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SYSTEMS & CONTROL LETTERS
www.elsevier.com/locate/sysconle

Multi-vehicle consensus with a time-varying reference state☆

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Received 21 May 2006; received in revised form 23 January 2007; accepted 27 January 2007
Available online 7 March 2007

Abstract

In this paper, we study the consensus problem in multi-vehicle systems, where the information states of all vehicles approach a time-varying reference state under the condition that only a portion of the vehicles (e.g., the unique team leader) have access to the reference state and the portion of the vehicles might not have a directed path to all of the other vehicles in the team. We first analyze a consensus algorithm with a constant reference state using graph theoretical tools. We then propose consensus algorithms with a time-varying reference state and show necessary and sufficient conditions under which consensus is reached on the time-varying reference state. The time-varying reference state can be an exogenous signal or evolve according to a nonlinear model. These consensus algorithms are also extended to achieve relative state deviations among the vehicles. An application example to multi-vehicle formation control is given as a proof of concept.

Keywords: Consensus seeking; Cooperative control; Multi-vehicle systems; Coordination

1. Introduction

Future autonomous vehicles will have the capability to significantly improve the operational effectiveness of both civilian and military applications. While autonomous vehicles performing solo missions can yield some benefits, greater benefits will come from having teams of autonomous vehicles operating in a coordinated fashion.

Requiring only local neighbor-to-neighbor information exchange among the vehicles, information consensus has recently received significant attention in the area of cooperative control of multi-vehicle systems (see e.g., [2,4–6,8,11,14,16,19,26]). The basic idea for information consensus is that each vehicle updates its information state based on the information states of its local (time-varying) neighbors in such a way that the final information states of all vehicles converge to a common value. This basic idea can be extended to deal with the case that all vehicles' information states converge to desired relative deviations or to incorporate different group behaviors into the consensus building process. Information consensus has applications in multi-vehicle rendezvous [10,12], formation control [3,5], flocking [15,24], attitude alignment [18], decentralized task assignment [1], sensor fusion [17,22,27], etc.

For most consensus algorithms studied in the literature, the final consensus value to be reached is inherently constant, which might not be appropriate when each vehicle's information state evolves over time, as occurs in formation control problems, where the formation evolves in two or three-dimensional space. In addition, most consensus algorithms guarantee that the information states converge to a common value but do not allow specification of a particular value. As a result, it is relevant to study consensus algorithms where the final consensus value is specified by a *reference state*, which may evolve as a function of vehicle/environmental dynamics. In practice, it is also possible that only a portion of the vehicles in the team (e.g., the unique team leader) have access to the reference state and those vehicles might not have a directed path to all of the other vehicles in the team. For example, in some formation control applications, the reference velocity for the whole team might only be available to a single or multiple team leaders.

The main objective of this paper is to propose and analyze consensus algorithms so that each vehicle in the team reaches consensus on a time-varying reference state that evolves over

☆ The work was supported in part by the Utah State University New Faculty Research Grant (2006–2007).
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doi:10.1016/j.sysconle.2007.01.002

time when only a portion of the vehicles have access to the time-varying reference state. Related to the current paper are [9,13,23]. In [23], necessary and sufficient conditions are derived so that a group of systems can be controlled by a team leader. In [13], a so-called forced consensus problem is considered, where it is assumed that only one vehicle that has a directed path to all of the other vehicles is driven by a constant setpoint. In [9], a similar problem is considered, where a constant reference state is available to one or more vehicles in the team.

In contrast, this paper considers the general case that the reference state is a time-varying exogenous signal or evolves according to a nonlinear model. It is also assumed that only a portion of the vehicles in the team have access to the time-varying reference state, and those vehicles might not have a directed path to all of the other vehicles in the team. It is assumed that the vehicles in the team have a globally defined reference frame (e.g., obtained from GPS). All of the analysis in this paper is based on the general case of directed information exchange. For example, some vehicles may have transceivers, while other less capable members only have receivers in heterogeneous teams. Also, in the case of information exchange through local sensing, vehicles may be equipped with sensors that only have a limited field of view (e.g., stereo vision system), which may result in unidirectional information flow. We first analyze a consensus algorithm with a constant reference state using graph theoretical tools and show that the existing algorithm for a constant reference state cannot guarantee consensus on a time-varying reference state. We then propose algorithms to deal with the time-varying case and show necessary and sufficient conditions under which consensus is reached on a time-varying reference state. Unlike the leader–follower topology, where information only flows from leaders to followers (e.g., [25]), the proposed algorithms allow information to flow from every vehicle to every other vehicle to introduce feedback or coupling among the vehicles and therefore increase redundancy and robustness for the whole team. In the case of a time-varying reference state, complexity issues result from the information feedback loops under the leader–follower framework. However, with the proposed algorithms, the information feedback loops do not adversely affect the stability of the whole team. Those algorithms for a time-varying reference state are also extended to achieve relative state deviations among the vehicles. It is worthwhile to mention that the extension of consensus algorithms from a constant reference to a time-varying reference is nontrivial. It is not straightforward how the internal model principle of control can be directly applied to consensus seeking with a time-varying reference state for multiple vehicle systems involving only local information exchange.

2. Background and preliminaries

A directed graph $\mathcal{G}$ consists of a node set $\mathcal{V}$ and an edge set $\mathcal{E} \in \mathcal{V} \times \mathcal{V}$ (see e.g. [21]). An edge (i, j) in a directed graph denotes that vehicle j can obtain information from vehicle i, but not necessarily vice versa. The pairs of nodes in an undirected graph are unordered, where the edge (i, j) denotes that vehicles i and j can obtain information from each other. Note that an undirected graph can be viewed as a special case of a directed graph, where an edge (i, j) in the undirected graph corresponds to edges (i, j) and (j, i) in the directed graph. For an edge (i, j) in a directed graph, i is the parent node and j is the child node. A directed path is a sequence of edges in a directed graph of the form $(i_1, i_2), (i_2, i_3), \ldots,$ where $i_j \in \mathcal{V}$. In a directed graph, a cycle is a directed path that starts and ends at the same node. A directed tree is a directed graph, where every node has exactly one parent except for one node, called the root, which has no parent, and the root has a directed path to every other node. A directed spanning tree of $\mathcal{G}$ is a directed tree that contains all nodes of $\mathcal{G}$. A directed graph has or contains a directed spanning tree if there exists a directed spanning tree as a subset of the directed graph, that is, there exists at least one node having a directed path to all of the other nodes.

Suppose that there are p nodes in the graph. The adjacency matrix $A=[a_{ij}] \in \mathbb{R}^{p\times p}$ of a weighted directed graph is defined as $a_{ii}=0$ and $a_{ij}>0$ if $(j, i) \in \mathcal{E}$, where $i \neq j$. The adjacency matrix of a weighted undirected graph is defined analogously except that $a_{ij} = a_{ji}$, for all $i \neq j$, since $(j, i) \in \mathcal{E}$ implies $(i, j) \in \mathcal{E}$.

Let the matrix $L=[\ell_{ij}] \in \mathbb{R}^{p\times p}$ be defined as $\ell_{ii}=\sum_{j\neq i} a_{ij}$ and $\ell_{ij} = -a_{ij}$, where $i \neq j$. The matrix L satisfies the conditions

$$\ell_{ij} \leqslant 0, \;\; i \neq j, \;\; \sum_{j=1}^{p} \ell_{ij} = 0, \;\; i = 1, \ldots, p. \tag{1}$$

For an undirected graph, the *Laplacian matrix* L has the property of symmetric positive semidefiniteness. However, L for a directed graph does not have this property. In both the undirected and directed cases, 0 is an eigenvalue of L with the associated eigenvector $\mathbf{1}_p$, where $\mathbf{1}_p$ is a $p \times 1$ column vector of all ones.

Let I_p denote the $p \times p$ identity matrix. Given a matrix $S = [s_{ij}] \in \mathbb{R}^{p\times p}$, the directed graph of S, denoted by $\Gamma(S)$, is the directed graph on p nodes $i, i \in \{1, 2, \ldots, p\}$, such that there is an edge in $\Gamma(S)$ from node j to node i if and only if $s_{ij} \neq 0$ (cf. [7]).

3. Consensus with a time-varying reference state

In this section we investigate consensus algorithms with a time-varying reference state for vehicles modeled by single-integrator dynamics. We assume a time-invariant, directed information-exchange topology throughout the paper.

Consider vehicles with dynamics given by

$$\dot{\xi}_i = u_i, \quad i = 1, \ldots, n, \tag{2}$$

where $\xi_i \in \mathbb{R}^m$ is the state of the ith vehicle, and $u_i \in \mathbb{R}^m$ is the control input. A consensus algorithm is proposed in [8,11,16,20] as

$$u_i = -\sum_{j=1}^{n} g_{ij}k_{ij}(\xi_i - \xi_j), \quad i = 1, \ldots, n, \tag{3}$$

where $k_{ij} > 0$, $g_{ii} \triangleq 0$, and g_{ij} is 1 if information flows from vehicle j to vehicle i and 0 otherwise, $\forall i \neq j$.

With (3), consensus is reached among the n vehicles if for all $\xi_i(0)$ and all $i, j = 1, \ldots, n$, $\xi_i(t) \to \xi_j(t)$ as $t \to \infty$. The final consensus value, which depends on both the information-exchange topologies and the weights k_{ij}, is a constant and might be *a priori* unknown. However, in some applications, it might be desirable that each state $\xi_i(t)$ approaches a (time-varying) reference state $\xi^r(t)$ and the reference state might only be available to a portion of the vehicles in the team.

In the following, we derive algorithms to achieve this objective. We say that the consensus problem with a reference state is solved if $\xi_i(t) \to \xi_j(t) \to \xi^r(t)$, $\forall i \neq j$, as $t \to \infty$.

3.1. Constant reference state

In this subsection, we consider the case of constant ξ_r, where the consensus algorithm can be summarized as

$$u_i = -\sum_{j=1}^{n} g_{ij} k_{ij} (\xi_i - \xi_j) - g_{i(n+1)} \alpha_i (\xi_i - \xi^r),$$
$$i = 1, \ldots, n, \quad (4)$$

where $k_{ij} > 0$, $\alpha_i > 0$, $g_{ii} \triangleq 0$, and g_{ij} is 1 if information flows from vehicle j to vehicle i and 0 otherwise, $\forall i, j \in \{1, \ldots, n\}$, and $g_{i(n+1)}$ is 1 if vehicle i has access to ξ^r and 0 otherwise. Note that in [8] ξ^r corresponds to the constant state of the group leader. Also note that [13] deals with the case where only one vehicle has access to the reference state. The vehicle, denoted as vehicle ℓ without loss of generality, must be the root of a directed spanning tree. As a result, $g_{\ell(n+1)} = 1$, and $g_{i(n+1)} = 0$, $\forall i \neq \ell$.

Next, we consider the general case where a portion of the vehicles in the team, denoted as a vehicle set $\mathscr{L}$, have access to the reference state under directed information exchange, that is, $g_{i(n+1)} = 1$, $\forall i \in \mathscr{L}$, and $g_{i(n+1)} = 0$, $\forall i \notin \mathscr{L}$.

We need the following lemmas from [20].

Lemma 3.1 (*Ren et al. [20]*). *Suppose that* $z = [z_1, \ldots, z_p]^T$ *with* $z_i \in \mathbb{R}$ *and* $L \in \mathbb{R}^{p\times p}$ *satisfies the property* (1). *Then the following four conditions are equivalent*: (i) *L has a simple zero eigenvalue with an associated eigenvector* $\mathbf{1}_p$ *and all of the other eigenvalues have positive real parts*; (ii) $Lz = 0$ *implies that* $z_1 = \cdots = z_p$; (iii) *consensus is reached asymptotically for a system* $\dot{z} = -Lz$; (iv) *the directed graph of L has a directed spanning tree.*

Lemma 3.2 (*Ren et al. [20]*). *Suppose that z and L are defined as in Lemma* 3.1. *Then the following four conditions are equivalent*: (i) *the directed graph of L has a directed spanning tree, and vehicle k has no incoming links*;[1] (ii) *the directed graph of L has a directed spanning tree, and every entry of the kth row of L is zero*; (iii) *consensus is reached asymptotically for a system* $\dot{z} = -Lz$ *with* $\xi_i(t) \to \xi_k(0)$, $\forall i$, *as* $t \to \infty$; (iv) *vehicle k is the only node that has a directed path to all of the other vehicles in the team.*

Note that Lemmas 3.1 and 3.2 are still valid for $z_i \in \mathbb{R}^m$ by introducing the notion of Kronecker product. We have the following theorem on consensus with a constant reference state.

Theorem 3.1. *Let* $G = [g_{ij}] \in \mathbb{R}^{(n+1)\times(n+1)}$ *be the adjacency matrix, where* g_{ij} *and* $g_{i(n+1)}$, $\forall i, j \in \{1, \ldots, n\}$, *are defined as in Eq.* (4) *and* $g_{(n+1)k} = 0$, $\forall k \in \{1, \ldots, n+1\}$. *Algorithm* (4) *solves the consensus problem with a constant reference state* ξ^r *if and only if the directed graph of G has a directed spanning tree.*[2]

Proof. Let $k_{i(n+1)} \triangleq \alpha_i$. Also let $L_{n+1} = [\ell_{ij}] \in \mathbb{R}^{(n+1)\times(n+1)}$ be defined as $\ell_{ij} = -g_{ij}k_{ij}$, $\ell_{ii} = \sum_{j\neq i} g_{ij}k_{ij}$, $\forall i \in \{1, \ldots, n\}$, $\forall j \in \{1, \ldots, n+1\}$, and $\ell_{(n+1)j} = 0$, $\forall j$. Letting $\xi_{n+1} \triangleq \xi^r$, gives $\dot{\xi}_{n+1} = 0$. With the consensus algorithm (4), Eq. (2) can be written in matrix form as

$$\dot{\xi} = -(L_{n+1} \otimes I_m)\xi,$$

where $\xi = [\xi_1^T, \ldots, \xi_{n+1}^T]^T$, and $\otimes$ denotes the Kronecker product. Note that L_{n+1} satisfies the property (1), and the directed graph of L_{n+1} is equivalent to that of G. Then from arguments (ii) and (iii) of Lemma 3.2 with L_{n+1} and ξ playing the roles of L and z, respectively, it follows that $\xi_i \to \xi_{n+1}(0)$, $\forall i$, if and only if the directed graph of G has a directed spanning tree. Equivalently, it follows that $\xi_i \to \xi^r$, $\forall i$, since $\xi_{n+1} \equiv \xi^r$. □

To illustrate, consider a team of $n = 4$ vehicles. Four subcases will be considered in this subsection, where $\xi^r \triangleq 1$ for each subcase. In Subcase (a), we let $g_{15} = 1$ and $g_{j5} = 0$, $\forall j \neq 1$, which corresponds to the case that only vehicle 1 has access to ξ^r. In Subcase (b), we let $g_{j5} = 1$, $\forall j$, which corresponds to the case that all of the vehicles have access to ξ^r. In Subcase (c), we let $g_{35} = g_{45} = 1$ and $g_{j5} = 0$, $\forall j \notin \{3, 4\}$, which corresponds to the case that only vehicles 3 and 4 have access to ξ^r. In Subcase (d), we let $g_{45} = 1$ and $g_{j5} = 0$, $\forall j \neq 4$, which corresponds to the case that only vehicle 4 has access to ξ^r. Fig. 1 shows the information-exchange topologies corresponding to each subcase.

Fig. 2 shows the actual states of all vehicles, denoted by dashed lines, and the reference state, denoted by a solid line, using the consensus algorithm (4). Note that ξ_i converges to ξ^r in each subcase except Subcase (d). Also note that node ξ^r has a directed path to all of the vehicles in Subcases (a)–(c) in Fig. 1. However, there does not exist a directed path from node ξ^r to all of the vehicles in Subcase (d) in Fig. 1. Note that Subcase (a) corresponds to the case discussed in [13]. Also note that in Subcase (c) in Fig. 1, neither 3 nor 4 is the root of a directed spanning tree, which implies that the results in [13]

[1] At most one such vehicle can exist when the directed graph has a directed spanning tree.

[2] Treat ξ^r as a virtual vehicle with index $n+1$. This condition is equivalent to the condition that ξ^r is the only node that has a directed path to all of the vehicles in the team from Lemma 3.2.

do not apply. However, as shown above, ξ_i still approaches ξ^r in this case.

3.2. Time-varying reference state

In this subsection, we assume that the reference state might be a time-varying exogenous signal or evolves according to certain nonlinear dynamics. Without loss of generality, suppose that ξ^r satisfies the dynamics given by

$$\dot{\xi}^r = f(t, \xi^r), \tag{5}$$

where $f(\cdot,\cdot)$ is piecewise continuous in t and locally Lipschitz in ξ^r.

We first show that Algorithm (4) is not sufficient for consensus on a time-varying reference state. As an example, let $\xi^r = \cos(t)$ and consider the four subcases as in Section 3.1. As shown in Fig. 3, the states do not converge to ξ^r in each subcase.

One might be tempted to apply the following algorithm in the case of a time-varying reference state

$$u_i = g_{i(n+1)} f(t, \xi^r) - \sum_{j=1}^{n} g_{ij} k_{ij} (\xi_i - \xi_j) - g_{i(n+1)} \alpha_i (\xi_i - \xi^r),$$
$$i = 1, \ldots, n, \tag{6}$$

where k_{ij}, α_i, g_{ij}, and $g_{i(n+1)}$ are defined as in Eq. (4).

As an example, similarly let $\xi^r = \cos(t)$ and consider the four subcases as in Section 3.1. As shown in Fig. 4, the states do not converge to ξ^r in each subcase except Subcase (b).

We have the following theorem for consensus on a time-varying reference state using Algorithm (6).

Theorem 3.2. *If* $g_{i(n+1)} = 1$, $i = 1, \ldots, n$, *then the consensus algorithm* (6) *solves the consensus problem with a time-varying reference state.*

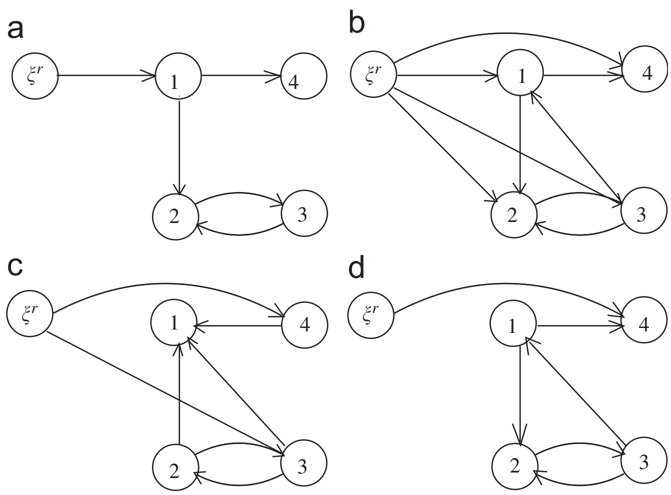

Fig. 1. Information-exchange topologies among the four vehicles, where one or more vehicles might have access to the constant reference state.

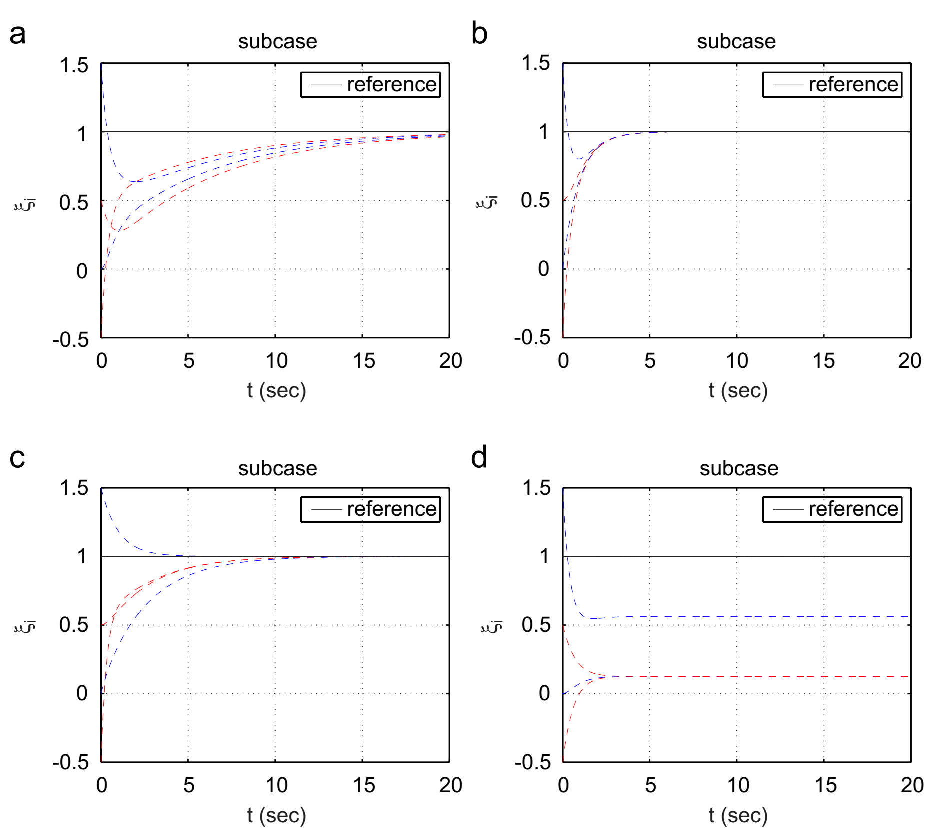

Fig. 2. Consensus seeking with a constant reference state using Algorithm (4).

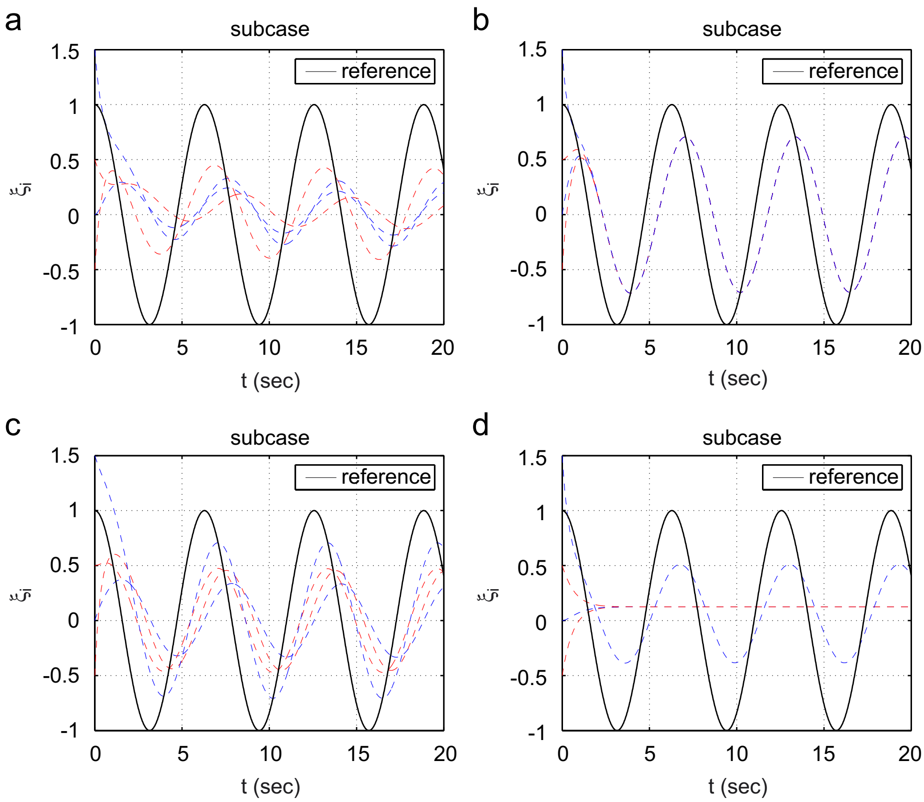

Fig. 3. Consensus seeking with a time-varying reference state using Algorithm (4).

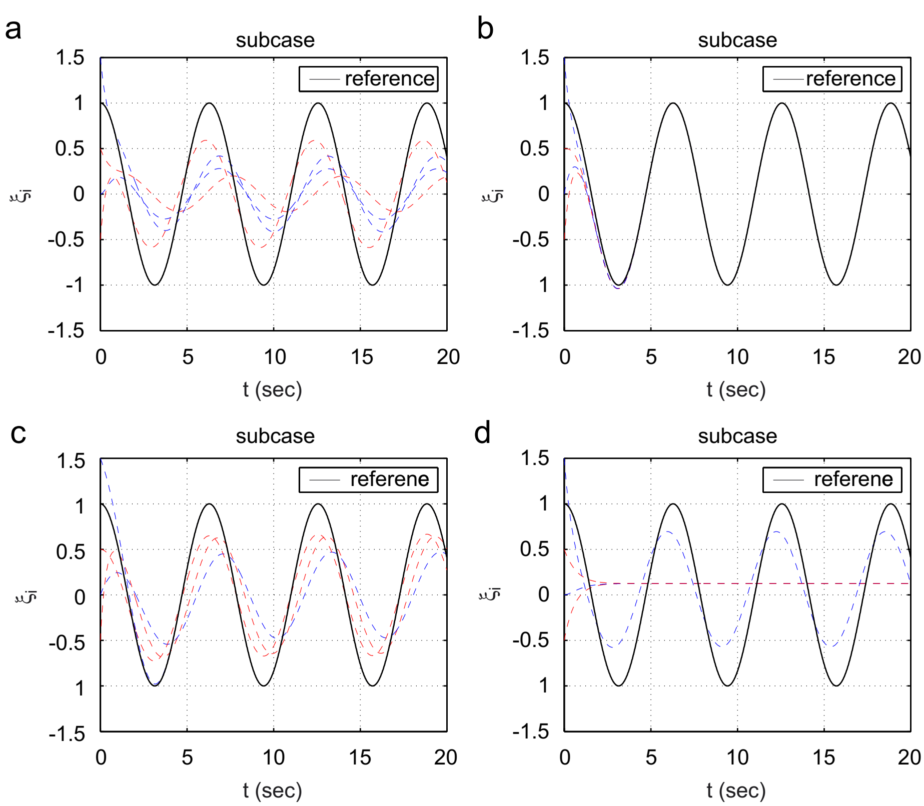

Fig. 4. Consensus seeking with a time-varying reference state using Algorithm (6).

Proof. With the consensus algorithm (6), Eq. (2) can be written in matrix form as $\dot{\tilde{\xi}} = -[(L_n + \Gamma) \otimes I_m]\tilde{\xi}$, where $\Gamma \in \mathbb{R}^{n\times n}$ is a diagonal matrix with α_i being the diagonal entries, $L_n = [\ell_{ij}] \in \mathbb{R}^{n\times n}$ is defined as $\ell_{ij} = -g_{ij}k_{ij}$ and $\ell_{ii} = \sum_{j\neq i} g_{ij}k_{ij}$, $\forall i, j \in \{1,\ldots,n\}$, and $\tilde{\xi} = [\tilde{\xi}_1^{\mathrm{T}},\ldots,\tilde{\xi}_n^{\mathrm{T}}]^{\mathrm{T}}$ with $\tilde{\xi}_i = \xi_i - \xi^r$. From Gershgorin disc theorem [7], it is straightforward to see that all of the eigenvalues of $-(L_n + \Gamma)$ have negative real parts. Therefore, it follows that $\tilde{\xi} \to 0$ asymptotically, that is, $\xi_i \to \xi^r$, $\forall i$, asymptotically. □

Note that the argument of Theorem 3.2 does not rely on the information-exchange topology among the vehicles. Even if there is no information exchange among the vehicles (i.e. $L=0$), the consensus algorithm (6) still solves the consensus problem with a time-varying reference state as long as each vehicle has access to ξ^r. However, this argument is rather restricted in the sense that each vehicle must have access to the time-varying reference state.

When only a portion of the vehicles have access to ξ^r, we propose the following consensus algorithm

$$u_i = \frac{1}{\eta_i}\sum_{j=1}^{n} g_{ij}k_{ij}[\dot{\xi}_j - \gamma_i(\xi_i - \xi_j)] + \frac{1}{\eta_i} g_{i(n+1)}\alpha_i[f(t,\xi^r) - \gamma_i(\xi_i - \xi^r)], \quad i = 1,\ldots,n, \tag{7}$$

where $k_{ij} > 0$, $\alpha_i > 0$, $\gamma_i > 0$, g_{ij} and $g_{i(n+1)}$ are defined as in Eq. (4), and $\eta_i = g_{i(n+1)}\alpha_i + \sum_{j=1}^{n} g_{ij}k_{ij}$. Note that information feedback is introduced to each vehicle through its local neighbors' information states and their derivatives.

In the special case where only one vehicle has access to ξ^r, the following consensus algorithm is also valid:

$$u_i = f(t,\xi^r) - \sum_{j=1}^{n} g_{ij}k_{ij}(\xi_i - \xi_j) - \alpha_i(\xi_i - \xi^r), \quad i = \ell,$$

$$u_i = \frac{1}{\sum_{j=1}^{n} g_{ij}k_{ij}}\sum_{j=1}^{n} g_{ij}k_{ij}[\dot{\xi}_j - \gamma_i(\xi_i - \xi_j)], \quad \forall i \neq \ell, \tag{8}$$

where $\alpha_i > 0$, $\gamma_i > 0$, ℓ denotes the index of the only vehicle that has access to ξ^r, and g_{ij} is defined as in Eq. (3).

Theorem 3.3. *Let $G = [g_{ij}] \in \mathbb{R}^{(n+1)\times(n+1)}$ be defined as in Theorem* 3.1. *Algorithms* (7) *and* (8) *solve the consensus problem with a time-varying reference state if and only if the directed graph of G has a directed spanning tree.*

Proof. For Algorithm (7), let $\xi_{n+1} \triangleq \xi^r$ and $k_{i(n+1)} \triangleq \alpha_i$. Noting that $\dot{\xi}_i = u_i$, we rewrite Eq. (7) as

$$\dot{\xi}_i = \frac{1}{\sum_{j=1}^{n+1} g_{ij}k_{ij}}\sum_{j=1}^{n+1} g_{ij}k_{ij}[\dot{\xi}_j - \gamma_i(\xi_i - \xi_j)], \quad i = 1,\ldots,n.$$

After some manipulation, we get that

$$\sum_{j=1}^{n+1} g_{ij}k_{ij}(\dot{\xi}_i - \dot{\xi}_j) = -\gamma_i \sum_{j=1}^{n+1} g_{ij}k_{ij}(\xi_i - \xi_j), \quad i = 1,\ldots,n,$$

which implies that

$$\sum_{j=1}^{n+1} g_{ij}k_{ij}(\xi_i - \xi_j) \to 0, \quad i = 1,\ldots,n. \tag{9}$$

Note that there are n equations but $n+1$ variables in Eq. (9). By adding an equation $0 = 0$, $i = n+1$, to Eq. (9), we can rewrite Eq. (9) in matrix form as $(L_{n+1} \otimes I_m)\xi \to 0$, where $\xi = [\xi_1^{\mathrm{T}},\ldots,\xi_{n+1}^{\mathrm{T}}]^{\mathrm{T}}$, the square matrix $L_{n+1} = [\ell_{ij}] \in \mathbb{R}^{(n+1)\times(n+1)}$ is defined as $\ell_{ii} = \sum_{j\neq i} g_{ij}k_{ij}$, $\ell_{ij} = -g_{ij}k_{ij}$, $\forall i \in \{1,\ldots,n\}$, $\forall j \in \{1,\ldots,n+1\}$, and $\ell_{(n+1)i} = 0$, $\forall i$. Note that all of the entries of the $n+1$th row of L are zero. Also note that L_{n+1} satisfies property (1) and the directed graph of L is equivalent to that of G, which has a directed spanning tree. Therefore, from arguments (ii) and (iv) of Lemma 3.1 with L_{n+1} and ξ playing the roles of L and z, respectively, $\xi_i \to \xi_j$, $\forall i, j \in \{1,\ldots,n+1\}$, if and only if the directed graph of G has a directed spanning tree. Equivalently, it follows that $\xi_i \to \xi^r$, $\forall i$, since $\xi_{n+1} \equiv \xi^r$.

For Algorithm (8), noting that $\dot{\xi}_i = u_i$, we rewrite the second equation in Eq. (8) as

$$\dot{\xi}_i = \frac{1}{\sum_{j=1}^{n} g_{ij}k_{ij}}\sum_{j=1}^{n} g_{ij}k_{ij}[\dot{\xi}_j - \gamma_i(\xi_i - \xi_j)], \quad \forall i \neq \ell.$$

After some manipulation, we get that

$$\sum_{j=1}^{n} g_{ij}k_{ij}(\dot{\xi}_i - \dot{\xi}_j) = -\gamma_i \sum_{j=1}^{n} g_{ij}k_{ij}(\xi_i - \xi_j), \quad \forall i \neq \ell,$$

which implies that $\sum_{j=1}^{n} g_{ij}k_{ij}(\xi_i - \xi_j) \to 0$, $\forall i \neq \ell$. Similarly, from arguments (ii) and (iv) of Lemma 3.1, it follows that $\xi_i \to \xi_j$, $i, j \in \{1,\ldots,n\}$, if and only if the directed graph of G has a directed spanning tree (with vehicle ℓ being the root). Noting that $\xi_i \to \xi_j$, $\forall i \neq j$, we know that $\xi_\ell \to \xi^r$ from the first equation in Eq. (8). Therefore, it follows that $\xi_i \to \xi^r$, $\forall i$, asymptotically. □

Compared to Algorithm (6), which requires that each vehicle has access to the time-varying reference state to reach consensus, Algorithms (7) and (8) allow only a portion of the vehicles to have access to the time-varying reference state.

To illustrate, consider two subcases in this subsection using the consensus algorithm (7), where $g_{35} = g_{45} = 1$ and $g_{j5} = 0$, $\forall j \notin \{3,4\}$ (Fig. 1(c)), and the consensus algorithm (8), where $\ell = 1$ (Fig. 1(a)), respectively. In Subcase (a), let $\xi^r = \cos(t)$. In Subcase (b), assume that ξ^r satisfies the nonlinear dynamics given by $\dot{\xi}^r = \sin(t)\sin(2\xi^r)$, where $\xi^r(0) = 0.5$. As shown in Figs. 5 and 6, the states of all vehicles converge to the exogenous signal $\cos(t)$ in Subcase (a) and to the solution of the nonlinear model $\dot{\xi}^r = \sin(t)\sin(2\xi^r)$ in Subcase (b) using both the algorithms (7) and (8).

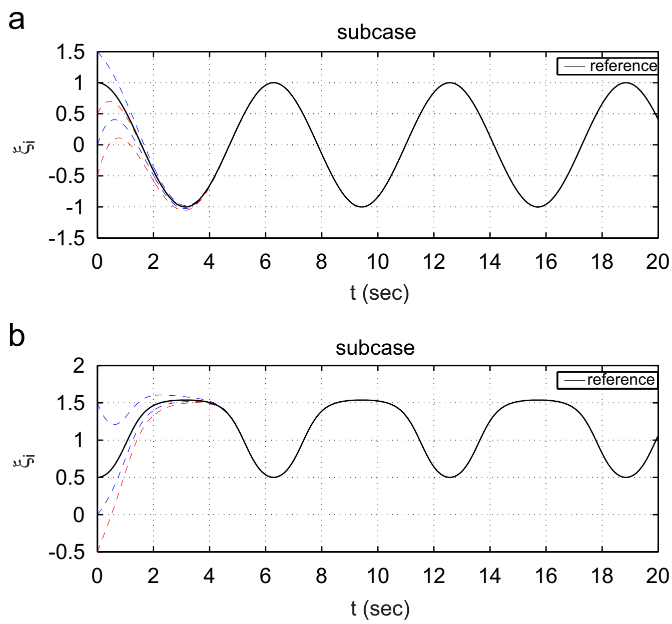

Fig. 5. Consensus seeking with a time-varying reference state using Algorithm (7).

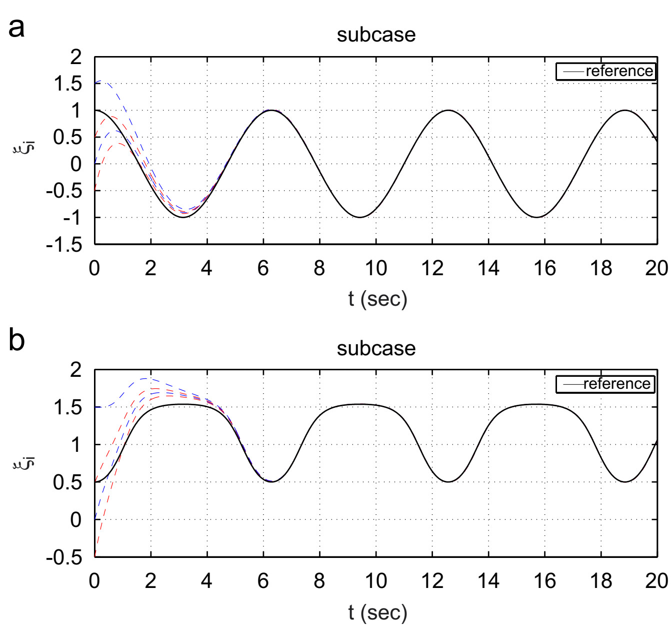

Fig. 6. Consensus seeking with a time-varying reference state using Algorithm (8).

Compared to the leader–follower strategy where information only flows from leaders to followers,[3] the consensus algorithm (8) takes into account the general case where information might flow from any vehicle to any other vehicle.

[3] The leader–follower topology corresponds to an information-exchange graph that is itself a directed spanning tree. Note that the condition that a graph has a directed spanning tree is not equivalent to the condition that a graph is itself a directed spanning tree. The latter condition is a special case of the former one.

3.3. Extensions to relative state deviations

The consensus algorithm (3) can be extended to guarantee that the differences of the vehicle states converge to desired values, i.e. $\xi_i(t) - \xi_j(t) \to \Delta_{ij}(t)$, where $\Delta_{ij}(t)$ denotes the desired (time-varying) separation between ξ_i and ξ_j. We propose the following algorithm for relative state deviations:

$$u_i = \dot{\delta}_i - \sum_{j=1}^{n} g_{ij}k_{ij}[(\xi_i - \xi_j) - (\delta_i - \delta_j)], \quad i = 1, \ldots, n, \tag{10}$$

where $\delta_i - \delta_j$, $\forall i \neq j$, denotes the desired separation between the information states. Note that the consensus algorithm (3) corresponds to the case that $\Delta_{ij} = 0$, $\forall i \neq j$.

We have the following theorem for relative state deviations.

Theorem 3.4. *With the consensus algorithm* (10), $\xi_i - \xi_j \to \delta_i - \delta_j$ *asymptotically if and only if the information-exchange topology has a directed spanning tree.*

Proof. With the consensus algorithm (10), Eq. (2) can be written in matrix form as

$$\dot{\hat{\xi}} = -(L_n \otimes I_m)\hat{\xi},$$

where $\hat{\xi} = [\hat{\xi}_1^{\mathrm{T}}, \ldots, \hat{\xi}_n^{\mathrm{T}}]^{\mathrm{T}}$ with $\hat{\xi}_i = \xi_i - \delta_i$ and $L_n = [\ell_{ij}] \in \mathbb{R}^{n\times n}$ with $\ell_{ij} = -g_{ij}k_{ij}$ and $\ell_{ii} = \sum_{j\neq i} g_{ij}k_{ij}$. Note that L_n satisfies property (1). From Lemma 3.1, we know that $\hat{\xi}_i \to \hat{\xi}_j$ asymptotically if and only if the information-exchange topology has a directed spanning tree. The rest of the proof then follows the fact that $\hat{\xi}_i \to \hat{\xi}_j$ is equivalent to $\xi_i - \xi_j \to \delta_i - \delta_j$. □

When a portion of the vehicles have access to ξ^r, we propose the following consensus algorithm for relative state deviations with a time-varying reference state:

$$u_i = \dot{\delta}_i + \frac{1}{\eta_i}\sum_{j=1}^{n} g_{ij}k_{ij}\{\dot{\xi}_j - \dot{\delta}_j - \gamma_i[(\xi_i - \xi_j) - (\delta_i - \delta_j)]\} + \frac{1}{\eta_i} g_{i(n+1)}\alpha_i[f(t,\xi^r) - \gamma_i(\xi_i - \delta_i - \xi^r)]. \tag{11}$$

In the special case that only one vehicle has access to ξ^r, we propose the following consensus algorithm for relative state deviations with a time-varying reference state:

$$u_i = \dot{\delta}_i + f(t,\xi^r) - \sum_{j=1}^{n} g_{ij}k_{ij}[(\xi_i - \xi_j) - (\delta_i - \delta_j)] - \alpha_i(\xi_i - \delta_i - \xi^r), \quad i = \ell$$

$$u_i = \dot{\delta}_i + \frac{1}{\sum_{j=1}^{n} g_{ij}k_{ij}}\sum_{j=1}^{n} g_{ij}k_{ij}\{\dot{\xi}_j - \dot{\delta}_j - \gamma_i[(\xi_i - \xi_j) - (\delta_i - \delta_j)]\}, \quad i \neq \ell, \tag{12}$$

where ℓ denotes the index of the vehicle that has access to ξ^r.

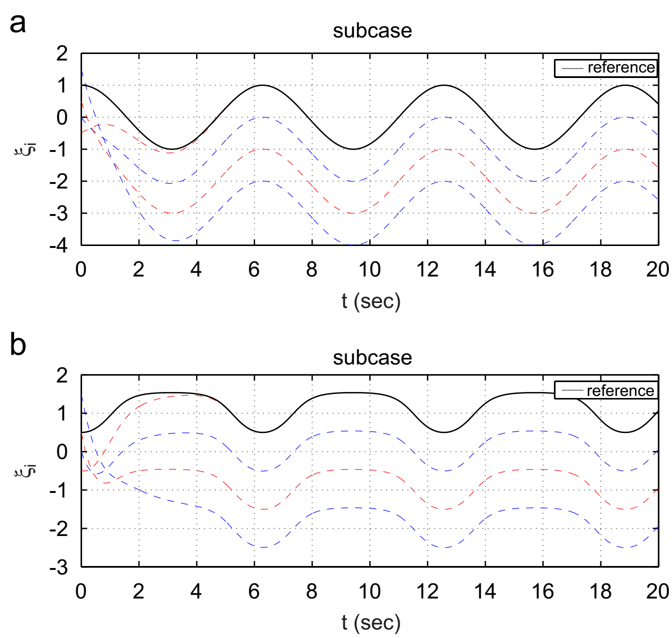

Fig. 7. Consensus seeking with a time-varying reference state using Algorithm (11).

Theorem 3.5. *Let $G=[g_{ij}]\in\mathbb{R}^{(n+1)\times(n+1)}$ be defined as in Theorem* 3.1. *With the Algorithms* (11) *and* (12), $\xi_i\to\xi^r+\delta_i$ *and* $\xi_i-\xi_j\to\delta_i-\delta_j$ *if and only if the directed graph of G has a directed spanning tree.*

Proof. Define $\tilde{\xi}_i=\xi_i-\delta_i$ and $\tilde{u}_i=u_i-\dot{\delta}_i$. Note that $\dot{\tilde{\xi}}_i=\tilde{u}_i$. Also note that Eqs. (11) and (12)) can be rewritten in the same form as Eqs. (7) and (8) with $\tilde{\xi}_i$ and $\tilde{u}_i$ playing the role of ξ_i and u_i, respectively. As a result, from Theorem 3.3, $\tilde{\xi}_i\to\tilde{\xi}_j\to\xi^r$, which implies that $\xi_i\to\xi^r+\delta_i$ and $\xi_i-\xi_j\to\delta_i-\delta_j$. □

To illustrate, consider two subcases in this subsection using the consensus algorithm (11), where $g_{35}=g_{45}=1$, $g_{j5}=0$, $\forall j\notin\{3,4\}$ (Fig. 1(c)), and $\delta_i=1-i$, $i=1,\ldots,4$. In Subcase (a), let $\xi^r=\cos(t)$. In Subcase (b), assume that ξ^r satisfies the nonlinear dynamics given by $\dot{\xi}^r=\sin(t)\sin(2\xi^r)$, where $\xi^r(0)=0.5$. As shown in Fig. 7, $\xi_1\to\xi^r$, $\xi_2\to\xi^r-1$, $\xi_3\to\xi^r-2$, and $\xi_4\to\xi^r-3$, where ξ^r is the exogenous signal $\cos(t)$ in Subcase (a) and is the solution of the nonlinear model $\dot{\xi}^r=\sin(t)\sin(2\xi^r)$ in Subcase (b).

Note that by appropriately defining $\delta_i(t)$, a desired formation geometry can be preserved among the vehicles using Algorithms (11) and (12).

4. Simulation

In this section, we simulate a scenario where four vehicles are required to maintain a desired formation geometry while the formation centroid needs to follow a reference trajectory. We will compare the existing leader–follower approach with a consensus algorithm with a time-varying reference state in the presence of disturbance.

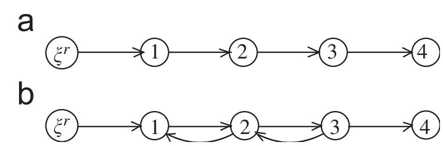

Fig. 8. Information-exchange topologies among the four vehicles. Fig. 8(a) denotes a leader–follower topology while Fig. 8(b) denotes a topology with information also flowing from followers to leaders.

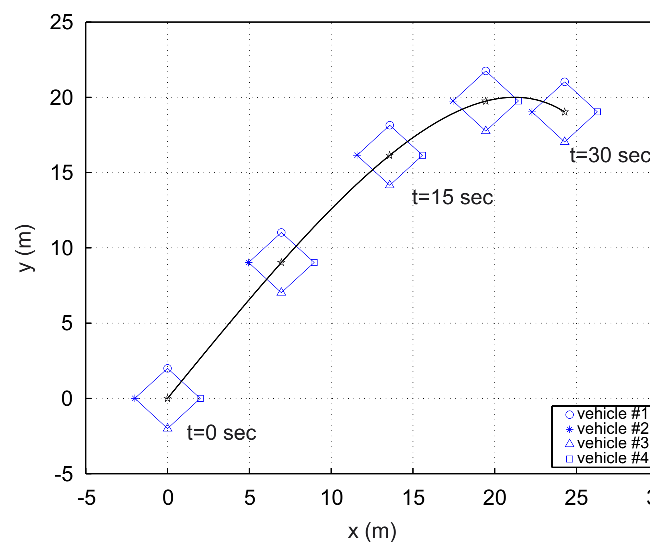

Fig. 9. Formation geometries of the four vehicles with the information-exchange topology given by Fig. 8(b) in the absence of disturbance.

Suppose that the vehicle dynamics are $\dot{r}_i=u_i$, where $r_i\in\mathbb{R}^2$ denotes the position of the ith vehicle and $u_i\in\mathbb{R}^2$ denotes the control input to the ith vehicle. We apply Eq. (11) as the control input. Let $\xi^r=[30\sin(\pi t/100), 20\sin(\pi t/50)]^{\mathrm{T}}$, $\delta_1=[0,2]^{\mathrm{T}}$, $\delta_2=[-2,0]^{\mathrm{T}}$, $\delta_3=[0,-2]^{\mathrm{T}}$, and $\delta_4=[2,0]^{\mathrm{T}}$. Also let $k_{ij}=\alpha_i=\gamma_i=1$. The desired formation geometry is a diamond shape.

Suppose that the information-exchange topologies among the vehicles are given by Fig. 8, where only vehicle 1 has access to ξ^r (i.e. $g_{15}=1$ and $g_{i5}=0$, $\forall i\neq 1$). Fig. 8(a) corresponds to a leader–follower topology where vehicle $j+1$ follows vehicle j, $j=1,2,3$, while Fig. 8(b) corresponds to a topology where information also flows from followers to leaders.

Fig. 9 shows the formation geometries of the four vehicles at $t\in\{0,7.5,15,22.5,30\}$ s with the information-exchange topology given by Fig. 8(b) in the absence of disturbance. The solid line represents $\xi^r(t)$, $t\in[0,30]$ s and stars represent $\xi^r(t)$ at $t\in\{0,7.5,15,22.5,30\}$ s. A similar result is achieved with the information-exchange topology given by Fig. 8(a).

Figs. 10 and 11 show the formation geometries of the four vehicles at $t\in\{0,7.5,15,22.5,30\}$ s with the information-exchange topologies given by Figs. 8(a) and (b), respectively, where vehicle 3 is disturbed at $t\in[10,20]$ s. With the leader–follower topology given by Fig. 8(a), vehicles 3 and 4 are left behind due to the disturbance to vehicle 3 while vehicles 1 and 2 keep moving forward by following $\xi^r(t)$ as shown in Fig. 10 at $t=15$ s. As a result, the desired diamond

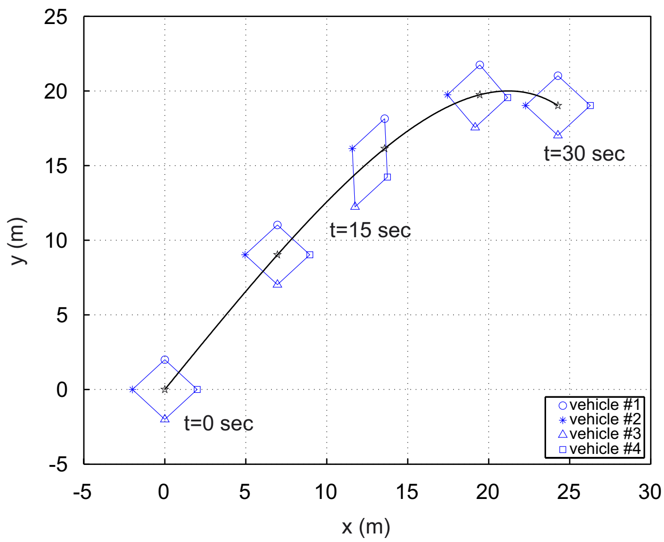

Fig. 10. Formation geometries of the four vehicles with the information-exchange topology given by Fig. 8(a) in the presence of disturbance.

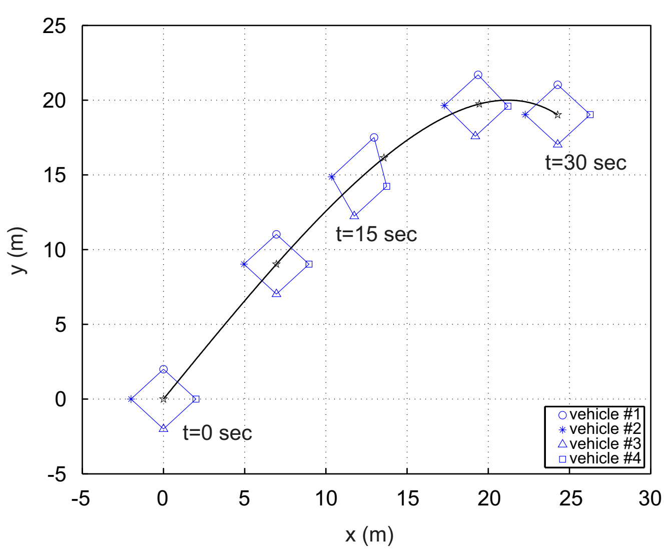

Fig. 11. Formation geometries of the four vehicles with the information-exchange topology given by Fig. 8(b) in the presence of disturbance.

shape does not maintain well at $t = 15$ s. In contrast, with the information-exchange topology given by Fig. 8(b), where information flows from vehicle 3 to vehicle 2 and from vehicle 2 to vehicle 1, vehicles 1 and 2 slow down for vehicles 3 and 4 to catch up as shown in Fig. 11 at $t = 15$ s, resulting in better formation maintenance. Note that the motion of ξ^r remains unaffected even if some vehicles are left behind due to disturbance as shown in Figs. 10 and 11. Although not shown here, it is possible to introduce feedback directly to the evolution law of ξ^r in Eq. (5) so that the reference state can adjust its motion according to vehicle performance.

5. Conclusion

The consensus problem with a time-varying reference state has been studied under the condition that only a portion of the vehicles have access to the reference state and those vehicles might not have a directed path to all of the other vehicles in the team. We have analyzed a consensus algorithm with a constant reference state using graph theoretical tools. We have also proposed and analyzed algorithms so that consensus is reached on a time-varying reference state. The consensus algorithms have also been extended to achieve relative state deviations among the vehicles. An application example has shown the effectiveness of our strategies. Note that although we focus on a directed fixed information-exchange topology in this paper, the analysis of the proposed algorithms can be extended to directed switching information-exchange topologies. This will be a topic of future research.

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