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Second-order Consensus Algorithm with Extensions to Switching Topologies and Reference Models

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Abstract— **In this paper, we extend the consensus algorithm for double integrator dynamics to the case that the information exchange topologies switch randomly with time and to the case that the final consensus value evolves according to a given nonlinear reference model. We show sufficient conditions under which consensus is reached under switching directed information exchange topologies. Unlike the consensus algorithm for single integrator dynamics, more stringent conditions are required to guarantee consensus under switching directed topologies in the case of the consensus algorithm for double integrator dynamics. In addition, we propose consensus algorithms so that the information variables of each vehicle approach the solution of a nonlinear reference model when only a portion of the vehicles in the team have access to the model.**

I. INTRODUCTION

As an inherently distributed strategy for multi-vehicle coordination, consensus algorithms have recently been studied extensively in the context of cooperative control of multi-vehicle systems [1]–[12], to name a few. Those algorithms only require local neighbor-to-neighbor information exchange between the vehicles. The basic idea for information consensus is that each vehicle updates its information state based on the information states of its local (possibly time-varying) neighbors in such a way that the final information state of each vehicle converges to a common value. This basic idea can be extended to deal with the case that each vehicle's information states converge to desired relative deviations or to incorporate different group behaviors into the consensus building process (see [13] for a survey).

Most work on consensus focuses on algorithms taking the form of first-order dynamics. Extensions of consensus algorithms to second-order dynamics are reported in [14]–[18], where formation keeping algorithms taking the form of second-order dynamics are addressed to guarantee attitude alignment, agreement of position deviations and velocities, and/or collision avoidance in a group of vehicles in the context of undirected information exchange. Taking into account the fact that information flow between the vehicles may be directed (e.g., local measurement by sensors with a limited field of view), [19] studies second-order consensus algorithms and performs a convergence analysis under a fixed directed information exchange topology. In contrast to first-order consensus algorithms, the convergence of the second-order consensus algorithms under directed information exchange relies not only the information exchange topology but also on the coupling strength between the derivatives of the information states.

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The main contributions of this paper are twofold. First, we provide a convergence analysis for a second-order consensus algorithm under switching directed information exchange topologies, which extends the convergence result under a fixed directed information exchange topology in [19]. Note that undirected information exchange is a special case of directed information exchange. We will show that in the case of switching directed information exchange topologies, the convergence result for the second-order consensus algorithm is much different from that of the first-order consensus algorithms. Second, we extend the second-order consensus algorithm to the case that the final consensus value evolves according to a given nonlinear reference model, where only a portion of the vehicles in the team have access to the model. This extension allows the vehicle/environmental dynamics or sensor measurement to be incorporated into the consensus building process as a form of feedback.

II. BACKGROUND AND PRELIMINARIES

It is natural to model information exchange between vehicles by directed/undirected graphs. A digraph (directed graph) consists of a pair $(\mathcal{N}, \mathcal{E})$, where $\mathcal{N}$ is a finite nonempty set of nodes and $\mathcal{E} \in \mathcal{N}^2$ is a set of ordered pairs of nodes, called edges. As a comparison, the pairs of nodes in an undirected graph are unordered. If there is a directed edge from node v_i to node v_j, then v_i is defined as the parent node and v_j is defined as the child node. A directed path is a sequence of ordered edges of the form $(v_{i_1}, v_{i_2}), (v_{i_2}, v_{i_3}), \cdots$, where $v_{i_j} \in \mathcal{N}$, in a digraph. An undirected path in an undirected graph is defined accordingly. In a digraph, a cycle is a path that starts and ends at the same node. A digraph is called strongly connected if there is a directed path from every node to every other node. An undirected graph is called connected if there is a path between any distinct pair of nodes. A directed tree is a digraph, where every node has exactly one parent except for one node, called root, which has no parent, and the root has a directed path to every other node. Note that in a directed tree, each edge has a natural orientation away from the root, and no cycle exists. In the case of undirected graphs, a tree is a graph in which every pair of nodes is connected by exactly one path. A directed spanning tree of a digraph is a directed tree formed by graph edges that connect all the nodes of the graph. A graph has (or contains) a directed spanning tree if there exists a directed spanning tree being a subset of the graph. Note that the condition that a digraph has a directed spanning tree is equivalent to the case that there exists a node having a directed path to all the other nodes. In the case of undirected graphs, having an undirected spanning tree is equivalent to being connected. However, in the case of directed graphs, having a directed spanning tree is a weaker condition than being strongly connected. The union of a group of digraphs is a digraph with nodes given by the union of the node sets and edges given by the union of the edge sets of those digraphs.

The adjacency matrix $A = [a_{ij}] \in \mathbb{R}^{n\times n}$ of a weighted digraph is defined as $a_{ii} = 0$ and $a_{ij} > 0$ if $(j, i) \in \mathcal{E}$ where $i \neq j$. The adjacency matrix of a weighted undirected graph is defined accordingly except that $a_{ij} = a_{ji}$, $\forall i \neq j$, since $(j, i) \in \mathcal{E}$ implies $(i, j) \in \mathcal{E}$. Let matrix $L = [\ell_{ij}]$ be defined as $\ell_{ii} = \sum_{j\neq i} a_{ij}$ and $\ell_{ij} = -a_{ij}$, where $i \neq j$. The matrix L satisfies the following conditions:

$$\ell_{ij} \leq 0, \quad i \neq j, \qquad \sum_{j=1}^{n} \ell_{ij} = 0, \quad i = 1, \cdots, n. \tag{1}$$

For an undirected graph, L is called the Laplacian matrix [20], which has the property that it is symmetric positive semi-definite. However, L for a digraph does not have this property.

Let $\mathbf{1}$ and $\mathbf{0}$ denote the $n \times 1$ column vector of all ones and all zeros respectively. In the case of an undirected information exchange graph, L has a simple zero eigenvalue with an associated eigenvector $\mathbf{1}$ and all of the other eigenvalues are positive if and only if the graph is connected [21]. In the case of a directed information exchange graph, L has a simple zero eigenvalue with an associated eigenvector $\mathbf{1}$ and all of the other eigenvalues have positive real parts if and only if the digraph has a directed spanning tree [22]. Let $x = [x_1, \cdots, x_n]^T$, where $x_j \in \mathbb{R}$, $j = 1, \cdots, n$, and $y = [y_1^T, \cdots, y_n^T]^T$, where $y_j \in \mathbb{R}^m$, $j = 1, \cdots, n$. Under the conditions of both cases, $Lx = 0$ implies that $x = \alpha\mathbf{1}$ (i.e., $x_1 = \cdots = x_n$), where $\alpha \in \mathbb{R}$, and $(L \otimes I_m)y = 0$, where $\otimes$ is the Kronecker product, implies that $y = \mathbf{1} \otimes \beta$ (i.e., $y_1 = \cdots = y_n$), where $\beta \in \mathbb{R}^m$.

Let I_n denote the $n \times n$ identity matrix and $0_{m\times n}$ denote the $m \times n$ matrix with all zero entries. Let $M_n(\mathbb{R})$ represent the set of all $n \times n$ real matrices. Given a matrix $S = [s_{ij}] \in M_n(\mathbb{R})$, the digraph of S, denoted by $\Gamma(S)$, is the digraph on n nodes v_i, $i \in \{1, \cdots, n\}$, such that there is a directed edge in $\Gamma(S)$ from v_j to v_i if and only if $s_{ij} \neq 0$ (c.f. [23]).

III. CONSENSUS ALGORITHM

Consider information variables with ℓ^{th}-order dynamics given by

$$\xi_i^{(\ell)} = u_i, \quad i \in \{1, \cdots, n\}, \tag{2}$$

where $\xi_i^{(\ell)} \in \mathbb{R}^m$ denotes the ℓ^{th} derivative of $\xi_i \in \mathbb{R}^m$ and $u_i \in \mathbb{R}^m$ is the control input.

A consensus algorithm is proposed in [24] as

$$u_i = -\sum_{j=1}^{n} g_{ij} k_{ij} [\sum_{k=0}^{\ell-1} \gamma_k (\xi_i^{(k)} - \xi_j^{(k)})], \tag{3}$$

where $k_{ij} > 0$, $\gamma_0 = 1$, $\gamma_k > 0$, $k = 1, \cdots, \ell - 1$, $\xi_i^{(k)}$ denotes the k^{th} derivative of ξ_i with $\xi_i^{(0)} = \xi_i$, $g_{ii} \triangleq 0$, and g_{ij} is 1 if information flows from vehicle j to vehicle i and 0 otherwise.

Consensus is said to be reached among the n vehicles if $\xi_i^{(k)} \to \xi_j^{(k)}$, $k = 0, 1, \cdots, \ell - 1$, $\forall i \neq j$. Note that the first-order and second-order linear consensus algorithms in the literature correspond to the case of $\ell = 1$ and $\ell = 2$ in Eq. (3) respectively. In this paper, we focus on the case of $\ell = 2$.

Under a fixed directed information exchange topology, (3) with $\ell = 1$ achieves consensus asymptotically if and only if the information exchange topology has a directed spanning tree [22].

In contrast, for (3) with $\ell = 2$, having a directed spanning tree is only a necessary condition for consensus under a fixed directed information exchange topology. Besides the information exchange topology, the value of γ_1 also plays a role for consensus seeking as shown below.

Let $L = [\ell_{ij}] \in \mathbb{R}^{n\times n}$ be given as $\ell_{ii} = \sum_{j\neq i} g_{ij} k_{ij}$ and $\ell_{ij} = -g_{ij} k_{ij}$, $\forall i \neq j$. Under a fixed information exchange topology, (3) with $\ell = 2$ achieves consensus asymptotically if the information exchange topology has a directed spanning tree and

$$\gamma_1 > \max_{i=2,\cdots,n} \sqrt{\frac{2}{|\mu_i| \cos(\frac{\pi}{2} - \tan^{-1} \frac{-\text{Re}(\mu_i)}{\text{Im}(\mu_i)})}}, \tag{4}$$

where μ_i, $i = 2, \cdots, n$, are the non-zero eigenvalues of $-L$, and $\text{Re}(\cdot)$ and $\text{Im}(\cdot)$ represent the real and imaginary parts of a number respectively.[1]

IV. EXTENSIONS TO SWITCHING TOPOLOGIES

In this section, we extend the convergence results under a fixed directed information exchange topology in [19] to those of switching topologies. In the case of switching directed information exchange topologies, the convergence analysis is more involved. Next, we will first show several examples for illustrative purpose and then state our main results.

Under switching information exchange topologies, (3) with $\ell = 1$ reaches consensus asymptotically if there exist infinitely many consecutive uniformly bounded time intervals such that the union of the information exchange graph across each interval has a directed spanning tree [7]. However, as shown in the following examples, this condition is generally not sufficient for information consensus in the case of $\ell = 2$.

Let $L_{(1)} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $L_{(2)} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -2 & 0 & 2 \end{bmatrix}$, and $L_{(3)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$. Note that the graphs of $L_{(1)}$ and $L_{(3)}$, denoted as $\mathcal{G}_1$ and $\mathcal{G}_3$ respectively, do not have a directed spanning tree while the graph of $L_{(2)}$, denoted as $\mathcal{G}_2$, does as shown in Fig. 1. Also let $\gamma_{1(i)} = 1$, $i = 1, 2, 3$, denote

[1]For details on how the lower bound for γ_1 is obtained, the readers are referred to [19] and references therein.

weighting factors γ_1 in Eq. (3) corresponding to each graph $\mathcal{G}_i$ respectively. Note that with fixed graph $\mathcal{G}_2$ consensus is reached asymptotically when $\gamma_{1(2)} = 1$ from Section III.

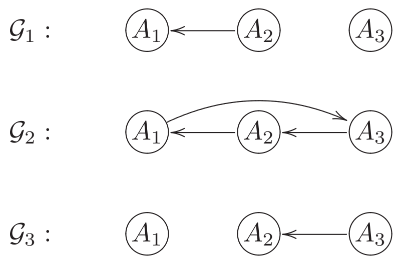

Fig. 1. Graphs of $L_{(1)}$, $L_{(2)}$, and $L_{(3)}$.

In the following, we sometimes denote $\dot{\xi}_i$ by ζ_i for simplicity. These two symbols will be used interchangeably. At each time interval of 5 seconds, we let the information exchange topology be $\mathcal{G}_1$ during 90 percent of the time and be $\mathcal{G}_2$ during the rest of the time. Note that at each time interval of 5 seconds the union of the information exchange topologies ($\mathcal{G}_1 \cup \mathcal{G}_2$) has a directed spanning tree. Using (3) with $\ell = 1$, consensus can be achieved as shown in Fig. 2. However, consensus cannot be achieved using (3) with $\ell = 2$ as shown in Fig. 3. As a comparison, if we increase the gain $\gamma_{1(2)}$ to be 10, consensus can be achieved asymptotically as shown in Fig. 4. Alternatively, if we reduce the length of each time interval to be 1 second, consensus can be achieved asymptotically with $\gamma_{1(2)} = 1$ as shown in Fig. 5. In addition, if we let the information exchange topology be $\mathcal{G}_1$ during 50 percent of the time and be $\mathcal{G}_2$ during the rest of the time, consensus can be achieved asymptotically as shown in Fig. 6. Next, at each time interval of 5 seconds, we let the information exchange topology be $\mathcal{G}_1$ during 90 percent of the time and be $\mathcal{G}_3$ during the rest of the time. Note that at each time interval of 5 seconds the union of the information exchange topologies ($\mathcal{G}_1 \cup \mathcal{G}_3$) has a directed spanning tree. Also note that graph $\mathcal{G}_3$ is only a subset of graph $\mathcal{G}_2$. Compared to Fig. 3, Fig. 7 shows that consensus can be achieved asymptotically even if graph $\mathcal{G}_3$ has less information exchange than graph $\mathcal{G}_2$.

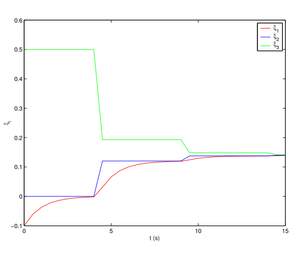

Fig. 2. Consensus of information under switching topologies using the first-order consensus algorithm ($\mathcal{G}_1 : 90\%$, $\mathcal{G}_2 : 10\%$).

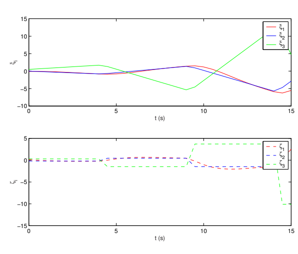

Fig. 3. Consensus of information under switching topologies using the second-order consensus algorithm ($\mathcal{G}_1 : 90\%$, $\mathcal{G}_2 : 10\%$).

In the special case that the information exchange topology between the vehicles is undirected and is based on their physical proximity, that is, there is information exchange

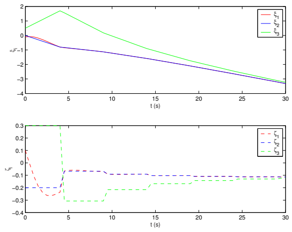

Fig. 4. Consensus of information under switching topologies using the second-order consensus algorithm with increased $\gamma_{1(2)}$ ($\mathcal{G}_1 : 90\%$, $\mathcal{G}_2 : 10\%$).

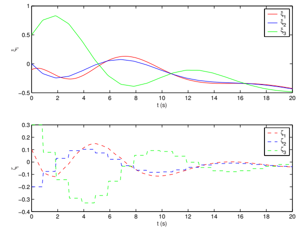

Fig. 5. Consensus of information under switching topologies using the second-order consensus algorithm with decreased interval length ($\mathcal{G}_1 : 90\%$, $\mathcal{G}_2 : 10\%$).

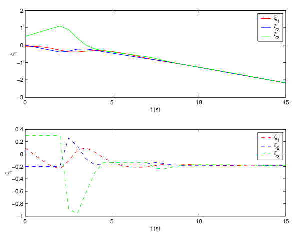

Fig. 6. Consensus of information under switching topologies using the second-order consensus algorithm ($\mathcal{G}_1 : 50\%$, $\mathcal{G}_2 : 50\%$).

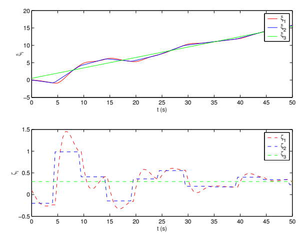

Fig. 7. Consensus of information under switching topologies using the second-order consensus algorithm ($\mathcal{G}_1 : 90\%$, $\mathcal{G}_3 : 10\%$).

between vehicle i and j if and only if the distance between them is below a certain threshold, we have the following theorem for information consensus motivated by [17].

Theorem 4.1: If the (time-varying) information exchange topology is undirected and connected at each time, the algorithm (3) with $\ell = 2$ achieves consensus asymptotically.

Proof: Let $V_{ij} = \frac{1}{2}k_{ij}(\xi_i - \xi_j)^2$, where $k_{ij} > 0$ is defined in Eq. (3). With (3), where $\ell = 2$, Eq. (2) can be rewritten as

$$\dot{\zeta}_i = -\sum_{j=1}^{n} g_{ij}\frac{\partial V_{ij}}{\partial \xi_i} - \sum_{j=1}^{n} g_{ij}k_{ij}\gamma_1(\zeta_i - \zeta_j). \tag{5}$$

Note that $k_{ij} = k_{ji}$ in the case of undirected information exchange. Also note that Eq. (5) can be written in matrix form as $\dot{\zeta} = -(L(t) \otimes I_m)\xi - \gamma_1(L(t) \otimes I_m)\zeta$, where $\xi = [\xi_1^T, \cdots, \xi_n^T]^T$, $\zeta = [\zeta_1^T, \cdots, \zeta_n^T]^T$, and $L(t) = [\ell_{ij}(t)] \in \mathbb{R}^{n\times n}$ is defined as $\ell_{ii}(t) = \sum_{j\neq i} g_{ij}(t)k_{ij}$ and $\ell_{ij} = -g_{ij}(t)k_{ij}$, $\forall i \neq j$, corresponding to the undirected (time-varying) information exchange topology at time t. Noting that Eq. (5) has the same form as Eq. (4) in [17], we can follow a similar proof to that of Theorem VI.2 in [17] to show that $\zeta_i \to \zeta_j$, $\forall i \neq j$, and $\dot{\zeta}_i \to 0$. As a result, we know that $(L(t) \otimes I_m)\xi \to 0$, which implies that $\xi_i \to \xi_j$, $\forall i \neq j$, since the information exchange topology is connected. ■

By applying (3) with $\ell = 2$, Eq. (2) can be written in matrix form as

$$\begin{bmatrix}\dot{\xi}\\ \dot{\zeta}\end{bmatrix} = \left(\underbrace{\begin{bmatrix}0_{n\times n} & I_n\\ -L & -\gamma_1 L\end{bmatrix}}_{\Gamma} \otimes I_m\right)\begin{bmatrix}\xi\\ \zeta\end{bmatrix}, \tag{6}$$

where $\xi = [\xi_1^T, \cdots, \xi_n^T]^T$ and $\zeta = [\zeta_1^T, \cdots, \zeta_n^T]^T$.

In the general case that the information exchange topology between the vehicles is directed and is switching randomly with time, we assume that Eq. (6) can be written as

$$\begin{bmatrix}\dot{\xi}\\ \dot{\zeta}\end{bmatrix} = (\Gamma_\sigma \otimes I_m)\begin{bmatrix}\xi\\ \zeta\end{bmatrix},$$

where $\sigma : [0,\infty) \to \mathcal{P}$ is a piecewise constant switching signal with switching times $t_0, t_1, \cdots$, and $\mathcal{P}$ denotes a set indexing the class of all possible directed information exchange topologies for the n vehicles that have a directed spanning tree. That is, we assume that $\Gamma(t)$ is piecewise constant and satisfies $\Gamma(t) = \Gamma(t_i)$, $t \in [t_i, t_{i+1})$.

Let $\xi_{ij} = \xi_i - \xi_j$ and $\zeta_{ij} = \zeta_i - \zeta_j$ be the consensus error variables. Note that $\xi_{ij} = \xi_{1j} - \xi_{1i}$ and $\zeta_{ij} = \zeta_{1j} - \zeta_{1i}$. Defining the consensus error vector as $\tilde{\xi} = [\xi_{12}^T, \xi_{13}^T, \cdots, \xi_{1n}^T]^T$ and $\tilde{\zeta} = [\zeta_{12}^T, \zeta_{13}^T, \cdots, \zeta_{1n}^T]^T$, we get the following equation:

$$\begin{bmatrix}\dot{\tilde{\xi}}\\ \dot{\tilde{\zeta}}\end{bmatrix} = (\Delta_\sigma \otimes I_m)\begin{bmatrix}\tilde{\xi}\\ \tilde{\zeta}\end{bmatrix}, \qquad (7)$$

where Δ_σ is a $2(n-1)\times 2(n-1)$ matrix that can be derived from Γ_σ. If Δ_σ is stable, we can find $a_\sigma \geq 0$ and $\chi_\sigma > 0$ such that $\left\|e^{\Delta_\sigma t}\right\| \leq e^{(a_\sigma - \chi_\sigma t)}$, $t \geq 0$.

We have the following theorem for information consensus under switching directed information exchange topologies.

Theorem 4.2: Let $t_0, t_1, \cdots$ be the times when the information exchange topology switches. Also let τ be the dwell time such that $t_{i+1} - t_i \geq \tau$, $\forall i = 0, 1, \cdots$. If the information exchange topology has a directed spanning tree for each $t \in [t_i, t_{i+1})$, the condition for γ_1 in Eq. (4) is satisfied for each Γ_σ, where $\sigma \in \mathcal{P}$, and the dwell time τ satisfies $\tau > \sup_{\sigma\in\mathcal{P}}\{\frac{a_\sigma}{\chi_\sigma}\}$, then (3) with $\ell = 2$ achieves consensus exponentially and is robust to information exchange noise under switching directed information exchange topologies.

Proof: Given a certain $\sigma_\ell \in \mathcal{P}$, suppose that the information exchange topology has a directed spanning tree for $t \in [t_\ell, t_{\ell+1})$ and the condition for γ_1 in Eq. (4) is satisfied for Γ_{σ_ℓ}. Then we know that consensus is achieved asymptotically if $\sigma(t) \stackrel{\triangle}{=} \sigma_\ell$, $\forall t \geq 0$, from Section III. That is, $\xi_i \to \xi_j$ and $\zeta_i \to \zeta_j$, $\forall i \neq j$, if $\sigma(t) \stackrel{\triangle}{=} \sigma_\ell$. Equivalently, we know that $\tilde{\xi} \to 0$ and $\tilde{\zeta} \to 0$ asymptotically if $\sigma(t) \stackrel{\triangle}{=} \sigma_\ell$, which implies that the switched system (7) is stable for each $\sigma \in \mathcal{P}$ under the conditions of the theorem. As a result, the switched system (7) is globally exponentially stable if the dwell time τ satisfies $\tau > \sup_{\sigma\in\mathcal{P}}\{\frac{a_\sigma}{\chi_\sigma}\}$ [25]. The stability of the switched system (7) implies that consensus can be achieved exponentially. The robustness of the consensus algorithm (3) to information exchange noise comes from the fact that Eq. (7) is globally exponentially stable. ■

V. EXTENSIONS TO REFERENCE MODELS

With (3), the final consensus value will depend on the information exchange topology as well as weighting factors k_{ij} and γ_k. As a result, the final consensus value may be *a priori* unknown. In some applications, it may be desirable that each information variable ξ_i approaches a (time-varying) reference state while reaching consensus during the transition. In the following, we assume that the reference state ξ^d satisfies the following nonlinear model

$$\ddot{\xi}^d = f(t, \xi^d, \dot{\xi}^d), \qquad (8)$$

where $\xi^d \in I\!R^m$ and $f(\cdot,\cdot,\cdot)$ is piecewise continuous in t and locally Lipschitz in ξ^d and $\dot{\xi}^d$.

Next, we first consider two special cases where either all the vehicles in the team or the unique team leader has access to the reference model and then consider the general case where only a portion of the vehicles have access to the reference model.

A. Full Access to the Reference Model

In this strategy, we incorporate the reference model to each vehicle's control law. The control law for each vehicle is designed as

$$u_i = f(t, \xi^d, \dot{\xi}^d) - \alpha[(\xi_i - \xi^d) + \gamma(\dot{\xi}_i - \dot{\xi}^d)] - \sum_{j=1}^{n} g_{ij}k_{ij}[(\xi_i - \xi_j) + \gamma(\dot{\xi}_i - \dot{\xi}_j)], \qquad (9)$$

where $\alpha > 0$, $\gamma > 0$, $k_{ij} > 0$, and g_{ij} is defined in Eq. (3).

With the control law (9) and Eq. (8), Eq. (2) can be written in matrix form as

$$\begin{bmatrix}\dot{\tilde{\xi}}\\ \ddot{\tilde{\xi}}\end{bmatrix} = \left(\underbrace{\begin{bmatrix}0_{n\times n} & I_n\\ -\alpha I_n - L & -\gamma(\alpha I_n + L)\end{bmatrix}}_{\Sigma}\otimes I_m\right)\begin{bmatrix}\tilde{\xi}\\ \dot{\tilde{\xi}}\end{bmatrix},$$

where $\tilde{\xi} = [\tilde{\xi}_1^T, \cdots, \tilde{\xi}_n^T]^T$ with $\tilde{\xi}_i = \xi_i - \xi^d$.

Note that if γ satisfies Inequality (4), then $\tilde{\xi}_i \to 0$ and $\dot{\tilde{\xi}}_i \to 0$, which implies that $\xi_i \to \xi^d$ and $\dot{\xi}_i \to \dot{\xi}^d$, $i = 1, \cdots, n$. In fact, even in the case that there is no information exchange between the vehicles (i.e., $L = 0$), $\xi_i \to \xi^d$ and $\dot{\xi}_i \to \dot{\xi}^d$, $i = 1, \cdots, n$. However, in the case that the graph of L has a directed spanning tree, the transient performance is guaranteed, that is, $\xi_i \to \xi_j$ during the transition [26].

To illustrate from a graphical point of view, consider the following information exchange topology given by Fig. 8 where each vehicle has access to the reference model. Here we treat ξ^d as a virtual vehicle. Note that there exists a link from node ξ^d to every vehicle in the team.

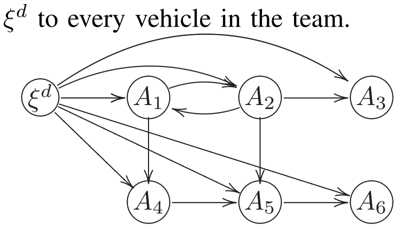

Fig. 8. Information exchange topology where each vehicle has access to the reference model.

B. Leader-follower Strategy

Leader-follower strategies are widely studied in the literature for multi-agent coordination (see e.g., [27]). Suppose that vehicle k is the unique team leader. The control law for vehicle k is designed as

$$u_k = f(t, \xi^d, \dot{\xi}^d) - K_{rk}(\xi_k - \xi^d) - K_{vk}(\dot{\xi}_k - \dot{\xi}^d), \quad (10)$$

where K_{rk} and K_{vk} are $m \times m$ symmetric positive definite matrices. The control law for follower vehicle i is designed as

$$u_i = \ddot{\xi}_j - K_{ri}(\xi_i - \xi_j) - K_{vi}(\dot{\xi}_i - \dot{\xi}_j), \quad (11)$$

where K_{ri} and K_{vi} are $m \times m$ symmetric positive definite matrices, and vehicle j is the leader of vehicle i.[2]

To illustrate, consider the following information exchange topology given by Fig. 9 where vehicle 1 is the unique team leader. In Fig. 9, vehicle j is the leader of vehicle $j+1$, $j = 2, 4, 5$, and vehicle 1 is the leader of vehicles 2 and 4. Note that there exists a link from node ξ^d to the team leader.

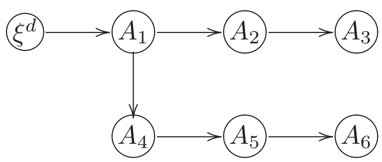

Fig. 9. A leader-follower topology where only the team leader has access to the reference model.

Note that in the leader-follower topology, information only flows from leaders to followers. In the case that a follower is perturbed by disturbance, the leaders are unaware of this disturbance and their motions remain unaffected.

C. General Case

In the general case that the information exchange topology may or may not have a directed spanning tree and one or more vehicles may have access to the reference model, we propose the following control law:

$$u_i = \frac{1}{\kappa_i}\sum_{j=1}^{n} g_{ij}[u_j - K_{ri}(\xi_i - \xi_j) - K_{vi}(\dot{\xi}_i - \dot{\xi}_j)] + \frac{1}{\kappa_i} g_{i(n+1)}[f(t, \xi^d, \dot{\xi}^d) - K_{ri}(\xi_i - \xi^d) - K_{vi}(\dot{\xi}_i - \dot{\xi}^d)], \quad (12)$$

where $g_{ii} \stackrel{\triangle}{=} 0$, g_{ij}, $\forall i, j \in \{1, \cdots, n\}$, is 1 if information flows from vehicle j to vehicle i and 0 otherwise, $g_{i(n+1)}$ is 1 if vehicle i has access to the reference model and 0 otherwise, $\kappa_i = \sum_{j=1}^{n+1} g_{ij}$, and K_{ri} and K_{vi} are $m \times m$ symmetric positive definite matrices. Note that in Eq. (12) each vehicle needs the information states, their derivatives, and the control inputs from its local neighbors.

We have the following theorem for consensus with a nonlinear reference model in the general case.

[2]That is, information flows from vehicle j to vehicle i in the information exchange topology. However, vehicle j may not be the team leader.

Theorem 5.1: Let $G = [g_{ij}] \in \mathbb{R}^{(n+1)\times(n+1)}$, where g_{ij} and $g_{i(n+1)}$, $1 \le i, j \le n$, are defined in Eq. (12) and $g_{(n+1)i} = 0$, $i = 1, \cdots, n+1$. With the control law (12), $\xi_i \to \xi^d$ and $\dot{\xi}_i \to \dot{\xi}^d$ asymptotically if and only if the graph of G has a directed spanning tree (with node ξ^d being the root).[3]

Proof: Let $\xi_{n+1} \equiv \xi^d$. Noting that $\ddot{\xi}^d = f(t, \xi^d, \dot{\xi}^d)$ and $\ddot{\xi}_j = u_j$, $j = 1, \cdots, n$, Eq. (12) can be rewritten as

$$\kappa_i \ddot{\xi}_i = \sum_{j=1}^{n+1} g_{ij}[\ddot{\xi}_j - K_{ri}(\xi_i - \xi_j) - K_{vi}(\dot{\xi}_i - \dot{\xi}_j)],$$

which implies that

$$\ddot{\sigma}_i = -K_{ri}\sigma_i - K_{vi}\dot{\sigma}_i,$$

where $\sigma_i = \sum_{j=1}^{n+1} g_{ij}(\xi_i - \xi_j)$. Then we know that $\sigma_i \to 0$ and $\dot{\sigma}_i \to 0$, $i = 1, \cdots, n$, since K_{ri} and K_{vi} are symmetric positive definite matrices. Let $L_\sigma = [\ell_{ij}]$ be an $(n+1)\times(n+1)$ matrix, where $\ell_{ii} = \sum_{j=1}^{n+1} g_{ij}$ and $\ell_{ij} = -g_{ij}$, $\forall i \neq j$. Note that matrix L_σ satisfies property (1). Also note that all entries of the n+1th row of L_σ are zero. In addition, note that $\sigma_i \to 0$ and $\dot{\sigma}_i \to 0$, $i = 1, \cdots, n$, can be written in matrix form as $(L_\sigma \otimes I_m)\xi \to 0$ and $(L_\sigma \otimes I_m)\dot{\xi} \to 0$ respectively, where $\xi = [\xi_1^T, \cdots, \xi_{n+1}^T]^T$. Therefore, we know that $\xi_i \to \xi_j$ and $\dot{\xi}_i \to \dot{\xi}_j$, $\forall i, j \in \{1, \cdots, n+1\}$, if and only if the graph of G has a directed spanning tree from Section II, which in turn implies that $\xi_i \to \xi^d$ and $\dot{\xi}_i \to \dot{\xi}^d$, $i = 1, \cdots, n$, since $\xi_{n+1} \equiv \xi^d$.[4] ■

To illustrate, consider the following information exchange topology given by Fig. 10 where only vehicles 1 and 5 have access to ξ^d. Note that although neither vehicle 1 nor vehicle 5 has a directed path to all the other vehicles in the team, there exists a directed path from node ξ^d to all the vehicles in the team. Also note that unlike the leader-follower topology where information only flows from leaders to followers, information in the general topology may flow from any vehicle to any other vehicle.

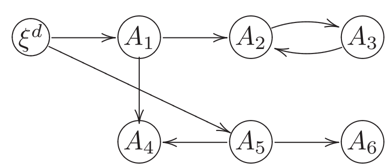

Fig. 10. A general information exchange topology where only a portion of the vehicles have access to the reference model and the original topology without node ξ^d does not have a directed spanning tree.

As an example, consider a nonlinear model given by $\ddot{\xi}^d = -\frac{\sin(\xi^d)}{1+e^{-t}}$, where $\xi^d(0) = \frac{\pi}{2}$ and $\dot{\xi}^d(0) = 0$. Assume that the information exchange topology between six vehicles is given by Fig. 10, where only vehicles 1 and 5 have access to ξ^d. Fig. 11 shows that the information states of the six vehicles reach consensus on the nonlinear model using (12).

[3]Equivalently, ξ^d is the only node that has a directed path to all the vehicles in the team.

[4]Note that no constraints are imposed on the nonlinear function $f(t, \xi^d, \dot{\xi}^d)$ in the proof as long as $f(\cdot, \cdot, \cdot)$ is piecewise continuous in t and locally Lipschitz in ξ^d and $\dot{\xi}^d$.

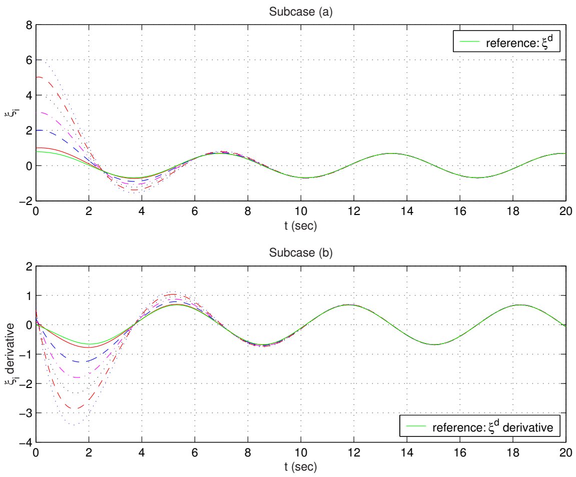

Fig. 11. Consensus on the reference model using (12) when only a portion of the vehicles have access to the reference model.

In the special case that $\dot{\xi}^d$ is constant, the control law is designed as

$$u_i = -\sum_{j=1}^{n} g_{ij}[K_{ri}(\xi_i - \xi_j) + K_{vi}(\dot{\xi}_i - \dot{\xi}_j)] - g_{i(n+1)}[K_{ri}(\xi_i - \xi^d) + K_{vi}(\dot{\xi}_i - \dot{\xi}^d)]. \quad (13)$$

It is straightforward to verify that Eq. (13) is only valid in the case that $\dot{\xi}^d$ is constant and cannot guarantee consensus on the reference model in the case that $\dot{\xi}^d$ is time-varying.

VI. CONCLUSION AND FUTURE WORK

We have extended the second-order consensus algorithm in [19] to the case of switching topologies and reference models. Sufficient conditions have been given to guarantee consensus under switching directed information exchange topologies. Consensus algorithms have also been proposed so that the information variables of each vehicle reach consensus on the solution of a nonlinear reference model in the case that only a portion of the vehicles have access to the reference model. Future work will address the effect of time delays on consensus seeking with reference models.

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