_i - r_j$ and $v_{ij} \triangleq v_i - v_j$. Consider a Lyapunov function candidate

$$V = \frac{1}{2} r^T (L \otimes I_m)^2 r + \frac{1}{2} v^T (L \otimes I_m) v + \frac{1}{2} \hat{x}^T (I_n \otimes P) \hat{x}.$$

Note that from the property of the Laplacian matrix L , V is positive definite and radially unbounded with respect to r_{ij} , v_{ij} , and \hat{x}_i when the graph of A is undirected connected.

Differentiating V , gives

$$\begin{aligned}
\dot{V} &= v^T(L \otimes I_m)^2 r + v^T(L \otimes I_m)u \\
&\quad + \frac{1}{2}\dot{\hat{x}}^T(I_n \otimes P)\dot{\hat{x}} + \frac{1}{2}\dot{\hat{x}}^T(I_n \otimes P)\ddot{\hat{x}} \\
&= v^T[(L \otimes I_m)^2 r + (L \otimes I_m)u] \\
&\quad + \frac{1}{2}\dot{\hat{x}}^T(I_n \otimes \Gamma^T)(I_n \otimes P)\dot{\hat{x}} + \frac{1}{2}v^T(L \otimes I_m)(I_n \otimes P)\dot{\hat{x}} \\
&\quad + \frac{1}{2}\dot{\hat{x}}^T(I_n \otimes P)(I_n \otimes \Gamma)\dot{\hat{x}} + \frac{1}{2}\dot{\hat{x}}^T(I_n \otimes P)(L \otimes I_m)v \\
&= v^T[(L \otimes I_m)^2 r + (L \otimes I_m)u] \\
&\quad - \frac{1}{2}\dot{\hat{x}}^T(I_n \otimes Q)\dot{\hat{x}} + v^T(L \otimes I_m)(I_n \otimes P)\dot{\hat{x}} \\
&= -\frac{1}{2}\dot{\hat{x}}^T(I_n \otimes Q)\dot{\hat{x}} \leq 0,
\end{aligned}$$

where we have used (12)-(13), $L = L^T$ when the graph of A is undirected, and properties of the Kronecker product. In particular, $(E \otimes F)^T = E^T \otimes F^T$ and $(E \otimes F)(G \otimes H) = EF \otimes GH$.

Let $S = \{r_{ij}, v_i, \hat{x}_i | \dot{V} = 0\}$. Note that $\dot{V} \equiv 0$ implies that $\dot{\hat{x}} \equiv 0$, which in turn implies that $\ddot{\hat{x}} \equiv 0$, $(L \otimes I_m)v \equiv 0$ by differentiating (12), and $y \equiv 0$ from (13). Because $(L \otimes I_m)v \equiv 0$, it follows that $(L \otimes I_m)\dot{v} \equiv 0$, which implies that $\dot{v} \in \text{span}(\mathbf{1}_n \otimes \eta)$, where $\eta \in \mathbb{R}^m$, when the graph of A is undirected connected. From (3) and (14), it follows that

$$\dot{v} \equiv -(L \otimes I_m)r. \quad (15)$$

Note that $(\mathbf{1}_n \otimes \eta)^T \dot{v} \equiv -(\mathbf{1}_n \otimes \eta)^T (L \otimes I_m)r \equiv -(\mathbf{1}_n^T L \otimes \eta^T I_m)r \equiv 0$ because $\mathbf{1}_n^T L = 0$ when the graph of A is undirected. Thus \dot{v} is orthogonal to $\mathbf{1}_n \otimes \eta$. We then conclude that $\dot{v} \equiv 0$, which in turn implies that $(L \otimes I_m)r \equiv 0$ from (15). If the graph of A is undirected connected, $(L \otimes I_m)r \equiv 0$ implies that $r_i \equiv r_j, \forall i \neq j$. By LaSalle's invariance principle, it follows that $r_i(t) \rightarrow r_j(t)$ and $v_i(t) \rightarrow v_j(t), \forall i \neq j$, as $t \rightarrow \infty$. ■

When it is desirable that $r_i(t) \rightarrow r_j(t)$ and $v_i(t) \rightarrow 0$ asymptotically as $t \rightarrow \infty$, we propose an algorithm as

$$\begin{aligned}
\dot{\hat{x}}_i &= \Gamma x_i + r_i \\
y_i &= P\Gamma \hat{x}_i + P r_i \\
u_i &= -\sum_{j=1}^n a_{ij}(r_i - r_j) - y_i, \quad i \in \mathcal{I}_n,
\end{aligned} \quad (16)$$

where Γ , P , and a_{ij} are defined as in (11).

Theorem 4.2: With (16), $r_i(t) \rightarrow r_j(t)$ and $v_i(t) \rightarrow 0$ asymptotically as $t \rightarrow \infty$ if the graph of A is undirected connected.

Proof: With (16), (3) can be written as

$$\dot{\hat{x}} = (I_n \otimes \Gamma)\hat{x} + r \quad (17)$$

$$y = (I_n \otimes P)\hat{x} \quad (18)$$

$$u = -(L \otimes I_m)r - y. \quad (19)$$

Consider a Lyapunov function candidate

$$V = \frac{1}{2}r^T(L \otimes I_m)r + \frac{1}{2}v^T v + \frac{1}{2}\dot{\hat{x}}^T(I_n \otimes P)\dot{\hat{x}},$$

which is positive definite and radially unbounded with respect to $r_{ij} \triangleq r_i - r_j$, v_i , and $\dot{\hat{x}}_i$. Following the proof of Theorem 4.1, the derivative of V is given as

$$\begin{aligned}
\dot{V} &= v^T[(L \otimes I_m)r + u] - \frac{1}{2}\dot{\hat{x}}^T(I_n \otimes Q)\dot{\hat{x}} + v^T(I_n \otimes P)\dot{\hat{x}} \\
&= -\frac{1}{2}\dot{\hat{x}}^T(I_n \otimes Q)\dot{\hat{x}} \leq 0.
\end{aligned}$$

Let $S = \{r_{ij}, v_i, \hat{x}_i | \dot{V} = 0\}$. Note that $\dot{V} \equiv 0$ implies that $\dot{\hat{x}} \equiv 0$, which in turn implies that $\ddot{\hat{x}} \equiv 0$, $v \equiv 0$ by differentiating (17), and $y \equiv 0$ from (18). Because $v \equiv 0$, it follows that $\dot{v} \equiv 0$, which implies that $-(L \otimes I_m)r \equiv 0$ from (3) and (19). If the graph of A is undirected connected, $(L \otimes I_m)r \equiv 0$ implies that $r_i \equiv r_j, \forall i \neq j$. By LaSalle's invariance principle, it follows that $r_i(t) \rightarrow r_j(t)$ and $v_i(t) \rightarrow 0, \forall i \neq j$, as $t \rightarrow \infty$. ■

V. CONSENSUS WITHOUT RELATIVE VELOCITY MEASUREMENT IN THE PRESENCE OF A GROUP REFERENCE VELOCITY

Note that (5) introduces a group reference velocity but requires measurement of relative velocities between neighboring vehicles. Next, we consider a consensus algorithm removing this requirement as

$$u_i = \dot{v}^d - \alpha(v_i - v^d) - \sum_{j=1}^n a_{ij}(r_i - r_j), \quad i \in \mathcal{I}_n, \quad (20)$$

where α is positive gain, a_{ij} is defined as in (6), and $v^d \in \mathbb{R}^m$ denotes the (time-varying) group reference velocity.

The following theorem considers the general case of directed information exchange among vehicles, which generalizes existing results in [15], [16].

Theorem 5.1: Let μ_i denotes the i^{th} eigenvalue of $-L$ with L given by (1), where $p = n$, and $\text{Re}(\cdot)$ and $\text{Im}(\cdot)$ represent, respectively, the real and imaginary parts of a number. With (20), $r_i(t) \rightarrow r_j(t)$ and $v_i(t) \rightarrow v^d(t)$ asymptotically as $t \rightarrow \infty$ if the directed graph of A has a directed spanning tree and

$$\alpha > \bar{\alpha}, \quad (21)$$

where $\bar{\alpha} \triangleq 0$ if all of the $n - 1$ nonzero eigenvalues of $-L$ are negative and

$$\bar{\alpha} \triangleq \max_{\forall \text{Re}(\mu_i) < 0 \text{ and } \text{Im}(\mu_i) > 0} |\mu_i| \sqrt{\frac{2}{-\text{Re}(\mu_i)}}$$

otherwise.

Proof: Let $r = [r_1^T, \dots, r_n^T]^T$, $\tilde{r} = r - \mathbf{1}_n \otimes \int_0^t v^d(\tau) d\tau$, $v = [v_1^T, \dots, v_n^T]^T$, and $\tilde{v} = v - \mathbf{1}_n \otimes v^d$. With (20), (3) can be written in matrix form as

$$\begin{bmatrix} \dot{\tilde{r}} \\ \dot{\tilde{v}} \end{bmatrix} = (\Gamma \otimes I_m) \begin{bmatrix} \tilde{r} \\ \tilde{v} \end{bmatrix},$$

where $\Gamma \triangleq \begin{bmatrix} 0 & I_n \\ -L & -\alpha I_n \end{bmatrix}$ with L given by (1), where $p = n$.

Noting that $L\mathbf{1}_n = 0$, it follows that $[\mathbf{1}_n^T, \mathbf{0}_n^T]^T$, where $\mathbf{0}_n$ denotes the $n \times 1$ column vector of all zeros, is an eigenvector for Γ associated with an eigenvalue 0. If Γ has a simple zero

eigenvalue and all of the other eigenvalues have negative real parts, then $\begin{bmatrix} \tilde{r}(t) \\ \tilde{v}(t) \end{bmatrix} \rightarrow \text{span} \left(\begin{bmatrix} \mathbf{1}_n \\ \mathbf{0}_n \end{bmatrix} \otimes \eta \right)$ asymptotically as $t \rightarrow \infty$, where η is an $m \times 1$ vector, which is equivalent to $\tilde{r}_i(t) \rightarrow \tilde{r}_j(t)$ and $\tilde{v}_i \rightarrow 0$ asymptotically as $t \rightarrow \infty$.

Next, we show that if the directed graph of A has a directed spanning tree and the inequality (21) is satisfied, then Γ has a simple zero eigenvalue and all of the other eigenvalues have negative real parts.

Let λ be an eigenvalue of Γ and $s = [p^T, q^T]^T$ be its associated eigenvector, where p and q are $n \times 1$ column vectors. Note that

$$\begin{aligned} \Gamma s &= \lambda s \\ \iff \begin{bmatrix} 0 & I_n \\ -L & -\alpha I_n \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix} &= \lambda \begin{bmatrix} p \\ q \end{bmatrix} \\ \iff q &= \lambda p \\ &-Lp - \alpha q = \lambda q \\ \iff -Lp &= (\lambda^2 + \alpha\lambda)p, \end{aligned}$$

which implies that $\lambda^2 + \alpha\lambda$ is an eigenvalue of $-L$ with an associated eigenvector p . Letting $\mu \triangleq \lambda^2 + \alpha\lambda$, gives $\lambda^2 + \alpha\lambda - \mu = 0$, which implies that given each μ , there are two roots for λ , denoted by $\lambda_{\pm} = \frac{-\alpha \pm \sqrt{\alpha^2 + 4\mu}}{2}$. As a result, each eigenvalue of $-L$, denoted by μ_i , $i = 1, \dots, n$, corresponds to two eigenvalues of Γ , denoted by λ_{2i-1} and λ_{2i} .

If the directed graph of A has a directed spanning tree, then L has a simple zero eigenvalue and all of the other eigenvalues have positive real parts, which implies that $-L$ has a simple zero eigenvalue and all of the other eigenvalues have negative real parts. Without loss of generality, let $\mu_1 = 0$ and $\text{Re}(\mu_i) < 0$, $i = 2, \dots, n$. Then it follows that $\lambda_1 = 0$ and $\lambda_2 = -\alpha$. Note that if $\mu_i < 0$, then $\text{Re}(\frac{-\alpha \pm \sqrt{\alpha^2 + 4\mu_i}}{2}) < 0$ for any $\alpha > 0$. It is left to show that the inequality (21) guarantees that all of the eigenvalues of Γ corresponding to μ_i that satisfies $\text{Re}(\mu_i) < 0$ and $\text{Im}(\mu_i) \neq 0$ have negative real parts. Motivated by [10], [14], we use Fig. 1 to show the notations used in the proof. We only need to consider μ_i that satisfies $\text{Re}(\mu_i) < 0$ and $\text{Im}(\mu_i) > 0$ since any μ_i that satisfies $\text{Re}(\mu_i) < 0$ and $\text{Im}(\mu_i) < 0$ is a complex conjugate of some μ_i that satisfies $\text{Re}(\mu_i) < 0$ and $\text{Im}(\mu_i) > 0$. Consider the triangle formed by vectors α^2 , $4\mu_i$, and $\alpha^2 + 4\mu_i$. According to law of cosines, $|\alpha^2 + 4\mu_i|^2 = (\alpha^2)^2 + (4|\mu_i|)^2 - 8\alpha^2|\mu_i|\cos(\phi_i)$, where $\cos(\phi_i) = \frac{-\text{Re}(\mu_i)}{|\mu_i|}$. Note that if $\alpha > |\mu_i|\sqrt{\frac{2}{-\text{Re}(\mu_i)}}$, then $|\alpha^2 + 4\mu_i|^2 < \alpha^4$, which implies that $|\sqrt{\alpha^2 + 4\mu_i}| < \alpha$. Therefore, it follows that $|\text{Re}(\sqrt{\alpha^2 + 4\mu_i})| < \alpha$, which in turn implies that $\text{Re}(\lambda_{2i-1, 2i}) = \text{Re}(\frac{-\alpha \pm \sqrt{\alpha^2 + 4\mu_i}}{2}) < 0$.

Combing the above arguments, it follows that if the directed graph of A has a directed spanning tree and the inequality (21) is valid, then $\tilde{r}_i(t) \rightarrow \tilde{r}_j(t)$ and $\tilde{v}_i(t) \rightarrow 0$ asymptotically as $t \rightarrow \infty$, which in turn implies that $r_i(t) \rightarrow r_j(t)$ and $v_i(t) \rightarrow v^d(t)$ asymptotically as $t \rightarrow \infty$. ■

Corollary 5.2: With (20), $r_i(t) \rightarrow r_j(t)$ and $v_i(t) \rightarrow$

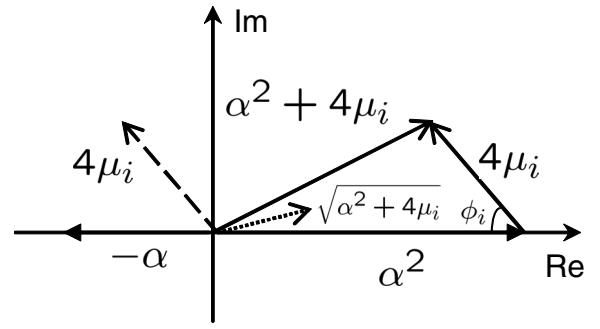


Fig. 1. Graphical view of notations used in the proof.

$v^d(t)$ asymptotically as $t \rightarrow \infty$ if the graph of A is undirected connected.

VI. CONSENSUS WITH A BOUNDED CONTROL INPUT AND WITH PARTIAL ACCESS TO A GROUP REFERENCE VELOCITY

Note that (5) and (20) require that the group reference velocity be available to each vehicle in the team. Next, we propose a consensus algorithm with a bounded control input that allows the group reference position r^d , velocity v^d , and acceleration \dot{v}^d to be available to only a subgroup of the team as

$$\begin{aligned} u_i &= \frac{1}{\kappa_i} \left(\sum_{j=1}^n a_{ij} \dot{v}_j + a_{i(n+1)} \dot{v}^d \right) \\ &- \frac{1}{\kappa_i} K_{r_i} \tanh \left[\sum_{j=1}^n a_{ij} (r_i - r_j) + a_{i(n+1)} (r_i - r^d) \right] \\ &- \frac{1}{\kappa_i} K_{v_i} \tanh \left[\sum_{j=1}^n a_{ij} (v_i - v_j) + a_{i(n+1)} (v_i - v^d) \right], \\ &i \in \mathcal{I}_n, \end{aligned} \quad (22)$$

where a_{ij} , $i, j \in \mathcal{I}_n$, is the (i, j) th entry of the weighted adjacency matrix A , $a_{i(n+1)} = 1$, $i \in \mathcal{I}_n$, if vehicle i has access to r^d , v^d , and \dot{v}^d , $\kappa_i \triangleq \sum_{j=1}^{n+1} a_{ij}$, \dot{v}^d is bounded, and K_{r_i} and K_{v_i} are $m \times m$ positive-definite diagonal matrices. Note that each control input not only depends on its local neighbors' positions and velocities but also depends on their accelerations. When the algorithm (22) is implemented, the term \dot{v}_j can be approximated by numerical differentiation. The algorithm (22) extends the result in [17] to explicitly account for actuator saturation.

Theorem 6.1: Let $A_{n+1} = [a_{ij}] \in \mathbb{R}^{(n+1) \times (n+1)}$, where a_{ij} , $i \in \mathcal{I}_n$, $j \in \mathcal{I}_{n+1}$, is defined in (22) and $a_{(n+1)j} = 0$, $j \in \mathcal{I}_{n+1}$. With (22), there exists a unique bounded solution for u_i and $r_i(t) \rightarrow r^d(t)$ and $v_i(t) \rightarrow v^d(t)$ asymptotically as $t \rightarrow \infty$ if and only if the directed graph of A_{n+1} has a directed spanning tree.

Proof: We first show that (22) has a unique solution if and only if the directed graph of A_{n+1} has a directed spanning tree and the solution is bounded.

Noting that all entries of the last row of A_{n+1} are zero, it follows that $\kappa_i = \sum_{j=1}^{n+1} a_{ij} \neq 0$, $i = 1, \dots, n$, if the directed graph of A_{n+1} has a directed spanning tree.

Define $W = [w_{ij}] \in \mathbb{R}^{n \times n}$ as $w_{ij} = -a_{ij}$, $i \neq j$, and $w_{ii} = \sum_{j=1, j \neq i}^{n+1} a_{ij}$. Also define $b = [b_1, \dots, b_n]^T \in \mathbb{R}^{n \times 1}$ with $b_i = -a_{i(n+1)}$, and $d = [d_1^T, \dots, d_n^T]^T \in \mathbb{R}^{mn \times 1}$ with $d_i = -K_{r_i} \tanh[\sum_{j=1}^n a_{ij}(r_i - r_j) + a_{i(n+1)}(r_i - r^d)] - K_{v_i} \tanh[\sum_{j=1}^n a_{ij}(v_i - v_j) + a_{i(n+1)}(v_i - v^d)]$. With (22), (3) can be written as $(W \otimes I_m)u = (-b \otimes I_m)\dot{v}^d + d$, where $u = [u_1^T, \dots, u_n^T]^T$. Note that b , \dot{v}^d , and d are all bounded. If W has full rank, then it is straightforward to show that there is a unique solution for u and the solution is bounded. Let $L_{n+1} = \begin{bmatrix} W|b \\ 0_{1 \times n}|0 \end{bmatrix} \in \mathbb{R}^{(n+1) \times (n+1)}$, which satisfies the property (2) with $p = n + 1$. Note that $\text{Rank}(L_{n+1}) = \text{Rank}(W|b)$ and $W\mathbf{1}_n + b = \mathbf{0}_n$ (i.e. b is a linear combination of the n columns of W). It follows that $\text{Rank}(W) = \text{Rank}(W|b) = \text{Rank}(L_{n+1})$. Also note that $\text{Rank}(L_{n+1}) = n$ if and only if the directed graph of A_{n+1} has a directed spanning tree. Therefore, $\text{Rank}(W) = n$ (i.e. full rank) if and only if the directed graph of A_{n+1} has a directed spanning tree. This proves the first argument of the theorem.

Note that with (22), (3) can be written as

$$\ddot{e}_i = -K_{r_i} \tanh(e_i) - K_{v_i} \tanh(\dot{e}_i), \quad (23)$$

where $e_i = \sum_{j=1}^n a_{ij}(r_i - r_j) + a_{i(n+1)}(r_i - r^d)$.

Consider a Lyapunov function candidate

$$V = \sum_{i=1}^n [\mathbf{1}_m^T K_{r_i} \log(\cosh(e_i)) + \frac{1}{2} \dot{e}_i^T \dot{e}_i],$$

which is positive definite and radially unbounded with respect to e_i and \dot{e}_i .

Differentiating V , gives

$$\begin{aligned} \dot{V} &= \sum_{i=1}^n (\dot{e}_i^T K_{r_i} \tanh(e_i) \\ &\quad + \dot{e}_i^T [-K_{r_i} \tanh(e_i) - K_{v_i} \tanh(\dot{e}_i)]) \\ &= - \sum_{i=1}^n \dot{e}_i^T K_{v_i} \tanh(\dot{e}_i) \leq 0. \end{aligned}$$

Let $S = \{e_i, \dot{e}_i | \dot{V} = 0\}$. Note that $\dot{V} \equiv 0$ implies that $\dot{e}_i \equiv 0$, which in turn implies that $\ddot{e}_i \equiv 0$. Because $\dot{e}_i \equiv 0$ and $\ddot{e}_i \equiv 0$, it follows that $e_i \equiv 0$ from (23). By LaSalle's invariance principle, it follows that $e_i(t) \rightarrow 0$ and $\dot{e}_i(t) \rightarrow 0$ asymptotically as $t \rightarrow \infty$. Note that $e = (W \otimes I_m)r + (b \otimes I_m)r^d$, where $e = [e_1^T, \dots, e_n^T]^T$ and $r = [r_1^T, \dots, r_n^T]^T$. Because $W\mathbf{1} + b = \mathbf{0}_n$ and $\text{Rank}(W) = n$ (i.e. $W^{-1}b = -\mathbf{1}_n$) if and only if the directed graph of A_{n+1} has a directed spanning tree, it follows that $e(t) \rightarrow 0$ asymptotically as $t \rightarrow \infty$ is equivalent to $r_i(t) \rightarrow r^d(t)$ asymptotically as $t \rightarrow \infty$ under the same assumption. Similarly, it follows that $\dot{e}(t) \rightarrow 0$ asymptotically as $t \rightarrow \infty$ is equivalent to $v_i(t) \rightarrow v^d(t)$ asymptotically as $t \rightarrow \infty$ under the same assumption. ■

VII. CONCLUSION AND FUTURE WORK

We have extended some existing consensus algorithms for double-integrator dynamics to account for actuator saturation, remove the requirement for relative velocity measurement, introduce a group reference velocity to each vehicle

without relative velocity measurement, and incorporate a group reference state to a subgroup of the team and account for actuator saturation. We have shown convergence conditions for consensus in each case. Future work will consider the effect of time delay in those algorithms.

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