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A Study of Grouping Effect On Mobile Actuator Sensor Networks for Distributed Feedback Control of Diffusion Process Using Central Voronoi Tessellations

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***Abstract*— When a team of networked mobile actuators (sprayers) are used to control the diffusion process in a region of interest with the help of the static mesh sensor networks, the question on how to group the mobile actuators into smaller subgroups is investigated in this paper to check the performance change under various grouping strategies. Our actuator path planning is based on the so-called Central Voronoi Tessellations (CVT) technique. Via extensive simulation studies, we found that, under the same total actuation resources, it is *not* definite to tell if the larger number of subgroups corresponds to a better performance.**

***Index Terms*— Robot grouping, mobile actuator networks, coordinated control, diffusion process, pollution neutralization, Centroidal Voronoi Tessellation.**

I. INTRODUCTION

The deployment of large groups of unmanned vehicles is rapidly becoming possible because of the advances in wireless networking and in miniaturization of electromechatronic systems. In the future, large number of robots will coordinate and perform challenging tasks including operation in dangerous environments, human health monitoring, and habitat monitoring for pollution detection which motivates "Mobile Actuator and Sensor Network (MAS-net) project" in CSOIS, Utah State University [2]. This project combines mobile robotics with the wireless sensor networks to control and monitor the spatially distributed diffusion process [4]. Each robot has limited sensing, computation and communication ability. But they can coordinate with each other to finish tasks like temporal-spatial feedback closed-loop control of a diffusing process. The application of this project can be in homeland security, where chemical, biological, radiological or nuclear (CBRN) terrorism can cause devastating damages. Some research challenges and opportunities are presented in [6].

In this paper, we consider the distributed control of a time-varying pollution diffusion process using groups of mobile actuators with an emphasis on how different grouping methods affect the control performance. The scenario is described as follows: A toxic diffusion source is releasing toxic gas/fog in a 2D domain. The diffusion process is modeled as a partial differential equation (PDE) system and we assume static mesh sensor networks are deployed in the polluted area to measure chemical concentration. Then, a few mobile robots equipped with controllable dispensers of neutralizing chemicals are sent out to counteract the pollution by properly releasing the neutralizing chemicals.

The technical approach proposed in this paper is related to a number of technological areas including coverage control [3], robot motion planning control [9] and the dynamic diffusion process control [8]. In [7], a gradient based algorithm is developed for chemical tracing with swarms of mobile robots and the diffusion process is assumed to be focused and smoke-like with wind blowing. A mobile robot equipped with sensing devices is used to estimate the parameters of gas releasing process in [8]. The techniques mentioned above could give us some ideas on how to model a diffusion process and find the source of pollution. However, the diffusion processes are not fully time-varying and no further solutions on how to counteract the pollution were investigated before. The motion planning of groups of actuators in a time-varying PDE system for feedback control still largely remains an open research question.

Motivated by the application of Centroidal Voronoi Tessellation (CVT) in coverage control of mobile sensing networks [3], we use a CVT-based algorithm to solve this problem. An application of CVT in feedback control system can be found in [14]. In [14], the sensor location problem in feedback control of partial differential equation system is solved by CVT. The functional gains are served as the density functions in CVT. We also need to point out that the CVT-based robot motion control is a distributed and scalable control algorithm. In our experiment, the pollution concentration is given by the sensors that cover the area and form a mesh. A simulation platform called `Diff-MAS2D` [18] has been developed for measurement scheduling and controls in distributed parameter systems with moving sensors and actuators. Simulation result shows the effectiveness of our algorithm for different group sizes.

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The remaining part of this paper is organized as follows. In Sec. II, the problem formulation is presented. In Sec. III, we give a brief introduction to Voronoi diagram and Centroidal Voronoi Tessellation. Section IV is devoted to introducing CVT-based optimal actuator location and path planning algorithms. In Sec. V, we analyze the tradeoff between group size and the efficiency in actuator control. The simulation results and comparisons are presented in Sec. VI. Finally, conclusions and future research directions are presented in Sec. VII.

II. Problem Formulation

In this section, the problem on how to control a diffusion process like pollution neutralization is introduced.

Suppose that Ω is a convex polytope in $\mathcal{R}^2$. $\rho(x,y) : \Omega \rightarrow \mathcal{R}_+$ is the concentration function that represents the pollutant concentration over Ω. To control a diffusion process, we assume using n actuators(robots). Let $P = (p_1, \cdots, p_n)$ be the location of n actuators and let $|\cdot|$ denote the Euclidean distance function. Every robot at p_i will move, sense the environment and release the neutralization chemical according to our control law. The objectives are to control the diffusion of the pollution to a confined area and to minimize the total polluted area as fast as we can .

To minimize the heavily affected area, actuators(sprays) should be sent to high polluted areas so that the pollution can be neutralized timely with no further diffusion. But they can be far from the lightly polluted areas and the diffused pollutants far away from the source need also to be neutralized timely. So putting all robots very close to the pollution source is not a good strategy. Considering both heavily and slightly polluted areas, we use the following cost function for minimization

$$\mathcal{K}(P, \mathcal{V}) = \sum_{i=1}^{n} \int_{V_i} \rho(q)|q - p_i|^2 \mathrm{d}q \text{ for } q \in \Omega. \quad (1)$$

n robots will partition Ω into a collection of n polytopes $\mathcal{V} = \{V_1, \cdots, V_n\}$, $p_i \in V_i$, $V_i \cap V_j = \emptyset$ for $i \neq j$ and $\cup_{i=1}^{n} \bar{V}_i = \bar{\Omega}$ ($\bar{V}_i = V_i \cup \partial V_i$ and $\bar{\Omega} = \Omega \cup \partial\Omega$). It is obvious that to minimize $\mathcal{K}$, the distance $|q - p_i|$ should be small in highly polluted areas. It is the concentration function $\rho(q)$ that determines the optimal positions of the robots. A necessary condition for $\mathcal{K}$ to be minimized is that $\{p_i, V_i\}_{i=1}^{k}$ is a Centroidal Voronoi Tessellation of Ω . Our algorithm is based on a discrete version of (1) and the concentration information comes from the measurements of the static mesh sensors.

III. Introduction to Voronoi Diagram and Centroidal Voronoi Tessellation

Here we give a brief introduction to the Voronoi diagram and Centroidal Voronoi Tessellation [19].The Voronoi diagram is to partition a plane with n points into convex polygons such that each polygon contains exactly one generating point and every point in a given polygon is closer to its generating point than to any other.

Given an open set $\Omega \subset \mathcal{R}^N$ and a set of points $\{z_i\}_{i=1}^{k}$ belonging to $\bar{\Omega}$, let $|\cdot|$ denote the Euclidean norm in $\mathcal{R}^N$ and let

$$V_i = \{q \in \Omega | |q - z_i| < |q - z_j| \text{ for } j = 1, \cdots, k, j \neq i\} \quad (2)$$
$$i = 1, \cdots, k.$$

The set $\{V_i\}_{i=1}^{k}$ is referred to as a Voronoi tessellation or Voronoi diagram of Ω and each V_i is referred to as the Voronoi region or Voronoi cell. The members of the set $\{z_i\}_{i=1}^{k}$ are referred to as generators of each cell V_i.

A centroidal Voronoi tessellation (CVT) is a Voronoi tessellation of a given set such that the associated generating points are centroids (centers of mass with respect to a given density function) of the corresponding Voronoi regions. It is defined like this: given a density function $\rho(q) \geq 0$ defined on $\bar{\Omega}$, we define the mass centroid z_i^* of V_i for each Voronoi cell V_i by:

$$z_i^* = \frac{\int_{V_i} r\rho(q)\mathrm{d}q}{\int_{V_i} \rho(q)\mathrm{d}q} \text{ for } i = 1, \cdots, k. \quad (3)$$

We call the tessellation defined by (2) a Centroidal Voronoi Tessellation if and only if

$$z_i = z_i^* \text{ for } i = 1, \cdots, k.$$

So, the points z_i are not only the generators for the Voronoi regions V_i but also the mass centroids of those regions.

Centroidal Voronoi Tessellation has broad applications in many fields [5]. It is the solution to optimal placement of resources, but in general, CVT can only be approximately constructed.

IV. CVT-based Optimal Actuator Motion Planning Algorithm

Although the CVT is used to solve the static resource location problem, if the diffusion process evolves slower than the convergence rate of the motion planning algorithm and the control efforts, CVT is still a valid solution to our problem, as verified in our simulation results presented in Sec. VI.

We use a modified Lloyd's method for actuator motion planning to get a CVT diagram. Lloyd's method is an iterative algorithm to generate a centroidal Voronoi diagram from any set of generating points. It is described as below:

Given a region Ω, a density function $\rho(x,y)$ defined for all $x \in \bar{\Omega}$, and a positive integer k

1) Select an initial set of k points $\{z_i\}_{i=1}^{k}$ (Actuator Starting Positions)as the generators.
2) Construct the Voronoi sets $\{V_i\}_{i=1}^{k}$ associated with generators $\{z_i\}_{i=1}^{k}$;
3) Determine the mass centroids of the Voronoi sets $\{V_i\}_{i=1}^{k}$; these centroids form the new set of points $\{z_i\}_{i=1}^{k}$;

4) Give the actuators (sprayers) command to move to the mass centroid points
5) If the new points meet some convergence criterion, terminate; other wise, return to step 2.

The Lloyd's method is iterative so that the motions of the robots can be adaptive to the evolving of the diffusion process.

We use the second-order dynamical equation to model the mobile actuator robots:

$$\ddot{p}_i = F_i = f_i - k_v \dot{p}_i \tag{4}$$

with F_i the control input and f_i a force input to control the robot motion determined by the following control law:

$$f_i = -k(p_i - \bar{p_i})$$

where $\bar{p_i}$ is the computed mass centroid of the current Voronoi cell.

The second term of (4) on the right hand side is the viscous friction artificially introduced [16]. k_v is the friction coefficient and $\dot{p}_i$ denotes the velocity of the robot i. This term is used to eliminate the oscillatory behavior of robots described in [11] when the robot is close to its destination. The viscous term assures that in the absence of the external force, the robot will come to a standstill state eventually.

We can also use proportional control for the neutralizing chemical releasing. The amount of chemicals each robot releases is proportional to the average pollutant concentration in the Voronoi cell belonging to that robot. Although our simulation is model-based, our control algorithms for each robot are not relying on the exact model information. They are based only on the sensor information that the robots can access.

V. Grouping Effect on Diffusion Control Performance

In this section, we discuss in detail if CVT-based algorithm could be used to large numbers of actuator groups and how to decide the appropriate grouping size according to the final performance requirements. In [1], we have shown that the CVT algorithm works well for 4 mobile actuators. It is obvious that we can achieve better control result for the pollution neutralization by using more mobile actuators. But a big group size also have tradeoffs like much more computation and communication requirements which will lower the efficiency and robustness of control system.

With the increasing of actuator numbers, we need to use the computational complexity theory to test if the CVT will bring big computation burden. There are many practical methods for constructing Voronoi Diagrams including the naive method [19], the flip method and the incremental method. Specifically, we chose the Delaunay triangulation method based on Qhull. According to [13], the computational complexity is

$$f_r = O\left(\frac{r^{d/2}}{(d/2)!}\right),$$

where d is the dimension; n the number of input points, r the number of processed points, and f_r the maximum number of facets of r vertices. For our problem, $d = 2$.

$$f_r = O\left(r\right),$$

.

For simulation purpose, we use `delaunay()` and `voronoi()` functions in MATLAB to get the CVT diagrams. We found that there is no big burden on computation.

Next, we will show how the CVT algorithm can be implemented in a distributed way. That is, the algorithm can be executed on a group of robots instead of a centralized one. In fact, we need only get p_i and $\overline{p_i} = C_{V_i}$ for every time step. To get a distributed implementation, each actuator needs to know the relative location of each Voronoi neighbor for computing its own Voronoi cell. We can use the method in [15] to get the Voronoi diagrams.

Given the above discussion on computational cost, it is feasible to consider more actuators for pollution neutralizing problem. However, we are interested in how many actuators in a subgroup or what the best grouping size is. In real circumstances, for a fair comparison, we must use the same amount of neutralizing chemicals for various numbers of groups. There should be an optimal group size given a specific performance metric.

VI. Simulation Results

`Diff-MAS2D` is used as the simulation platform for our implementation. The area concerned is given by $\Omega = \{(x, y)|0 \le x \le 1, 0 \le y \le 1\}$. The system with control input is modeled as

$$\frac{\partial\rho(x,y,t)}{\partial t} = k(\frac{\partial^2\rho(x,y,t)}{\partial x^2} + \frac{\partial^2\rho(x,y,t)}{\partial y^2}) + f_c(x,y,t) + f_d(x,y,t), \tag{5}$$

where $k = 0.01$ and the Neumann boundary condition is given by

$$\frac{\partial u}{\partial n} = 0.$$

where n is the outward direction normal to the boundary.

The stationary pollution source is modeled as a point disturbance f_d to the the PDE system (5) with its position at $(0.75, 0.35)$ and

$$f_d(t) = 20e^{-t}|_{(x=0.75,y=0.35)}.$$

In our simulation, we assume that once deployed, the mesh sensors remain static. There are 29×29 sensors evenly distributed in a square area $(0, 1)^2$ and they form a mesh over the area. There are 4 mobile robots that can release the neutralizing chemicals. For the robot motion control, the viscous coefficient is given by $k_v = 1$ and the control input is given by

$$F_i = -3(p_i - \bar{p_i}) - \dot{p}_i.$$

The pollution source begins to diffuse at $t = 0$ to the area Ω and initially the mobile actuator robots are evenly distributed within the domain Ω (one by one square) at the following specific positions: $(0.5, 0.5)$ for $1*1$ grouping case; for $2*2$ grouping case, $(0.33, 0.33), (0.33, 0.66), (0.66, 0.33), (0.66, 0.66)$, respectively, and so on and so forth. Figure 1 shows the initial positions of the robot groups (2*2 grouping), the positions of the sensors and the position of the pollution source as a reference. Figure 2 shows the typical trajectories of the 2*2 actuator group.

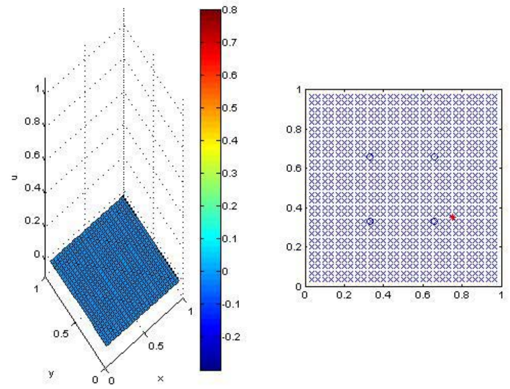


Fig. 1. Initial layout of actuators and sensors (2*2 grouping).

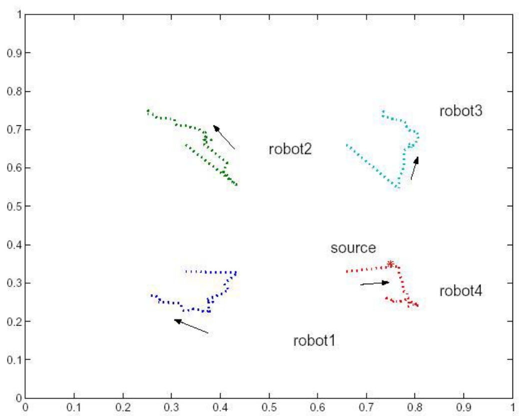


Fig. 2. 2*2 robot group trajectories for controlling the diffusion process.

We choose the simulation time to $t = 5$ sec. and the time step as $\Delta t = 0.002$ sec. The robot recomputes its desired position every 0.2s. To show how the robots can control the diffusion of the pollutants, the robots begin to react at $t = 0.4$s. The system evolves under the effects of diffusion of pollutants and diffusion of neutralizing chemicals released by robots. To show the scalability of the CVT algorithm for bigger groups, the control results are shown by using 5*5 and 9*9 mobile actuators respectively. Table I shows the time to do simulations on the PC (P4-2.6G, 256M RAM) and the remaining pollutants at the end of the simulation. It can be seen that the computational load does not increase much with the increase of the number of actuator groups. In Fig. 3, the y axis is the sum of the mesh sensor measurements.

TABLE I

COMPUTATIONAL TIME FOR SIMULATION AND CONTROL RESULTS

Grouping	Time for simulation	Remaining pollutants
2*2	510 s	3.3994
3*3	582 s	0.7046
4*4	615 s	0.3372

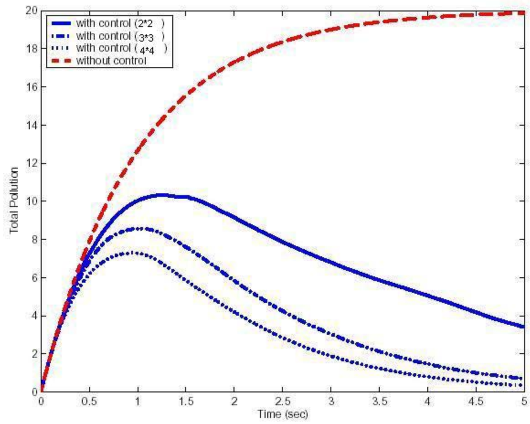


Fig. 3. Evolution of the amount of pollutants (2*2, 3*3 and 4*4 robots).

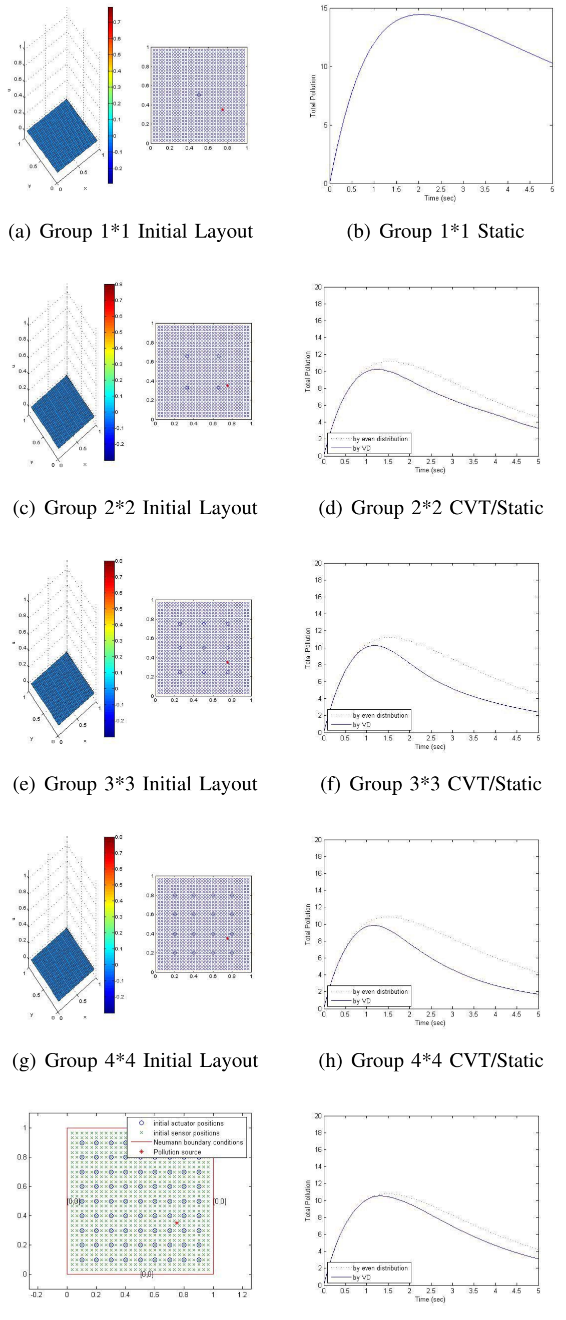
To compare the performance of different groupings of actuators, we compare our CVT-based algorithm with the uniformly distributed case. The control laws for chemical releasing are the same. But Table II shows the different output parameter for neutralizing chemical releasing so that each actuator group has the similar total control input. In Fig. 4, the y axis represents the total pollution all the mesh sensors could detect. Detailed results including $Pollution_{max}$, T_{max} (when the pollution has a peak), and $Pollution_{final}$ are shown in Table III. The case with exactly one static actuator and one pollution source is provided as a baseline for comparison.

TABLE II

RUN TIME FOR SIMULATION AND CONTROL RESULTS

Grouping	Actuator Numbers	Neutralizing parameter
1*1 Actuator	1	320
2*2 Actuator	4	81
3*3 Actuator	9	36
4*4 Actuator	16	20.25
9*9 Actuator	81	4

From the above results, we can find out our CVT algorithm is distributed, scalable and with high performance.



(a) Group 1*1 Initial Layout (b) Group 1*1 Static

(c) Group 2*2 Initial Layout (d) Group 2*2 CVT/Static

(e) Group 3*3 Initial Layout (f) Group 3*3 CVT/Static

(g) Group 4*4 Initial Layout (h) Group 4*4 CVT/Static

(i) Group 9*9 Initial Layout (j) Group 9*9 CVT/Static

Fig. 4. Grouping Performance for mobile CVT/Static algorithm

All the mobile CVT methods achieve better results than those of the static evenly distributed ones. However, the optimal grouping size for diffusion control is not always corresponding to the largest size. In other words, under the same total actuation resources, it is *not* definite to tell if the larger number of subgroups corresponds to a better performance. A mathematical model is needed for quantitative analysis of the effect of the grouping size on the efficiency in diffusion control.

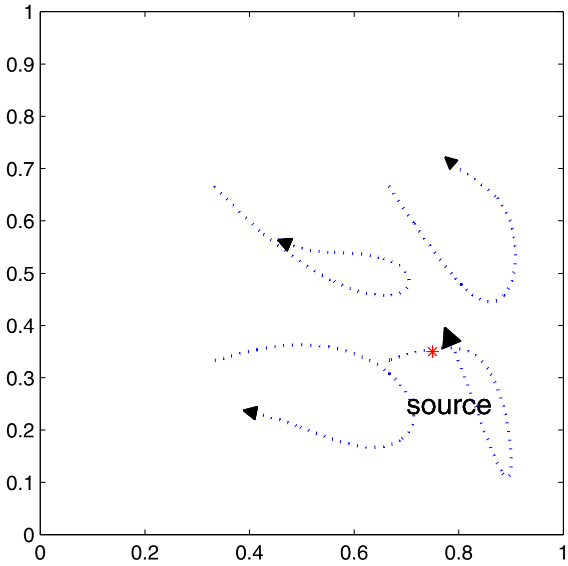


Fig. 5. Trajectories of 2*2 robots using CVT algorithm.

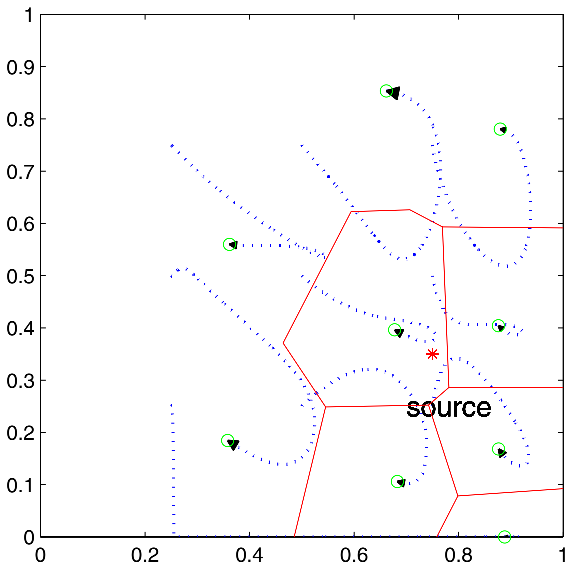


Fig. 6. Trajectory of 3*3 robots using CVT algorithm and the final Voronoi diagram.

VII. Conclusion

In this paper, we extended the application of Centroidal Voronoi Tessellation to the case of large number of mobile actuators for diffusion control. Computational complexity and distributed algorithm are discussed for scalability testing. Through our extensive simulation studies, we demonstrated the effect of the grouping size on the efficiency in diffusion control. Unfortunately, under the same total actuation resources, it is *not* definite to tell if the larger number of subgroups corresponds to a better performance.

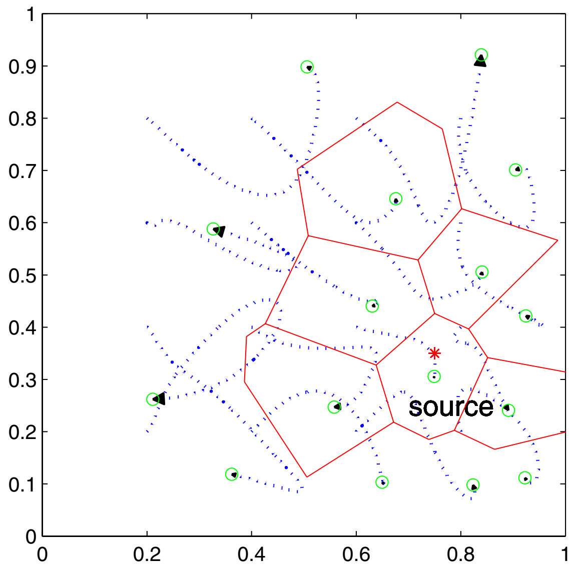


Fig. 7. Trajectory of 4*4 robots using CVT algorithm and the final Voronoi diagram.

TABLE III
COMPARISON OF PERFORMANCE FOR DIFFERENT GROUP SIZE

Grouping	P_{max}	t_{max}	P_{final}	P_{integ}
1*1 Static	100%	100%	100%	100%
2*2 (Static)	77.4%	74.9%	44.4%	68.5%
2*2 (Mobile)	71.0%	59.8%	32.0%	57.4%
3*3 (Static)	77.8%	74.3%	44.3%	68.5%
3*3 (Mobile)	71.1%	58.4%	23.5%	50.1%
4*4 (Static)	75.3%	72.3%	41.6%	65.9%
4*4 (Mobile)	68.2%	57.6%	16.7%	46.3%
9*9 (Static)	74.8%	73.1%	41.2%	65.5%
9*9 (Mobile)	72.3%	63.7%	30.2%	58.7%

In the future, we will investigate how to perform quantitative analysis of the effect of the grouping size on the efficiency in diffusion control. Furthermore, we will extend our research for pollution feedback control by using mobile sensors and take into account the sensor noise and unreliable communication induced uncertainties.

REFERENCES

[1] YangQuan Chen, Zhongmin Wang and Jinsong Liang, "Optimal Dynamic Actuator Location in Distributed Feedback Control of A Diffusion Process," in *Proceedings of the 44th IEEE Conference on Decision and Control, and European Control Conference 2005*, Seville, Spain, December 12-15 2005.

[2] Zhongmin Wang, Zhen Song, Peng-Yu Chen, Anisha Arora, Dan Stormout, and YangQuan Chen, "MASmote - a mobility node for MAS-net (mobile actuator sensor networks)," in *Proceedings of 2004 IEEE International Conference on Robotics and Biomimetics*, Shengyang, China, August 22-25 2004, Robio04.

[3] Jorge Cortés, Sonia Matínez, Timur Karatas, and Francesco Bullo, "Coverage Control for Mobile Sensing Networks," *IEEE Transactions on Robotics and Automation*, vol. 20, no. 20, pp. 243–255, April 2004.

[4] YangQuan Chen, Kevin L. Moore, and Zhen Song, "Diffusion boundary determination and zone control via mobile actuator-sensor networks (MAS-net): Challenges and opportunities," in *Intelligent Computing: Theory and Applications II, part of SPIE's Defense and Security*, Orlando, FL, Apr. 2004.

[5] Qiang Du, Vance Faber, and Max Gunzburger, "Centroidal Voronoi Tessellations: Applications and Algorithms," *SIAM REVIEW*, vol. 41, no. 4, pp. 637–676, 1999.

[6] K. Moore and Y. Chen, "Model-based approach to characterization of diffusion processes via distributed control of actuated sensor networks," in *IFAC Symposium of Telematics Applications in Automation and Robotics, Helsinki University of Technology Espoo, Finland*, 2004.

[7] Dimitri Zarzhitsky, Diana F. Spears and william M. Spears, "Swarms for chemical plume tracing," in *Swarm Intelligence Symposium, 2005. SIS 2005. Proceedings 2005 IEEE*,June 2005.

[8] Christopoulos, V.N., Roumeliotis, S., "Adaptive Sensing for Instantaneous Gas Release Parameter Estimation," in *Robotics and Automation, 2005. Proceedings of the 2005 IEEE International Conference on* .

[9] Butler, Z., Rus, D., "Controlling mobile sensors for monitoring events with coverage constraints," in *2004 IEEE International Conference on Robotics and Automation*,Apr.26-May 1, 2004, pp. 1568–1573.

[10] Ioannis Rekleitis, Vincent Lee-Shue, Ai Peng New, and Howie Choset, "Limited communication, multi-robot team based coverage," in *Proceedings of the 2004 IEEE International Conference on Robotics and Automation*, New Orleans, LA, April 2004, pp. 3462–3468.

[11] Nojeong Heo and P. K. Varshney, "Energy-efficient deployment of intelligent mobile sensor networks," *IEEE Transactions on Man and CyberneticsSystems*, vol. Part A, pp. 78 – 92, January 2005.

[12] Jeanne A. Atwell and Belinda B. King, "Reduced order controllers for spatially distributed systems via proper orthogonal decomposition," *SIAM Journal on Scientific Computing*, vol. 26, no. 1, pp. 128–151, 2004.

[13] C. Bradford Barber, David P. Dobkin, and Hannu Huhdanpaa, "The Quickhull Algorithm for Convex Hulls," *ACM Transaction on Math. Software*, vol. 22, no. 4, 469-483, December 1996.

[14] A. L. Faulds and B. B. King, "Sensor location for feedback control of partial differential equation systems," in *Proceedings of the 2000 IEEE CCA/CACSD*, Anchorage, AK, September 2000, pp. 536 – 541.

[15] M. Cao and C. Hadjicostis, "Distributed algorithms for Voronoi diagrams and application in ad-hoc networks," in *Preprint* Oct. 2002.

[16] Andrew Howard, Maja J Mataric, and Gaurav S Sukhatme, "Mobile sensor network deployment using potential fields: A distributed, scalable solution to the area coverage problem," in *Proceedings of the 6th International Symposium on Distributed Autonomous Robotics Systems*, Fukuoka, Japan, June 2002.

[17] Timothy H. Chung, Vijay Gupta, Joel W. Burdick, and Richard M. Murray, "On a decentralized active sensing strategy using mobile sensor platforms in a network," in *Proceedings of 43th IEEE Conference on Decision and Control*, December 2004, pp. 1914–1919.

[18] Jingsong Liang and YangQuan Chen, "Diff/Wave-MAS2D: A simulation platform for measurement and actuation scheduling in distributed parameter systems with mobile actuators and sensors," in *Proceedings of the IEEE International Conference on Mechatronics and Automation*, Niagara Falls, Canada,July 2005, pp. 2228–2233.

[19] Atsuyuki Okabe, Barry Boots, Kokichi Sugihara and Sung Nok Chiu, "Spatial Tessellations: Concepts and Applications of Voronoi Diagrams," *John Wiley & Sons, LTD.*, chapter 4, pp. 229–287, 2000.

[20] "MATLAB," `http://www.mathworks.com/`.

[21] "Voronoi," `http://www.msi.umn.edu/~schaudt/voronoi/voronoi.html`.