

# Consensus Based Formation Control Strategies for Multi-vehicle Systems

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**Abstract**—In this paper we first introduce a fundamental consensus algorithm for systems modeled by second-order dynamics. We then apply variants of the consensus algorithm to tackle formation control problems by appropriately choosing information states on which consensus is reached. Even in the absence of centralized leadership, the consensus based formation control strategies can guarantee accurate formation maintenance in the general case that information flow is unidirectional. We also show that existing leader-follower, behavioral, and virtual structure / virtual leader formation control approaches can be unified in the general framework of consensus building. A multi-vehicle formation control example is shown in simulation to illustrate our strategies.

## I. INTRODUCTION

Formation control of multi-vehicle systems has been studied extensively in the literature with the hope that through efficient coordination many inexpensive, simple vehicles, can achieve better performance than a single monolithic vehicle.

Typical approaches for formation control can be roughly categorized as leader-follower [1], behavioral [2], [3], virtual leader / virtual structure [4], [5], [6], [7], [8], and graph rigidity based approaches [9], [10], [11], to name a few.

The study of information consensus is aimed at guaranteeing that each vehicle in a team converges to a consistent view of their information states. Consensus algorithms have recently been studied in [12], [13], [14], [15], to name a few. Those algorithms take the form of first-order dynamics. Extensions to second-order dynamics under undirected information flow are discussed in [16], [17]. The basic idea for consensus is that each vehicle updates its information state based on the information states of its local (time-varying) neighbors in such a way that the final information state of each vehicle converges to a common value.

Through appropriately choosing information states on which consensus is reached, consensus algorithms can be applied to tackle formation control problems. We propose consensus based formation control strategies that guarantee accurate formation keeping with only local information exchange. In particular, we show that the leader-follower, behavioral, and virtual structure approaches in the literature can be thought of as special cases of consensus based formation control strategies. In the consensus building framework, the leader-follower approach (e.g. [1]) corresponds to the case that consensus is reached on each sum of a position vector and an inter-vehicle separation vector and the information flow is itself a (directed) spanning tree. The behavioral

approach (e.g. [3]) corresponds to the case that consensus is reached on each deviation vector between the actual vehicle location and the desired vehicle location and the information flow forms a bidirectional ring topology. The decentralized virtual structure approach (e.g. [18]) corresponds to the case that consensus is reached on each instantiation of the virtual structure states and the information flow forms a bidirectional ring topology. As a comparison, in the more general consensus based formation control framework, formation keeping is guaranteed as long as a subset of the (possibly directed) information flow topology forms a (directed) spanning tree. As a result, group robustness can be introduced to the leader-follower approach by allowing information flow from the followers to the leaders. For the behavioral and decentralized virtual structure approach, more flexibility in intervehicle information exchange is introduced in the sense that (i) there is no need to identify two adjacent neighbors, (ii) sensors with limited fields of views can be employed, and (iii) packet loss for some information exchange links may be accounted for.

The main contribution of this paper is to introduce a consensus algorithm for systems modeled by second-order dynamics, which extends the first-order consensus algorithm in the literature, and apply the algorithm to formation control problems by appropriately choosing the information states on which consensus is reached. We show that existing leader-follower, behavioral, and virtual structure formation control approaches can be unified in the general framework of consensus building. Taking into account the case that sensors may have limited fields of views, we consider the general case that information flow is unidirectional. As a result, bidirectional (undirected) information flow is a special case of the unidirectional one.

## II. BACKGROUND AND PRELIMINARIES

It is natural to model information exchange between vehicles by directed/undirected graphs. A digraph (directed graph) consists of a pair  $(\mathcal{N}, \mathcal{E})$ , where  $\mathcal{N}$  is a finite nonempty set of nodes and  $\mathcal{E} \in \mathcal{N}^2$  is a set of ordered pairs of nodes, called edges. As a comparison, the pairs of nodes in an undirected graph are unordered. If there is a directed edge from node  $v_i$  to node  $v_j$ , then  $v_i$  is defined as the parent node and  $v_j$  is defined as the child node. A directed path is a sequence of ordered edges of the form  $(v_{i_1}, v_{i_2}), (v_{i_2}, v_{i_3}), \dots$ , where  $v_{i_j} \in \mathcal{N}$ , in a digraph. An undirected path in an undirected graph is defined accordingly. A digraph is called strongly connected if there is a directed path from every node to every other node. An undirected graph is called connected if there is a path between any

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distinct pair of nodes. A directed tree is a digraph, where every node, except the root, has exactly one parent. A (directed) spanning tree of a digraph is a directed tree formed by graph edges that connect all the nodes of the graph. We say that a graph has (or contains) a (directed) spanning tree if there exists a (directed) spanning tree being a subset of the graph. Note that the condition that a digraph has a (directed) spanning tree is equivalent to the case that there exists at least one node having a directed path to all the other nodes. In the case of undirected graphs, having an undirected spanning tree is equivalent to being connected. However, in the case of directed graphs, having a directed spanning tree is a weaker condition than being strongly connected.

The adjacency matrix  $A = [a_{ij}]$  of a weighted graph is defined as  $a_{ii} = 0$  and  $a_{ij} > 0$  if  $(j, i) \in \mathcal{E}$  where  $i \neq j$ . The Laplacian matrix of the weighted graph is defined as  $L = [l_{ij}]$ , where  $l_{ii} = \sum_{j \neq i} a_{ij}$  and  $l_{ij} = -a_{ij}, \forall i \neq j$ . For an undirected graph, the Laplacian matrix is symmetric positive semi-definite. This property does not hold for a digraph (nonsymmetric) Laplacian matrix.

Let  $\mathbf{1}$  and  $\mathbf{0}$  denote the  $n \times 1$  column vector of all ones and all zeros respectively. Let  $I_n$  denote the  $n \times n$  identity matrix and  $0_{m \times n}$  denote the  $m \times n$  matrix with all zero entries. Let  $M_n(\mathbb{R})$  represent the set of all  $n \times n$  real matrices. Given a matrix  $A = [a_{ij}] \in M_n(\mathbb{R})$ , the digraph of  $A$ , denoted by  $\Gamma(A)$ , is the digraph on  $n$  nodes  $v_i, i \in \{1, 2, \dots, n\}$ , such that there is a directed edge in  $\Gamma(A)$  from  $v_j$  to  $v_i$  if and only if  $a_{ij} \neq 0$  (c.f. [19]).

### III. CONSENSUS BASED FORMATION CONTROL STRATEGIES

#### A. Theoretical Results

To achieve information consensus, we propose the following consensus algorithm:

$$\begin{aligned} \dot{\xi}_i &= \zeta_i \\ \dot{\zeta}_i &= \dot{\zeta}^* - \alpha(\zeta_i - \zeta^*) - \sum_{j=1}^n g_{ij} k_{ij} [(\xi_i - \xi_j) + \gamma(\zeta_i - \zeta_j)], \end{aligned} \quad (1)$$

where  $\xi_i \in \mathbb{R}^q, \zeta_i \in \mathbb{R}^q, \zeta^* \in \mathbb{R}^q, \alpha > 0, k_{ij} > 0, \gamma > 0, g_{ii} \triangleq 0$ , and  $g_{ij} = 1$  if vehicle  $i$  receives information from vehicle  $j$  and 0 otherwise.

The goal of consensus algorithm (1) is to guarantee that  $\xi_i \rightarrow \xi_j$  and  $\zeta_i \rightarrow \zeta_j \rightarrow \zeta^*(t), \forall i, j$ . This algorithm generalizes the second-order consensus algorithm proposed in [20]. In the following, we assume  $q = 1$  for simplicity. However, all the results hereafter remain valid for  $q > 1$ .

Let  $\xi = [\xi_1, \dots, \xi_n]^T$  and  $\zeta = [\zeta_1, \dots, \zeta_n]^T$ . Also let  $\tilde{\xi} = \xi - \mathbf{1} \int_0^t \zeta^*(\tau) d\tau$  and  $\tilde{\zeta} = \zeta - \mathbf{1} \zeta^*$ . Eq. (1) can be written in matrix form as

$$\begin{bmatrix} \dot{\tilde{\xi}} \\ \dot{\tilde{\zeta}} \end{bmatrix} = \underbrace{\begin{bmatrix} 0_{n \times n} & I_n \\ -L & -\alpha I_n - \gamma L \end{bmatrix}}_{\Sigma} \begin{bmatrix} \tilde{\xi} \\ \tilde{\zeta} \end{bmatrix},$$

where  $L = [l_{ij}] \in M_n(\mathbb{R})$  denotes the digraph Laplacian matrix with  $l_{ii} = \sum_{j \neq i} g_{ij} k_{ij}$  and  $l_{ij} = -g_{ij} k_{ij}, \forall i \neq j$ .

We can solve the equation  $\det(\lambda I_{2n} - \Sigma) = 0$  to find the eigenvalues of  $\Sigma$ . Note that

$$\begin{aligned} \det(\lambda I_{2n} - \Sigma) &= \det(\lambda^2 I_n + \gamma \lambda L + \alpha \lambda I_n + L) \\ &= \det((\lambda^2 + \alpha \lambda) I_n + (1 + \gamma \lambda) L). \end{aligned} \quad (2)$$

Also note that

$$\det(\lambda I_n + L) = \prod_{i=1}^n (\lambda - \mu_i), \quad (3)$$

where  $\mu_i$  is the  $i^{\text{th}}$  eigenvalue of  $-L$ .

By comparing Eqs. (2) and (3), we see that the roots of Eq. (2) can be obtained by solving  $\lambda^2 + \alpha \lambda = \mu_i(1 + \gamma \lambda)$ . Therefore, it is straightforward to see that the eigenvalues of  $\Sigma$  are given by

$$\rho_{i\pm} = \frac{\gamma \mu_i - \alpha \pm \sqrt{(\gamma \mu_i - \alpha)^2 + 4 \mu_i}}{2},$$

where  $\rho_{i\pm}$  are called eigenvalues of  $\Sigma$  that are associated with  $\mu_i$ .

Note that  $\Sigma$  has  $m$  zero eigenvalues if and only if  $-L$  has  $m$  zero eigenvalues. Without loss of generality, we let  $\mu_1 = 0$ , which implies that  $\rho_{1+} = 0$  and  $\rho_{1-} = -\alpha$ . We have the following lemma regarding a necessary and sufficient condition for information consensus using consensus algorithm (1).

*Lemma 3.1:* Let  $p$  be a left eigenvector of  $-L$  associated with eigenvalue 0 and  $p^T \mathbf{1} = 1$ . With consensus algorithm (1),  $\xi_i(t) \rightarrow p^T \xi(0) + \frac{1}{\alpha} p^T (\zeta(0) - \mathbf{1} \zeta^*(0)) + \int_0^t \zeta^*(\tau) d\tau$  and  $\zeta_i(t) \rightarrow \zeta_j(t) \rightarrow \zeta^*(t), \forall i$ , asymptotically as  $t \rightarrow \infty$  if and only if matrix  $\Sigma$  has a simple zero eigenvalue and all the other eigenvalues have negative real parts.

*Proof:* (Sufficiency.) Note that  $\Sigma$  can be written in Jordan canonical form as

$$\Sigma = \underbrace{[w_1, \dots, w_{2n}]^T}_P \underbrace{\begin{bmatrix} 0 & 0_{1 \times (2n-1)} \\ 0_{(2n-1) \times 1} & J' \end{bmatrix}}_J \underbrace{\begin{bmatrix} \nu_1^T \\ \vdots \\ \nu_{2n}^T \end{bmatrix}}_{P^{-1}},$$

where  $w_j, j = 1, \dots, 2n$ , can be chosen to be the right eigenvectors or generalized eigenvectors of  $\Sigma$ ,  $\nu_j^T, j = 1, \dots, 2n$ , can be chosen to be the left eigenvectors or generalized eigenvectors of  $\Sigma$ , and  $J'$  is the Jordan upper diagonal block matrix corresponding to the  $2n - 1$  eigenvalues that have negative real parts.

Without loss of generality, we choose  $\nu_1 = [p^T, \frac{1}{\alpha} p^T]^T$  and  $w_1 = [\mathbf{1}^T, \mathbf{0}^T]^T$ . Note that  $e^{J't} \rightarrow 0$  as  $t \rightarrow \infty$ . Then we can show that  $\xi(t) \rightarrow \mathbf{1} p^T \xi(0) + \frac{1}{\alpha} \mathbf{1} p^T \zeta(0)$  and  $\zeta(t) \rightarrow \mathbf{0}$  by straightforward computation, which in turn gives the sufficient part.

(Necessity.) If  $\xi_i(t) \rightarrow p^T \xi(0) + \frac{1}{\alpha} p^T (\zeta(0) - \mathbf{1} \zeta^*(0)) + \int_0^t \zeta^*(\tau) d\tau$  and  $\zeta_i(t) \rightarrow \zeta_j(t) \rightarrow \zeta^*(t)$  asymptotically as  $t \rightarrow \infty$ , we know that  $\tilde{\xi}_i \rightarrow \tilde{\xi}_j$  and  $\tilde{\zeta}_i \rightarrow 0$ , where  $\tilde{\xi}_i$  and  $\tilde{\zeta}_i$  are the  $j^{\text{th}}$  components of  $\tilde{\xi}$  and  $\tilde{\zeta}$  respectively. As a result, we know that  $\lim_{t \rightarrow \infty} e^{\Sigma t} = \lim_{t \rightarrow \infty} P e^{J't} P^{-1}$  has a rank one, which in turn implies that  $\lim_{t \rightarrow \infty} e^{J't}$  has a rank one. However, if the sufficient condition does not hold, then

$\Sigma$  has either more than one zero eigenvalue or exactly one zero eigenvalue but at least one eigenvalue with positive real part. Then we know that  $\lim_{t \rightarrow \infty} e^{Jt}$  has a rank larger than one. This results in a contradiction. ■

We also have the following sufficient condition for information consensus using consensus algorithm (1).

*Theorem 3.1:* Let  $p$  be a left eigenvector of  $-L$  associated with eigenvalue 0 and  $p^T \mathbf{1} = 1$ . Consensus algorithm (1) guarantees that  $\xi_i(t) \rightarrow p^T \xi(0) + \frac{1}{\alpha} p^T (\zeta(0) - \mathbf{1} \zeta^*(0)) + \int_0^t \zeta^*(\tau) d\tau$  and  $\zeta_i(t) \rightarrow \zeta_j(t) \rightarrow \zeta^*(t)$ ,  $\forall i$ , asymptotically if the information exchange topology has a (directed) spanning tree and

$$\gamma > \max_{i=2, \dots, n} \sqrt{\frac{2}{|\mu_i| \cos(\frac{\pi}{2} - \tan^{-1} \frac{-\text{Re}(\mu_i)}{\text{Im}(\mu_i)})}}, \quad (4)$$

where  $\mu_i$ ,  $i = 2, \dots, n$ , are the non-zero eigenvalues of  $-L$ , and  $\text{Re}(\cdot)$  and  $\text{Im}(\cdot)$  represent the real and imaginary parts of a number respectively.

*Proof:* If the information exchange topology has a (directed) spanning tree, we know that  $\mu_1 = 0$  and  $\text{Re}(\mu_i) < 0$ ,  $i = 2, \dots, n$  [15]. If  $\gamma > \max_{i=2, \dots, n} \sqrt{\frac{2}{|\mu_i| \cos(\frac{\pi}{2} - \tan^{-1} \frac{-\text{Re}(\mu_i)}{\text{Im}(\mu_i)})}}$ , it can be verified that  $|(\gamma \mu_i)^2| > |(\gamma \mu_i)^2 + 4\mu_i|$ ,  $i = 2, \dots, n$ , by following the argument of Theorem 6 in [21]. That is, for the triangle composed of vectors  $s_1 = (\gamma \mu_i)^2$ ,  $s_2 = 4\mu_i$ , and  $s_3 = (\gamma \mu_i)^2 + 4\mu_i$ , we know that  $\eta_1 > \eta_3$  using the law of cosines, where  $\eta_i$  is the inner angle of the triangle that faces edge  $s_i$ . Represent  $\gamma \mu_i$  and  $\gamma \mu_i - \alpha$  in polar coordinates as  $(r_1, \theta_1)$  and  $(r_2, \theta_2)$  respectively. As argued in Theorem 6 in [21], we only need to consider  $\mu_i$  located in the second quadrant of the complex plane since any  $\mu_j$  located in the third quadrant is a complex conjugate of some  $\mu_i$  located in the second quadrant. Let  $\theta_i \in (\frac{\pi}{2}, \pi]$ , where  $i = 1, 2$ . Then  $(\gamma \mu_i)^2$  and  $(\gamma \mu_i - \alpha)^2$  can be represented in polar coordinates as  $(r_1^2, 2\theta_1)$  and  $(r_2^2, 2\theta_2)$  respectively. Consider another triangle composed of vectors  $q_1 = (\gamma \mu_i - \alpha)^2$ ,  $q_2 = 4\mu_i$ , and  $q_3 = (\gamma \mu_i - \alpha)^2 + 4\mu_i$  with inner angles given by  $\phi_i$ , where  $\phi_i$  faces edge  $q_i$ . Noting that  $\theta_2 > \theta_1$ , we can show that  $\phi_1 > \eta_1$  and  $\phi_3 < \eta_3$  by comparing the triangles composed of  $q_i$  and  $s_i$  respectively, which implies that  $\phi_1 > \phi_3$ . Using the law of cosines, we show that  $|(\gamma \mu_i - \alpha)^2| > |(\gamma \mu_i - \alpha)^2 + 4\mu_i|$ ,  $i = 2, \dots, n$ . As a result, we know that  $\rho_{i+}$  and  $\rho_{i-}$ ,  $i = 2, \dots, n$ , have negative real parts by following a similar argument to that of Theorem 6 in [21]. In addition, we know that  $\rho_{1+} = 0$  and  $\rho_{1-} = -\alpha$ . Therefore, we see that consensus can be achieved asymptotically from Lemma 3.1. ■

Note that algorithm (1) represents the fundamental form of second-order consensus algorithms. These algorithms can be extended to achieve different convergence results as shown in the next subsection.

## B. Design Methodologies

In this section, we apply consensus algorithm (1) to design formation control strategies. The consensus algorithm and its variants will be used in the context of leader-follower,

behavioral, and virtual structure approaches. We will show that these three approaches can be unified in the general framework of consensus building.

We assume that the dynamics of each vehicle are

$$\dot{r}_i = v_i, \quad \dot{v}_i = u_i, \quad (5)$$

where  $r_i \in \mathbb{R}^m$  and  $v_i \in \mathbb{R}^m$  represent the position and velocity of vehicle  $i$ , and  $u_i \in \mathbb{R}^m$  is the control input.

In the first case, a variant of consensus algorithm (1) is applied to guarantee that  $r_i - r_j \rightarrow \delta_{ij}$  and  $v_i \rightarrow v_j$ , where  $\delta_{ij}$  denotes the desired separation between vehicle  $i$  and vehicle  $j$ . Let  $\delta_i \in \mathbb{R}^m$  be constants. The control input is designed as

$$u_i = \dot{v}^d - \alpha(v_i - v^d) - \sum_{j=1}^n g_{ij} k_{ij} [(r_i - \delta_i) - (r_j - \delta_j) + \gamma(v_i - v_j)]. \quad (6)$$

Note that  $r_i - \delta_i$  and  $v_i$  satisfy consensus algorithm (1) with  $r_i - \delta_i$ ,  $v_i$ , and  $v^d$  playing the role of  $\xi_i$ ,  $\zeta_i$ , and  $\zeta^*$  respectively. Then we know that  $r_i - \delta_i \rightarrow r_j - \delta_j$  and  $v_i \rightarrow v_j \rightarrow v^d(t)$  if the conditions in Theorem 3.1 are satisfied. That is,  $r_i - r_j \rightarrow \delta_i - \delta_j$  and  $v_i \rightarrow v_j \rightarrow v^d(t)$ . Therefore,  $\delta_i$  can be chosen such that desired separations between vehicles are guaranteed. Note that Eq. (6) can be extended to account for the case that  $\delta_i$  is time-varying. In particular, the leader-follower approach corresponds to the case that information only flows from leaders to followers. In a leader-follower scenario as shown in Fig. 1 (a), if the information flow from a leader to a follower breaks, the team fails. However, in the general framework of consensus building, information flow can be introduced from the followers to the leader without affecting formation keeping but improving team robustness as shown in Fig. 1 (b).

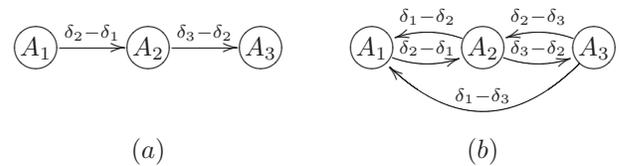


Fig. 1. Formation keeping with desired separations.

In the second case, a variant of consensus algorithm (1) is used to achieve formation maintenance via the behavioral approach (e.g. [2], [3]). Let  $\{r_i^* | i = 1, \dots, n\}$  specify the desired (possibly time-varying) formation shape. Fig. 2 shows a scenario where multiple vehicles reach consensus on  $r_i^* - r_i$ .

Note that if  $r_i^* - r_i$  reaches a common value, then the desired formation shape is guaranteed to be preserved. The

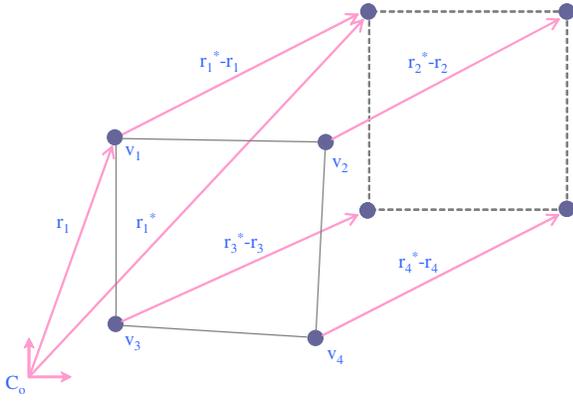


Fig. 2. Consensus reached on deviation vectors.

control input in this case is designed as

$$\begin{aligned}
 u_i = & \ddot{r}_i^* + \dot{v}^d - \alpha(v_i - \dot{r}_i^* - v^d) \\
 & - \sum_{j=1}^n g_{ij} k_{ij} [(r_i - r_i^*) - (r_j - r_j^*)] \\
 & - \sum_{j=1}^n g_{ij} k_{ij} \gamma [(v_i - \dot{r}_i^*) - (v_j - \dot{r}_j^*)], \quad (7)
 \end{aligned}$$

where  $v^d$  specifies the nominal formation velocity, the third term is a damping term, the last two terms are used to guarantee that the desired formation shape between vehicles is preserved (formation keeping behavior). Note that  $r_i - r_i^*$  and  $v_i - \dot{r}_i^*$  satisfy consensus algorithm (1) with  $r_i - r_i^*$ ,  $v_i - \dot{r}_i^*$ , and  $v^d$  playing the role of  $\xi_i$ ,  $\zeta_i$ , and  $\zeta^*$  respectively. Then we know that  $r_i - r_i^* \rightarrow r_j - r_j^*$  and  $v_i - \dot{r}_i^* \rightarrow v_j - \dot{r}_j^* \rightarrow v^d$ .

In particular, if  $r_i^* - r_i \rightarrow 0$ , then each vehicle reaches its desired location. The control input in this case is designed as

$$\begin{aligned}
 u_i = & \ddot{r}_i^* - \beta(r_i - r_i^*) - \gamma\beta(v_i - \dot{r}_i^*) \\
 & - \sum_{j=1}^n g_{ij} k_{ij} [(r_i - r_i^*) - (r_j - r_j^*)] \\
 & - \sum_{j=1}^n g_{ij} k_{ij} \gamma [(v_i - \dot{r}_i^*) - (v_j - \dot{r}_j^*)], \quad (8)
 \end{aligned}$$

where the second and third terms are used to guarantee that each vehicle arrives at its destination (goal seeking behavior).

Letting  $r = [r_1^T, \dots, r_n^T]^T$ ,  $v = [v_1^T, \dots, v_n^T]^T$ ,  $r^* = [r_1^{*T}, \dots, r_n^{*T}]^T$ , and  $v^* = [\dot{r}_1^{*T}, \dots, \dot{r}_n^{*T}]^T$ , we get

$$\begin{bmatrix} \dot{\tilde{r}} \\ \dot{\tilde{v}} \end{bmatrix} = \left( \underbrace{\begin{bmatrix} 0_{n \times n} & I_n \\ -(\beta I_n + L) & -\gamma(\beta I_n + L) \end{bmatrix}}_{\Delta} \otimes I_m \right) \begin{bmatrix} \tilde{r} \\ \tilde{v} \end{bmatrix},$$

where  $\tilde{r} = r^* - r$ ,  $\tilde{v} = v^* - v$ , and  $\otimes$  denotes the Kronecker product.

Note that all eigenvalues of  $-\beta I_n - L$  have negative real parts for any information exchange topology. If Inequality (4)

is true, then all eigenvalues of  $\Delta$  also have negative real parts from Theorem 6 in [21]. Therefore, we see that  $\tilde{r} \rightarrow 0$  and  $\tilde{v} \rightarrow 0$ , i.e.,  $r_i \rightarrow r_i^*$  and  $v_i \rightarrow \dot{r}_i^*$ . That is, even if the information exchange topology does not have a (directed) spanning tree (even the worse case that  $L = 0_{n \times n}$ ), all vehicles will eventually reach their (possibly time-varying) destinations. However, the desired formation shape is not guaranteed to be preserved during the transition. The last two terms in (8) with the graph of  $L$  having a (directed) spanning tree are important to guarantee formation keeping during the transition.

In the third case, a variant of consensus algorithm (1) is used to decentralize the virtual structure approach. In a centralized scheme, the state of the virtual coordinate frame is implemented at a central location (e.g., a ground station or a leader vehicle) and broadcast to every vehicle in the team. While this implementation may be feasible in the case that a robust central location exists and high bandwidth communication is available, issues such as a single point of failure or stringent intervehicle communication constraints will significantly degrade over system performance. One remedy to these limitations is to instantiate a local copy of the state of the virtual coordinate frame on each vehicle. If each vehicle implements the same cooperation algorithm, we expect that the decentralized scheme achieves the same cooperation as the centralized one. However, due to dynamically changing local situation awareness (e.g., disturbances, noise, environmental changes, etc.) for each vehicle, there exist discrepancies among each instantiation of the state of the virtual coordinate frame. In this case, our approach is to apply a consensus algorithm to drive each instantiation of the state of the virtual coordinate frame to converge to a sufficiently common value as well as design local control strategies such that the actual states of each vehicle track its desired ones.

In the context of virtual structures, we continuously update the virtual structure instantiation on each vehicle according to a consensus strategy of the form

$$\begin{aligned}
 \dot{r}_{Fi} &= v_{Fi} \\
 \dot{v}_{Fi} &= \dot{v}_F^d - \alpha(v_{Fi} - v_F^d) + \kappa_i(x_i, x_i^d) \\
 & - \sum_{j=1}^n g_{ij} k_{ij} [(r_{Fi} - r_{Fj}) + \gamma(v_{Fi} - v_{Fj})],
 \end{aligned}$$

where  $[r_{Fi}^T, v_{Fi}^T]^T$  is the  $i^{\text{th}}$  instantiation of the virtual structure state (i.e., formation center, formation velocity),  $v_F^d \in \mathbb{R}^m$  specifies the nominal formation velocity,  $x_i = [r_i^T, v_i^T]^T$  is the local state of vehicle  $i$ ,  $x_i^d = \psi_i(t, x_i, r_{Fi}, v_{Fi})$  is the desired local state of vehicle  $i$ , and  $\kappa_i(\cdot, \cdot)$  denotes the group feedback term introduced from the  $i^{\text{th}}$  vehicle to the  $i^{\text{th}}$  instantiation of the virtual structure state. The introduction of  $\kappa_i$  is to adjust the evolution speed of the formation according to formation performance (e.g. formation accuracy).

The local control of the  $i^{\text{th}}$  vehicle is designed as  $u_i = \chi_i(t, x_i, x_i^d)$  such that  $x_i \rightarrow x_i^d$ . Note that  $u_i$  only depends on the  $i^{\text{th}}$  vehicle's local state information and its virtual structure instantiation.

Without much effort, it can be shown that if  $\|\kappa_i - \kappa_j\|$ ,  $\forall i \neq j$ , is bounded, so are  $\|r_{Fi} - r_{Fj}\|$  and  $\|v_{Fi} - v_{Fj}\|$ . In particular, if  $\|\kappa_i\|$ ,  $\forall i$ , is bounded, so are  $\|r_{Fi} - r_{Fj}\|$  and  $\|v_{Fi} - v_{Fj}\|$ .

As an alternative, a variant of consensus algorithm (1) can be used to derive local control laws for each vehicle directly such that they agree on a common formation center. Let  $r_j$  be the  $j^{\text{th}}$  vehicle's position. Let  $r_{0j}$  be the  $j^{\text{th}}$  vehicle's understanding of the formation center. Also let  $r_{jF}$  be the desired deviation of the  $j^{\text{th}}$  vehicle from its understanding of the formation center. Note that  $r_j = r_{0j} + r_{jF}$ . Fig. 3 shows a scenario where multiple vehicles reach consensus on a (possibly time-varying) formation center. If  $r_{0j}$  reaches a common value, denoted as  $r_0$ , then the desired formation shape is preserved since  $r_j \rightarrow r_0 + r_{jF}$ . In this case, the control input is designed as

$$\begin{aligned} u_i = & \ddot{r}_{iF} + \dot{v}_F^d - \alpha(v_i - \dot{r}_{iF} - v_F^d) \\ & - \sum_{j=1}^n g_{ij} k_{ij} [(r_i - r_{iF}) - (r_j - r_{jF})] \\ & - \sum_{j=1}^n g_{ij} k_{ij} \gamma [(v_i - \dot{r}_{iF}) - (v_j - \dot{r}_{jF})]. \end{aligned}$$

Note that  $r_i - r_{iF}$  and  $v_i - \dot{r}_{iF}$  satisfy consensus algorithm (1) with  $r_i - r_{iF}$ ,  $v_i - \dot{r}_{iF}$ , and  $v_F^d$  playing the role of  $\xi_i$ ,  $\zeta_i$ , and  $\zeta^*$  respectively. Then we know that  $r_i - r_{iF} \rightarrow r_j - r_{jF}$  and  $v_i - \dot{r}_{iF} \rightarrow v_j - \dot{r}_{jF} \rightarrow v_F^d$ . That is,  $r_{0i} \rightarrow r_{0j}$  and  $\dot{r}_{0i} \rightarrow \dot{r}_{0j} \rightarrow v_F^d$ .

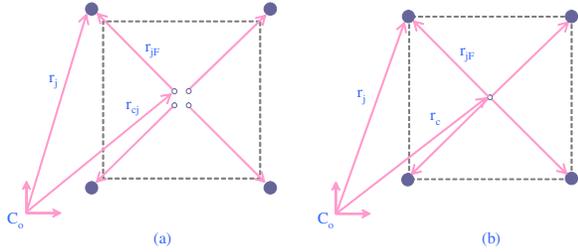


Fig. 3. Consensus reached on a (possibly time-varying) formation center.

#### IV. APPLICATIONS TO MULTI-VEHICLE FORMATION MANEUVERS

In this section, we apply the consensus based formation control strategies to coordinate the movement of multiple vehicles with dynamics given by Eq. (5) with  $m = 2$ . Due to space limitations, we only show applications via the behavioral approach.

We apply control law (7) to direct four vehicles to move from their initial locations to pre-defined directions. During the movements, these four vehicles are required to preserve a square formation. Note that although we require a fixed formation shape here for simplicity (i.e., constant  $r_i^*$ ), a time-varying formation shape can also be achieved by use of time-varying  $r_i^*(t)$  in Eq. (7). The information exchange topologies for the vehicles are given by Fig. 4, where a

directed edge from the  $i^{\text{th}}$  vehicle to the  $j^{\text{th}}$  vehicle means that the  $j^{\text{th}}$  vehicle can receive information from the  $i^{\text{th}}$  vehicle. Note that subplot (a) in Fig. 4 has a (directed) spanning tree while subplot (b) does not have a (directed) spanning tree.

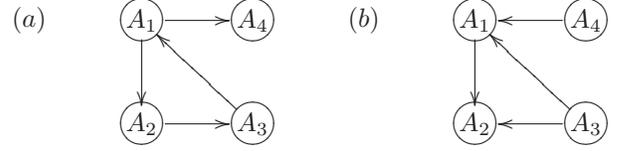


Fig. 4. Information exchange topologies between four vehicles.

We assume that the control effort is saturated and satisfies  $|u_{xi}| \leq 1$  and  $|u_{yi}| \leq 1$ , where  $u_{xi}$  and  $u_{yi}$  are the two components of  $u_i$ . In the following we assume that no collision avoidance occurs between vehicles. Note that a collision avoidance behavior can be added to Eq. (7). We will consider three cases. Table I gives control parameters for each case.

TABLE I  
CONTROL PARAMETERS FOR DIFFERENT CASES.

<b>Case 1:</b>	Control law: Eq. (7), Interaction graph: Fig. 4 (a) $k_{ij} = 5$ , $\alpha = 1$ , $\gamma = 1$ , $v^d = [2, 2]^T$
<b>Case 2:</b>	Control law: Eq. (7), Interaction graph: Fig. 4 (b) $k_{ij} = 5$ , $\alpha = 1$ , $\gamma = 1$ , $v^d = [2, 2]^T$
<b>Case 3:</b>	Control law: Eq. (7), Interaction graph: Fig. 4 (a) $k_{ij} = 5$ , $\alpha = 1$ , $\gamma = 0.05$ , $v^d = [2, 2]^T$

Figs. 5, 6, and 7 show the trajectories of the four vehicles in Cases 1, 2, and 3 respectively, where circles and diamonds represent the actual and desired starting positions of each vehicle respectively ( $t = 0$  sec), squares represent the actual ending positions of each vehicle ( $t = 10$  sec), stars specify the desired formation shape, and triangles represent the actual positions of each vehicle at  $t = \{2.5; 5.0; 7.5\}$  sec. Note that the team preserves the desired square formation and moves with a nominal formation velocity given by  $v^d$  in Case 1. However, the desired square formation is not preserved in Case 2 due to the lack of a (directed) spanning tree in Fig. 4 (b). Similarly, the desired square formation is not preserved in Case 3 due to small  $\gamma$ .

#### V. CONCLUSION AND FUTURE WORK

We have introduced a consensus algorithm for systems modeled by second-order dynamics and applied variants of the algorithm to formation control problems by appropriately choosing the information states on which consensus is reached. We have shown that existing leader-follower, behavioral, and virtual structure formation control approaches can be thought of as special cases of the consensus based strategies. An application to multi-vehicle formation control has been given to demonstrate the effectiveness of our strategies.

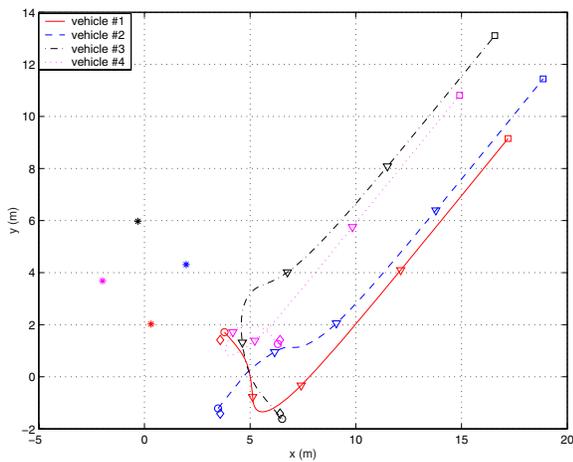


Fig. 5. Trajectories of the four vehicles in Case 1.

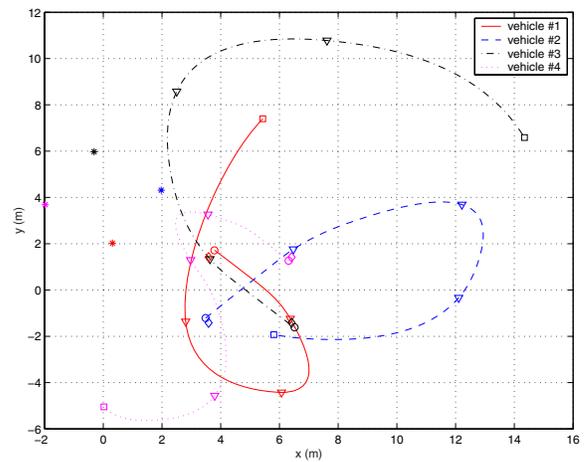


Fig. 7. Trajectories of the four vehicles in Case 3.

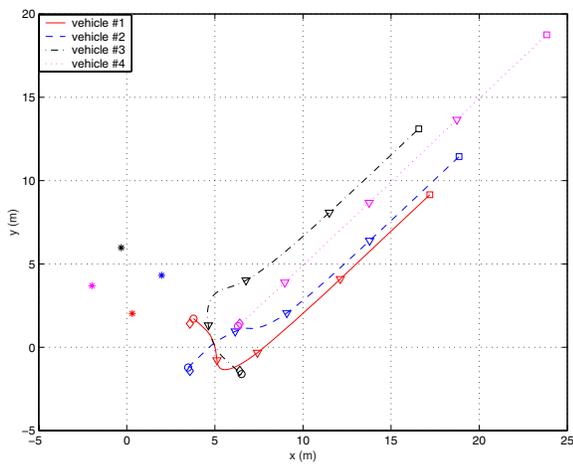


Fig. 6. Trajectories of the four vehicles in Case 2.

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