Proceedings of the 2007 American Control Conference
Marriott Marquis Hotel at Times Square
New York City, USA, July 11-13, 2007

Consensus Seeking in Multi-vehicle Systems with a Time-varying Reference State

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Abstract— **In this paper, we study the consensus problem in multi-vehicle systems where the information states of each vehicle approach a common time-varying reference state. We first analyze consensus algorithms with a constant reference state using graph theoretical tools. We then propose consensus algorithms with a time-varying reference state and show necessary and sufficient conditions under which consensus is reached on the time-varying reference state when only a portion of the vehicles (e.g., the unique team leader) have access to the reference state and those vehicles might not have directed paths to the other vehicles in the team. The reference state may be a time-varying exogenous signal or evolves according to a nonlinear model. These consensus algorithms are also extended to achieve relative state deviations between the vehicles.**

I. INTRODUCTION

Future autonomous vehicles will have the capability to improve significantly the operational effectiveness of both civilian and military applications. While autonomous vehicles performing solo missions will yield some benefits, greater benefits will come from having teams of autonomous vehicles operating in a coordinated fashion.

Consensus problems have recently received significant attention in the area of cooperative control of multi-vehicle systems due to their potential applications for designing distributed multi-vehicle coordination strategies (see [1] for an overview).

For most consensus algorithms studied in the literature, the final consensus value to be reached is inherently constant. For example, in [2]–[5], the final consensus value, which depends on both the information-exchange topologies and the weights, is a weighted average of the vehicles' initial states. In the leader following case of [2], the final consensus value is the constant state of the group leader, where convergence analysis is given under undirected switching information-exchange topologies. In [6], [7], a constant setpoint is introduced to the consensus algorithm in the case of a directed fixed information-exchange topology. However, a constant final consensus value might not be appropriate in the case that the information states of each vehicle are dynamically evolving in time according to some inherent dynamics, as happens in some formation control problems where the formation is moving through space. As a result, it is relevant to study consensus problems where the final consensus value evolves with time or as a function of vehicle/environmental dynamics or sensor measurement, called the reference state hereafter. In practice, it is also possible that only a portion of the vehicles in the team have access to the time-varying reference state and those vehicles might not have directed paths to the other vehicles in the team.

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The main objective of this paper is to propose and analyze consensus algorithms so that each vehicle in the team reaches consensus on a time-varying reference state that evolves with time or according to certain nonlinear dynamics in the case that only a portion of the vehicles have access to the reference state and those vehicles might not have directed paths to the other vehicles in the team. All of the analysis in this paper is based on directed information exchange. We first analyze consensus algorithms with a constant reference state using graph theoretical tools and show that the existing algorithms for a constant reference state cannot guarantee consensus on a time-varying reference state. We then propose algorithms to deal with the time-varying case and show conditions under which consensus is reached on the time-varying reference state. Unlike the leader-follower topology where information only flows from leaders to followers (e.g., [8]), the proposed algorithms allow information to flow from followers to leaders to introduce feedback. We show that with the time-varying reference state complexity results from the information feedback loops. Those results are also extended to achieve relative state deviations between the vehicles. It is worthwhile to mention that the extension of consensus algorithms from a constant reference to a time-varying reference is nontrivial. It is not straightforward how the internal model principle of control can be directly applied to consensus seeking with a time-varying reference state for multiple vehicle systems involving only local information exchange.

II. BACKGROUND AND PRELIMINARIES

It is natural to model information exchange between vehicles by directed or undirected graphs. A digraph (directed graph) consists of a pair $(\mathcal{N}, \mathcal{E})$, where $\mathcal{N}$ is a finite nonempty set of nodes, and $\mathcal{E} \in \mathcal{N} \times \mathcal{N}$ is a set of ordered pairs of nodes, called edges. An edge (i, j) in a digraph denotes that vehicle j can obtain information from vehicle i, but not necessarily vice versa. As a comparison, the pairs of nodes in an undirected graph are unordered, where an edge (i, j) denotes that vehicles i and j can obtain information from one another. Note that an undirected graph can be considered a special case of a digraph, where an edge (i, j) in the undirected graph corresponds to edges (i, j) and (j, i) in the directed graph. If there is a directed edge from node i to node j, then i is defined as the parent node, and j is defined as the child node. A directed path is a sequence of ordered

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edges of the form $(v_{i_1}, v_{i_2}), (v_{i_2}, v_{i_3}), \cdots$, where $v_{i_j} \in \mathcal{N}$, in a digraph. An undirected path in an undirected graph is defined accordingly. In a digraph, a cycle is a path that starts and ends at the same node. A digraph is called strongly connected if there is a directed path from every node to every other nodes. An undirected graph is called connected if there is a path between any distinct pair of nodes. A directed tree is a digraph, where every node has exactly one parent except for one node, called root, which has no parent, and the root has a directed path to every other node. Note that in a directed tree, each edge has a natural orientation away from the root, and no cycle exists. In the case of undirected graphs, a tree is a graph in which every pair of nodes is connected by exactly one path. A directed spanning tree of a digraph is a directed tree formed by graph edges that connect all of the nodes of the graph. A graph has (or contains) a directed spanning tree if there exists a directed spanning tree being a subset of the graph. Note that the condition that a digraph has a directed spanning tree is equivalent to the case that there exists at least one node having a directed path to all of the other nodes. In the case of undirected graphs, having an undirected spanning tree is equivalent to being connected. However, in the case of directed graphs, having a directed spanning tree is a weaker condition than being strongly connected.

The adjacency matrix $A = [a_{ij}] \in I\!R^{n\times n}$ of a weighted digraph is defined as $a_{ii} = 0$ and $a_{ij} > 0$ if $(j,i) \in \mathcal{E}$ where $i \neq j$. The adjacency matrix of a weighted undirected graph is defined accordingly except that $a_{ij} = a_{ji}$, $\forall i \neq j$, since $(j,i) \in \mathcal{E}$ implies $(i,j) \in \mathcal{E}$. Let matrix $L = [\ell_{ij}] \in I\!R^{n\times n}$ be defined as $\ell_{ii} = \sum_{j\neq i} a_{ij}$ and $\ell_{ij} = -a_{ij}$, where $i \neq j$. The matrix L satisfies the following conditions:

$$\begin{aligned} &\ell_{ij} \leq 0, \quad i \neq j \\ &\sum_{j=1}^{n} \ell_{ij} = 0, \quad i = 1, \cdots, n. \end{aligned} \tag{1}$$

For an undirected graph, L is called the Laplacian matrix, which has the property that it is symmetric positive semi-definite. However, L for a digraph does not have this property.

Let $\mathbf{1}$ and $\mathbf{0}$ denote the $n\times 1$ column vector of all ones and all zeros respectively. Let I_n denote the $n\times n$ identity matrix. Let $M_n(I\!R)$ represent the set of all $n\times n$ real matrices. Given a matrix $S = [s_{ij}] \in M_n(I\!R)$, the digraph of S, denoted by $\Gamma(S)$, is the digraph on n nodes v_i, $i \in \{1, 2, \cdots, n\}$, such that there is a directed edge in $\Gamma(S)$ from v_j to v_i if and only if $s_{ij} \neq 0$ (c.f. [9]).

III. CONSENSUS WITH A REFERENCE STATE

In this section we investigate consensus algorithms with a time-varying reference state for vehicles modeled by single-integrator dynamics. Consider vehicles with dynamics given by

$$\dot{\xi}_i = u_i, \quad i = 1, \ldots, n, \tag{2}$$

where $\xi_i \in I\!R^m$ is the state of the i^{th} vehicle, and $u_i \in I\!R^m$ is the control input. A consensus algorithm is proposed in [2], [3], [10] as

$$u_i = -\sum_{j=1}^{n} g_{ij} k_{ij} (\xi_i - \xi_j), \quad i = 1, \ldots, n, \tag{3}$$

where $k_{ij} > 0$, $g_{ii} \stackrel{\triangle}{=} 0$, and g_{ij} is 1 if information flows from vehicle j to vehicle i and 0 otherwise, $\forall i \neq j$.

With the consensus algorithm (3), consensus is reached among the n vehicles if $\xi_i(t) \to \xi_j(t)$, $\forall i \neq j$, as $t \to \infty$. The final consensus value, which depends on both the information-exchange topologies and the weights k_{ij}, is a weighted average of the vehicles' initial states. However, in some applications, it might be desirable that each state $\xi_i(t)$ approaches a (time-varying) reference state $\xi^r(t)$ while the reference state might be available to only a portion of the vehicles in the team.

In the following, we derive algorithms to achieve this objective. We say that the consensus problem with a reference state is solved if $\xi_i(t) \to \xi_j(t) \to \xi^r(t)$, $\forall i \neq j$, as $t \to \infty$.

Before moving on, we need the following lemmas from [10].

Lemma 3.1: [10] Suppose that $z = [z_1, \cdots, z_p]^T$ with $z_i \in I\!R$ and $L \in I\!R^{p\times p}$ satisfies the property (1). Then the following four conditions are equivalent: (i) L has a simple zero eigenvalue with an associated eigenvector $\mathbf{1}$ and all of the other eigenvalues have positive real parts; (ii) $Lz = 0$ implies that $z_1 = \cdots = z_p$; (iii) Consensus is reached asymptotically for a system $\dot{z} = -Lz$; (iv) The directed graph of L has a directed spanning tree.

Lemma 3.2: [10] Suppose that z and L are defined in Lemma 3.1. Then the following four conditions are equivalent: (i) The directed graph of L has a directed spanning tree and vehicle k has no incoming links[1]; (ii) The directed graph of L has a directed spanning tree and every entry of the k^{th} row of L is zero; (iii) Consensus is reached asymptotically for a system $\dot{z} = -Lz$ with $\xi_i(t) \to \xi_k(0)$, $\forall i$, as $t \to \infty$; (iv) Vehicle k is the only node that has a directed path to all of the other vehicles in the team.

A. Constant Reference State

In this subsection, we consider the case that the reference state ξ_r is constant, where the consensus algorithm can be summarized as

$$u_i = -\sum_{j=1}^{n} g_{ij} k_{ij} (\xi_i - \xi_j) - g_{i(n+1)} \alpha_i (\xi_i - \xi^r), \tag{4}$$

where $k_{ij} > 0$, $\alpha_i > 0$, $g_{ii} \stackrel{\triangle}{=} 0$, g_{ij} is 1 if information flows from vehicle j to vehicle i and 0 otherwise, and $g_{i(n+1)}$ is 1 if vehicle i has access to ξ^r and 0 otherwise. Note that in [2] ξ^r corresponds to the constant state of the group leader. Also note that [6] deals with the case that only one vehicle has access to the reference state. The vehicle, denoted as vehicle ℓ without loss of generality, must be the root of a directed

[1] At most one such vehicle can exist in the case that the directed graph has a directed spanning tree.

spanning tree. As a result, $g_{\ell(n+1)} = 1$, and $g_{i(n+1)} = 0$, $\forall i \neq \ell$.

Next, we consider the case that a portion of the vehicles in the team, denoted as a vehicle set $\mathcal{L}$, have access to the reference state under directed information exchange, that is, $g_{i(n+1)} = 1$, $\forall i \in \mathcal{L}$, and $g_{i(n+1)} = 0$, $\forall i \notin \mathcal{L}$. We have the following theorem on consensus with a constant reference state over a directed information-exchange topology.

Theorem 3.1: Let $G = [g_{ij}] \in \mathbb{R}^{(n+1)\times(n+1)}$ be the adjacency matrix, where g_{ij} and $g_{i(n+1)}$, $\forall i, j \in \{1, \ldots, n\}$, are defined in Eq. (4) and $g_{(n+1)k} = 0$, $\forall k \in \{1, \ldots, n+1\}$. The algorithm (4) solves the consensus problem with a constant reference state ξ^r if and only if the directed graph of G has a directed spanning tree.[2]

Proof: Let $k_{i(n+1)} \stackrel{\triangle}{=} \alpha_i$. Also let $L_{n+1} = [\ell_{ij}] \in \mathbb{R}^{(n+1)\times(n+1)}$ be defined as $\ell_{ij} = -g_{ij}k_{ij}$, $\ell_{ii} = \sum_{j\neq i} g_{ij}k_{ij}$, $\forall i \in \{1, \ldots, n\}$, $\forall j \in \{1, \ldots, n+1\}$, and $\ell_{(n+1)j} = 0$, $\forall j$. Letting $\xi_{n+1} \stackrel{\triangle}{=} \xi^r$, gives $\dot{\xi}_{n+1} = 0$. With (4), Eq. (2) can be written in matrix form as

$$\dot{\xi} = -(L_{n+1} \otimes I_m)\xi,$$

where $\xi = [\xi_1^T, \ldots, \xi_{n+1}^T]^T$, and $\otimes$ denotes the Kronecker product. Note that L_{n+1} satisfies the property (1), and the directed graph of L_{n+1} is equivalent to that of G. Then from arguments (ii) and (iii) of Lemma 3.2 with L_{n+1} and ξ playing the roles of L and z respectively, it follows that $\xi_i \to \xi_{n+1}(0)$, $\forall i$, if and only if the directed graph of G has a directed spanning tree. Equivalently, it follows that $\xi_i \to \xi^r$, $\forall i$, since $\xi_{n+1} \equiv \xi^r$. ■

To illustrate, consider a team of $n = 4$ vehicles. Four subcases will be considered in this subsection, where $\xi^r \stackrel{\triangle}{=} 1$ for each subcase. In Subcase (a), we let $g_{15} = 1$ and $g_{j5} = 0$, $\forall j \neq 1$, which corresponds to the case that only vehicle 1 has access to ξ^r. In Subcase (b), we let $g_{j5} = 1$, $\forall j$, which corresponds to the case that all of the vehicles have access to ξ^r. In Subcase (c), we let $g_{35} = g_{45} = 1$ and $g_{j5} = 0$, $\forall j \notin \{3, 4\}$, which corresponds to the case that only vehicles 3 and 4 have access to ξ^r. In Subcase (d), we let $g_{45} = 1$ and $g_{j5} = 0$, $\forall j \neq 4$, which corresponds to the case that only vehicle 4 has access to ξ^r. Fig. 1 shows the information-exchange topologies corresponding to each subcase.

Fig. 2 shows the states of each vehicle using (4). Note that ξ_i converges to ξ^r in each subcase except Subcase (d). Also note that node ξ^r has a directed path to all of the vehicles in Subcases (a), (b), and (c) in Fig. 1. However, there does not exist a directed path from node ξ^r to all of the vehicles in Subcase (d) in Fig. 1. Note that Subcase (a) corresponds to the case discussed in [6]. Also note that in Subcase (c) in Fig. 1, node 4 is not the root of a directed spanning tree, which implies that the results in [6] do not apply. However, as shown above, ξ_i still approaches ξ^r in this case.

[2]Treat ξ^r as a virtual vehicle with index $n+1$. This condition is equivalent to the condition that ξ^r is the only node that has a directed path to all of the vehicles in the team from Lemma 3.2.

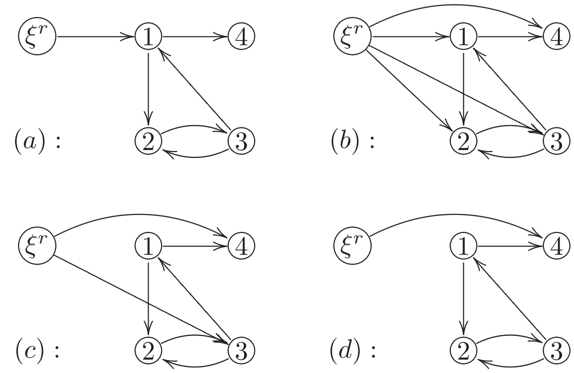

Fig. 1. Information-exchange topologies between the four vehicles, where one or more vehicles might have access to the constant reference state.

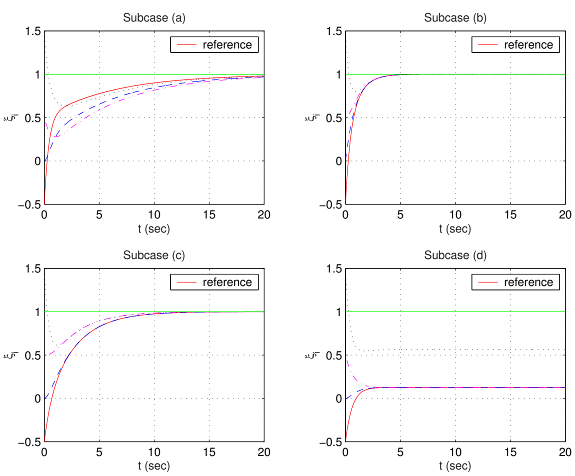

Fig. 2. Consensus seeking with a constant reference state using (4).

B. Time-varying Reference State

In this subsection, we assume that the reference state might be a time-varying exogenous signal or evolve according to certain nonlinear dynamics. Without loss of generality, suppose that ξ^r satisfies the dynamics given by

$$\dot{\xi}^r = f(t, \xi^r), \tag{5}$$

where $f(\cdot, \cdot)$ is piecewise continuous in t and locally Lipschitz in ξ^r.

We first show that the algorithm (4) is not sufficient for consensus on a time-varying reference state. As an example, let $\xi^r = \cos(t)$ and consider the four subcases as in Subsection III-A. As shown in Fig. 3, the states of each vehicle do not converge to ξ^r in each subcase.

One might be tempted to apply the following algorithm in the case of a time-varying reference state

$$u_i = g_{i(n+1)} f(t, \xi^r) - \sum_{j=1}^{n} g_{ij}k_{ij}(\xi_i - \xi_j) - g_{i(n+1)}\alpha_i(\xi_i - \xi^r),$$
$$i = 1, \ldots, n, \tag{6}$$

where k_{ij}, α_i, g_{ij}, and $g_{i(n+1)}$ are defined as in Eq. (4).

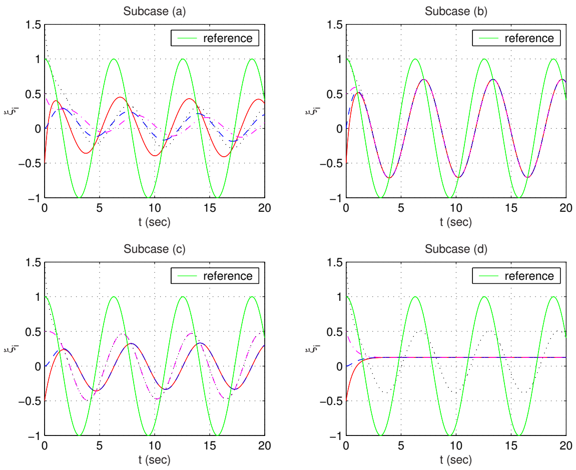

Fig. 3. Consensus seeking with a time-varying reference state using (4).

As an example, similarly let $\xi^r = \cos(t)$ and consider the four subcases as in Subsection III-A. As shown in Fig. 4, the states of each vehicle do not converge to ξ^r in each subcase except Subcase (b).

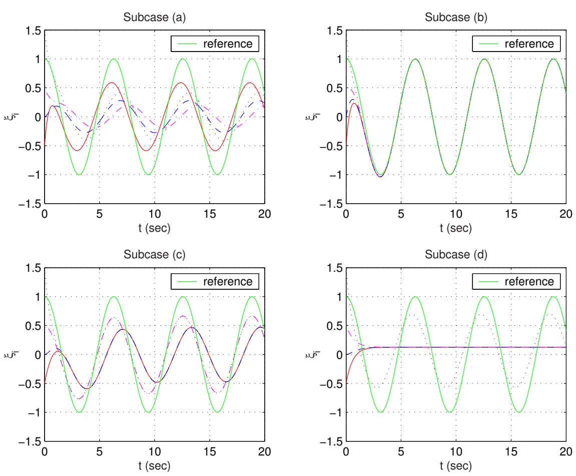

Fig. 4. Consensus seeking with a time-varying reference state using (6).

We have the following theorem for consensus on a time-varying reference state using the algorithm (6).

Theorem 3.2: If $g_{i(n+1)} = 1$, $i = 1, \ldots, n$, then the consensus algorithm (6) solves the consensus problem with a time-varying reference state.

Proof: With (6), Eq. (2) can be written in matrix form as $\dot{\tilde{\xi}} = -[(L_n + \Gamma) \otimes I_m]\tilde{\xi}$, where $\Gamma \in I\!R^{n\times n}$ is a diagonal matrix with α_i being the diagonal entries, $L_n = [\ell_{ij}] \in I\!R^{n\times n}$ is defined as $\ell_{ij} = -g_{ij}k_{ij}$ and $\ell_{ii} = \sum_{j\neq i} g_{ij}k_{ij}$, $\forall i, j \in \{1, \ldots, n\}$, and $\tilde{\xi} = [\tilde{\xi}_1^T, \ldots, \tilde{\xi}_n^T]^T$ with $\tilde{\xi}_i = \xi_i - \xi^r$. From Gershgorin disc theorem [9], it is straightforward to see that all of the eigenvalues of $-(L_n + \Gamma)$ have negative real parts. Therefore, we know that $\tilde{\xi} \to 0$ asymptotically, that is, $\xi_i \to \xi^r$, $\forall i$, asymptotically. ■

Note that the argument of Theorem 3.2 does not rely on the information-exchange topology between the vehicles. Even if there is no information exchange between the vehicles (i.e. $L = 0$), the algorithm (6) still solves the consensus problem with a time-varying reference state as long as each vehicle has access to ξ^r. However, this argument is rather restricted in the sense that each vehicle must have access to the time-varying reference state.

When only a portion of the vehicles have access to ξ^r, we propose the following consensus algorithm

$$\begin{aligned} u_i =& \frac{1}{\eta_i}\sum_{j=1}^{n} g_{ij}k_{ij}[\dot{\xi}_j - \gamma_i(\xi_i - \xi_j)] \\ &+ \frac{1}{\eta_i} g_{i(n+1)}\alpha_i[f(t, \xi^r) - \gamma_i(\xi_i - \xi^r)], \quad i = 1, \ldots, n, \end{aligned} \tag{7}$$

where $k_{ij} > 0$, $\alpha_i > 0$, $\gamma_i > 0$, g_{ij} and $g_{i(n+1)}$ are defined as in Eq. (4), and $\eta_i = g_{i(n+1)}\alpha_i + \sum_{j=1}^{n} g_{ij}k_{ij}$. Note that information feedback is introduced to each vehicle through its local neighbors' information states and their derivatives.

In the special case that only one vehicle has access to ξ^r, the following consensus algorithm is also valid

$$\begin{aligned} u_i =& f(t, \xi^r) - \sum_{j=1}^{n} g_{ij}k_{ij}(\xi_i - \xi_j) - \alpha_i(\xi_i - \xi^r), \quad i = \ell \\ u_i =& \frac{1}{\sum_{j=1}^{n} g_{ij}k_{ij}} \sum_{j=1}^{n} g_{ij}k_{ij}[\dot{\xi}_j - \gamma_i(\xi_i - \xi_j)], \quad \forall i \neq \ell, \end{aligned} \tag{8}$$

where $\alpha_i > 0$, $\gamma_i > 0$, ℓ denotes the index of the only vehicle that has access to ξ^r, and g_{ij} is defined as in Eq. (3).

Theorem 3.3: Let $G = [g_{ij}] \in I\!R^{(n+1)\times(n+1)}$ be defined in Theorem 3.1. The algorithms (7) and (8) solve the consensus problem with a time-varying reference state if and only if the directed graph of G has a directed spanning tree.

Proof: For (7), let $\xi_{n+1} \stackrel{\triangle}{=} \xi^r$ and $k_{i(n+1)} \stackrel{\triangle}{=} \alpha_i$. Noting that $\dot{\xi}_i = u_i$, we rewrite Eq. (7) as

$$\dot{\xi}_i = \frac{1}{\sum_{j=1}^{n+1} g_{ij}k_{ij}} \sum_{j=1}^{n+1} g_{ij}k_{ij}[\dot{\xi}_j - \gamma_i(\xi_i - \xi_j)], \quad i = 1, \ldots, n.$$

After some manipulation, we get that

$$\sum_{j=1}^{n+1} g_{ij}k_{ij}(\dot{\xi}_i - \dot{\xi}_j) = -\gamma_i \sum_{j=1}^{n+1} g_{ij}k_{ij}(\xi_i - \xi_j) \quad i = 1, \ldots, n,$$

which implies that

$$\sum_{j=1}^{n+1} g_{ij}k_{ij}(\xi_i - \xi_j) \to 0, \quad i = 1, \ldots, n. \tag{9}$$

By adding a dummy equation $0 = 0$, $i = n + 1$, to Eq. (9), we can rewrite Eq. (9) in matrix form as $(L_{n+1} \otimes I_m)\xi \to 0$, where $\xi = [\xi_1^T, \ldots, \xi_{n+1}^T]^T$, $L_{n+1} = [\ell_{ij}] \in I\!R^{(n+1)\times(n+1)}$ is defined as $\ell_{ii} = \sum_{j\neq i} g_{ij}k_{ij}$, $\ell_{ij} = -g_{ij}k_{ij}$, $\forall i \in \{1, \ldots, n\}$, $\forall j \in \{1, \ldots, n + 1\}$, and $\ell_{(n+1)i} = 0$, $\forall i$. Note that all of the entries of the $n + 1^{\text{th}}$ row of L are zero. Also note that L_{n+1} satisfies the property (1) and the directed graph of L is equivalent to that of G, which has a directed spanning tree. Therefore, from the arguments (ii) and (iv) of Lemma 3.1 with L_{n+1} and ξ playing the roles of L and z respectively, we know that $\xi_i \to \xi_j$, $\forall i, j \in \{1, \ldots, n+1\}$, if and only if the directed graph of G has a directed spanning tree. Equivalently, it follows that $\xi_i \to \xi^r$, $\forall i$, since $\xi_{n+1} \equiv \xi^r$.

For (8), noting that $\dot{\xi}_i = u_i$, we rewrite the second equation in Eq. (8) as

$$\dot{\xi}_i = \frac{1}{\sum_{j=1}^n g_{ij}k_{ij}} \sum_{j=1}^n g_{ij}k_{ij}[\dot{\xi}_j - \gamma_i(\xi_i - \xi_j)], \quad \forall i \neq \ell.$$

After some manipulation, we get that

$$\sum_{j=1}^n g_{ij}k_{ij}(\dot{\xi}_i - \dot{\xi}_j) = -\gamma_i \sum_{j=1}^n g_{ij}k_{ij}(\xi_i - \xi_j), \quad \forall i \neq \ell,$$

which implies that $\sum_{j=1}^n g_{ij}k_{ij}(\xi_i - \xi_j) \to 0$, $\forall i \neq \ell$. Similarly, from the arguments (ii) and (iv) of Lemma 3.1, we know that $\xi_i \to \xi_j$, $i, j \in \{1, \ldots, n\}$, if and only if the directed graph of G has a directed spanning tree (with vehicle ℓ being the root). Noting that $\xi_i \to \xi_j$, $\forall i \neq j$, we know that $\xi_\ell \to \xi^r$ from the first equation in Eq. (8). Therefore, it follows that $\xi_i \to \xi^r$, $\forall i$, asymptotically. ∎

Compared to the algorithm (6), which requires that each vehicle have access to the time-varying reference state to reach consensus, the algorithms (7) and (8) allow only a portion of the vehicles to have access to the time-varying reference state.

To illustrate, consider two subcases in this subsection using (7), where $g_{35} = g_{45} = 1$ and $g_{j5} = 0$, $\forall j \notin \{3, 4\}$ (Fig. 1c), and (8), where $\ell = 1$ (Fig. 1a), respectively. In Subcase (a), let $\xi^r = \cos(t)$. In Subcase (b), assume that ξ^r satisfies the nonlinear dynamics given by $\dot{\xi}^r = \sin(t)\sin(2\xi^r)$, where $\xi^r(0) = 0.5$. As shown in Figs. 5 and 6, the states of each vehicle converge to the exogenous signal $\cos(t)$ in Subcase (a) and to the solution of the nonlinear model $\dot{\xi}^r = \sin(t)\sin(2\xi^r)$ in Subcase (b) using both (7) and (8).

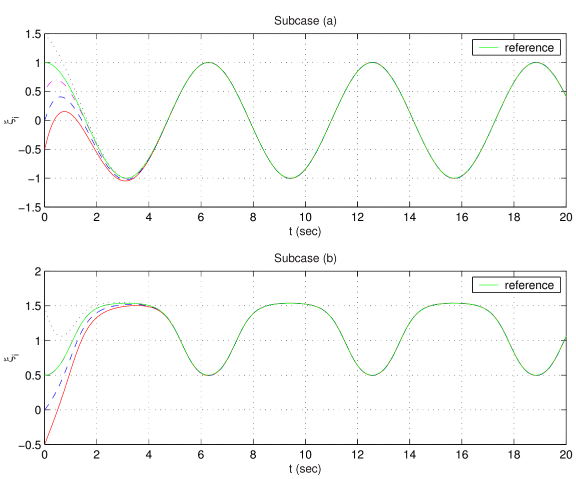

Fig. 5. Consensus seeking with a time-varying reference state using (7).

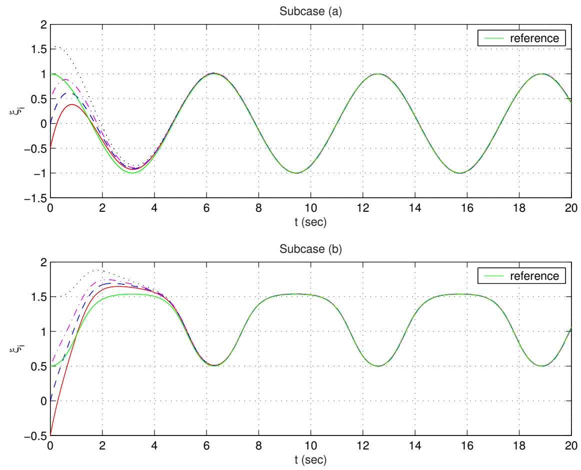

Fig. 6. Consensus seeking with a time-varying reference state using (8).

Compared to the leader-follower strategy where information only flows from leaders to followers[3], the algorithm (8) takes into account the general case that information might also flow from every vehicle to every other vehicle.

C. Extensions to Relative State Deviations

The consensus algorithm (3) can be extended to guarantee that the differences of the vehicle states converge to desired values, i.e., $\xi_i - \xi_j \to \Delta_{ij}(t)$, where $\Delta_{ij}(t)$ denotes the desired (time-varying) separation between ξ_i and ξ_j. We propose the following algorithm for relative state deviations

$$u_i = \dot{\delta}_i - \sum_{j=1}^n g_{ij}k_{ij}[(\xi_i - \xi_j) - (\delta_i - \delta_j)], \quad i = 1, \ldots, n, \tag{10}$$

where $\delta_i - \delta_j$, $\forall i \neq j$, denotes the desired separation between the states. Note that the consensus algorithm (3) corresponds to the case that $\Delta_{ij} = 0$, $\forall i \neq j$.

We have the following theorem for relative state deviations.

Theorem 3.4: With (10), $\xi_i - \xi_j \to \delta_i - \delta_j$ asymptotically if and only if the information-exchange topology has a directed spanning tree.

Proof: With (10), Eq. (2) can be written in matrix form as

$$\dot{\hat{\xi}} = -(L_n \otimes I_m)\hat{\xi},$$

where $\hat{\xi} = [\hat{\xi}_1^T, \ldots, \hat{\xi}_n^T]^T$ with $\hat{\xi}_i = \xi_i - \delta_i$ and $L_n = [\ell_{ij}] \in \mathbb{R}^{n\times n}$ with $\ell_{ij} = -g_{ij}k_{ij}$ and $\ell_{ii} = \sum_{j\neq i} g_{ij}k_{ij}$. Note that L_n satisfies the property (1). From Lemma 3.1, we know that $\hat{\xi}_i \to \hat{\xi}_j$ asymptotically if and only if the information-exchange topology has a directed spanning tree. The rest of the proof then follows the fact that $\hat{\xi}_i \to \hat{\xi}_j$ is equivalent to $\xi_i - \xi_j \to \delta_i - \delta_j$. ∎

[3]The leader-follower topology corresponds to an information-exchange graph that is itself a directed spanning tree. Note that the condition that a directed graph has a directed spanning tree is not equivalent to the condition that a directed graph is itself a directed spanning tree. The latter condition is a special case of the former one.

When a portion of the vehicles have access to ξ^r, we propose the following consensus algorithm for relative state deviations with a time-varying reference state

$$u_i = \dot{\delta}_i + \frac{1}{\eta_i}\sum_{j=1}^{n} g_{ij}k_{ij}\{\dot{\xi}_j - \dot{\delta}_j - \gamma_i[(\xi_i - \xi_j) - (\delta_i - \delta_j)]\} + \frac{1}{\eta_i} g_{i(n+1)}\alpha_i[f(t,\xi^r) - \gamma_i(\xi_i - \delta_i - \xi^r)]. \quad (11)$$

In the special case that only one vehicle has access to ξ^r, we propose the following consensus algorithm for relative state deviations with a time-varying reference state

$$\begin{aligned} u_i &= \dot{\delta}_i + f(t,\xi^r) - \sum_{j=1}^{n} g_{ij}k_{ij}[(\xi_i - \xi_j) - (\delta_i - \delta_j)] \\ &\quad - \alpha_i(\xi_i - \delta_i - \xi^r), \quad i = \ell \\ u_i &= \dot{\delta}_i + \frac{1}{\sum_{j=1}^{n} g_{ij}k_{ij}} \sum_{j=1}^{n} g_{ij}k_{ij}\{\dot{\xi}_j - \dot{\delta}_j \\ &\quad - \gamma_i[(\xi_i - \xi_j) - (\delta_i - \delta_j)]\}, \quad i \neq \ell, \end{aligned} \quad (12)$$

where ℓ denotes the index of the vehicle that has access to ξ^r.

Theorem 3.5: Let $G = [g_{ij}] \in I\!R^{(n+1)\times(n+1)}$ be defined as in Theorem 3.1. With the algorithms (11) and (12), $\xi_i \to \xi^r + \delta_i$ and $\xi_i - \xi_j \to \delta_i - \delta_j$ if and only if the directed graph of G has a directed spanning tree.

Proof: Define $\tilde{\xi}_i = \xi_i - \delta_i$ and $\tilde{u}_i = u_i - \dot{\delta}_i$. Note that $\dot{\tilde{\xi}}_i = \tilde{u}_i$. Also note that Eqs. (11) and (12) can be rewritten in the same form as Eqs. (7) and (8) with $\tilde{\xi}_i$ and $\tilde{u}_i$ playing the roles of ξ_i and u_i respectively. As a result, from Theorem 3.3, $\tilde{\xi}_i \to \tilde{\xi}_j \to \xi^r$, which implies that $\xi_i \to \xi^r + \delta_i$ and $\xi_i - \xi_j \to \delta_i - \delta_j$. ■

To illustrate, consider two subcases in this subsection using (11), where $g_{35} = g_{45} = 1$, $g_{j5} = 0$, $\forall j \notin \{3,4\}$ (Fig. 1c), and $\delta_i = 1 - i$, $i = 1,\ldots,4$. In Subcase (a), let $\xi^r = \cos(t)$. In Subcase (b), assume that ξ^r satisfies the nonlinear dynamics given by $\dot{\xi}^r = \sin(t)\sin(2\xi^r)$, where $\xi^r(0) = 0.5$. As shown in Fig. 7, $\xi_1 \to \xi^r$, $\xi_2 \to \xi^r - 1$, $\xi_3 \to \xi^r - 2$, and $\xi_4 \to \xi^r - 3$, where ξ^r is the exogenous signal $\cos(t)$ in Subcase (a) and is the solution of the nonlinear model $\dot{\xi}^r = \sin(t)\sin(2\xi^r)$ in Subcase (b).

Note that by appropriately defining $\delta_i(t)$, a desired formation geometry can be preserved between the vehicles using the algorithms (11) and (12).

IV. CONCLUSION AND FUTURE WORK

The consensus problem with a time-varying reference state has been studied. We have analyzed consensus algorithms with a constant reference state using graph theoretical tools. We have also proposed and analyzed algorithms so that consensus is reached on a time-varying reference state when only a portion of the vehicles have access to the reference state and those vehicles might not have directed paths to the other vehicles in the team. The consensus algorithms have also been extended to achieve relative state deviations between the vehicles. An expanded version of the paper is available in [11]. An experimental demonstration of the proposed algorithms on a team of four Amigobots can be found at `http://www.engineering.usu.edu/ece/faculty/wren/research.php`. Note that although we focus on a directed fixed information-exchange topology in this paper, the analysis of the proposed algorithms can be extended to directed switching information-exchange topologies via similar techniques in [5]. This will be a topic of future research.

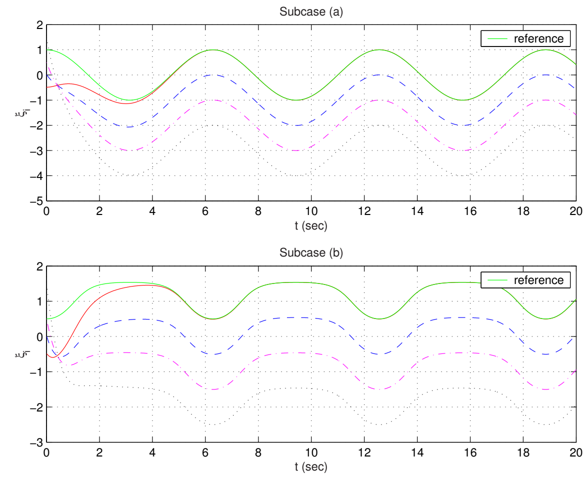

Fig. 7. Consensus seeking with a time-varying reference state using (11).

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