Decentralization of Virtual Structures in Formation Control of Multiple Vehicle Systems *via* Consensus Strategies

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In this paper, we propose a coordination framework that decentralizes virtual structures in multi-vehicle systems via consensus strategies. We instantiate a local copy of the virtual structure state on each vehicle as well as the same coordination algorithm. Then we develop consensus strategies to guarantee that each instantiation of the virtual structure state comes into consensus when the virtual structure state is driven by a common input or inputs with bounded inconsistency under both fixed and switching interaction topologies. We show conditions under which consensus can be achieved for each instantiation of the virtual structure state and provide boundedness analyses for the inconsistency of different instantiations when inconsistent inputs exist. The decentralized framework is then applied in simulations to a multi-vehicle formation control scenario as a proof of concept.

Keywords: Formation control, virtual structures, information consensus, multiple vehicle systems.

1. Introduction

Formation control of multiple vehicle systems has been studied extensively in the literature with the hope that through efficient coordination many inexpensive, simple vehicles, can achieve better performance than a single monolithic vehicle. One solution for formation control is the virtual structure approach [1,9]. Similar ideas include the action reference scheme [7] and the virtual leader approach [3,8]. The basic idea is to specify a virtual leader or a virtual coordinate frame, called *virtual structure* hereafter for simplicity, as a reference for the whole group such that each vehicle's desired states can be defined relative to the virtual structure. As a result, single vehicle path planning and trajectory generation techniques can be employed for the virtual structure while trajectory tracking strategies can be employed for each vehicle.

In the current literature, the state of the virtual structure is generally implemented at a central location and broadcast to each vehicle (e.g., [1,2,11,16]). One drawback for this centralized implementation is that the central location is a single point of failure. Another drawback is that this centralized implementation does not scale well with the number of vehicles in the group. Furthermore, due to communication range constraints, the central location may not be able to exchange information with every vehicle in the group when the formation covers a large area. One remedy to these drawbacks is to instantiate a local copy of the state of the virtual structure and implement the same coordination algorithm on each vehicle. However, due to dynamically changing local situational awareness, discrepancies may appear for different instantiations of the virtual structure state. As a result, we need to develop strategies to ensure

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that all instantiations of the virtual structure state converge to a consistent value.

The study of information consensus is aimed at guaranteeing that each vehicle in a team converges to a consistent view of their information states. Consensus strategies have recently been studied in [4,6,10, 12,17], to name a few (see [18] for a survey of consensus algorithms in multivehicle coordination). The basic idea for information consensus is that each vehicle updates its information state based on the information states of its local (time-varying) neighbors, denoted as $\mathcal{N}_i(t)$, in such a way that the final information state of each vehicle converges to a common value.

The main contribution of this paper is to decentralize the virtual structures via consensus strategies in the presence of noises and/or exogenous inputs, expanding on our earlier work [14]. We propose consensus strategies to guarantee that consensus is reached for each instantiation of the virtual structure state when the virtual structure state is driven by a common input or inputs with bounded inconsistency. Each instantiation of the virtual structure state is then used as a basis to derive coordination algorithms for each vehicle. The decentralization of the virtual structures has several advantages. First, the issue of a single point of failure can be avoided. Second, the decentralized framework is scalable to a large number of vehicles in terms of communication constraints. Third, only local information exchange is required for its implementation. Finally, unlike the leader-follower approach, there is no need to identify specific vehicles using the decentralized coordination framework. In contrast to the decentralized scheme in [15], the current framework via consensus strategies allows random packet loss for each communication link as well as dynamically changing, sparse, and intermittent inter-vehicle communication topologies. In addition, in the current framework via consensus strategies, there is no need to identify two adjacent neighbors in order to form a ring topology as in [15] since at each time each vehicle simply communicates with any available local neighbors.

The organization of this paper is as follows. In Section 2 we propose a decentralized coordination framework for virtual structures via consensus strategies. In Section 3 we propose consensus strategies that are implemented in the decentralized coordination framework. We show conditions under which consensus can be achieved for different instantiations of the virtual structure state and provide boundedness analyses for inconsistency of the instantiations under fixed and switching interaction topologies. In Section 4 we apply the concept of decentralization of virtual structures to a multi-vehicle formation control application, while Section 5 concludes the paper.

2. Decentralized Coordination Framework for Virtual Structures Via Consensus Strategies

In this section, we first review the idea of virtual structures and then propose a decentralized coordination framework for the virtual structures via consensus strategies.

2.1. Virtual Structures

In formation control, the entire desired formation can be treated as a single rigid body or structure called the *virtual structure*. The centroid of the desired formation is often chosen as the *virtual center* of the virtual structure. A *formation frame* is located at the virtual center with a certain orientation to represent the configuration of the desired formation. The desired position and orientation for each vehicle can then be represented relative to the formation frame.

Fig. 1 shows an illustrative example of a virtual structure composed of three vehicles with planar motions, where $\mathcal{C}_I$ represents the inertial frame and $\mathcal{C}_F$ represents the formation frame located at virtual center (x_F, y_F) with an orientation θ_F relative to $\mathcal{C}_I$. Note that here we choose an example in 2D for illustrative purposes. The approaches hereafter can be easily extended to 3D. In Fig. 1, V_i denotes the actual position of the ith vehicle while V_i' denotes the desired position of the ith vehicle, $i = 1, 2, 3$. Note that V_1', V_2', and V_3' form the virtual structure and $\xi = [x_F, y_F, \theta_F]^T$ represents the virtual structure state. Also note that the desired position and orientation of each vehicle in $\mathcal{C}_I$ can be specified by ξ and the vehicle's desired deviations from $\mathcal{C}_F$. Letting (x_i^d, y_i^d) and θ_i^d represent
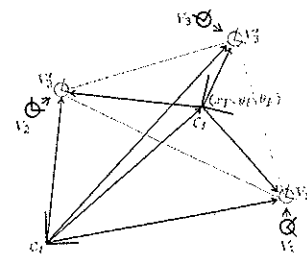

Fig. 1. A virtual structure composed of three vehicles.

the desired position and orientation of the ith vehicle relative to C_I, respectively, we have

$$\begin{bmatrix} x_i^d(t) \\ y_i^d(t) \end{bmatrix} = \begin{bmatrix} x_F(t) \\ y_F(t) \end{bmatrix} + \begin{bmatrix} \cos(\theta_F(t)) & -\sin(\theta_F(t)) \\ \sin(\theta_F(t)) & \cos(\theta_F(t)) \end{bmatrix}$$
$$\times \begin{bmatrix} x_{iF}^d(t) \\ y_{iF}^d(t) \end{bmatrix}$$
$$\theta_i^d(t) = \theta_F(t) + \theta_{iF}^d(t), \qquad (1)$$

where $(x_{iF}^d(t), y_{iF}^d(t))$ and θ_{iF}^d are the desired position and orientation of the ith vehicle relative to formation frame C_F. If each vehicle can track its desired position and orientation accurately, then the desired formation shape can be preserved accurately.

2.2. Decentralized Coordination Framework

One way to apply the virtual structure approach to formation control is through a centralized coordination scheme, where the virtual structure state is implemented at a central station, and then the virtual structure state is broadcast to each vehicle in the team [1]. However, as the number of vehicles in the team increases, this scheme may result in degraded overall system performance due to heavy communication at the central station, a single point of failure for the whole system.

As an alternative, each vehicle in the team can instantiate a local copy of the virtual structure state. Fig. 2 shows a decentralized coordination framework. In Fig. 2 each vehicle instantiates a local copy of a consensus module, denoted as C_i. The consensus module C_i obtains instantiations of the virtual structure state, denoted as $\xi_i = [x_{Fi}, y_{Fi}, \theta_{Fi}]^T$ for the ith instantiation, from its local (time-varying) neighbors and implements consensus strategies to guarantee that each instantiation of the virtual structure state approaches a common value (i.e., $\xi_i(t) \to \xi_j(t)$ as $t \to \infty$). Note that each vehicle can initialize the initial condition of

the virtual structure state $\xi_i(0) = [x_{Fi}(0), y_{Fi}(0), \theta_{Fi}(0)]^T$ in a distributed manner. That is, each vehicle can view its initial position and orientation as its desired position and orientation and solve $\xi_i(0)$ from (1). The consensus strategies used in C_i will be introduced in Section 3. The module K_i is the local vehicle controller, which receives ξ_i from the consensus module, converts ξ_i to the desired states for the ith vehicle, and then controls the actual state for the ith vehicle to track its desired state (i.e., $x_i(t) \to x_i^d(t)$, $y_i(t) \to y_i^d(t)$, and $\theta_i(t) \to \theta_i^d(t)$ as $t \to \infty$). Formation feedback from the ith local vehicle controller to the ith consensus module C_i is achieved through performance measure z_i. The module V_i represents the ith physical vehicle, with control input u_i and output y_i.

With the decentralized coordination framework, formation control problems can be decoupled into two subtasks. One is to design consensus strategies such that each instantiation of the virtual structure state comes into consensus. The other is to develop local control algorithms such that each vehicle can track its desired state. The second subtask is a single-vehicle trajectory tracking problem and has been extensively studied in the literature. Therefore, we will only focus on the first subtask in this paper. In [15], a decentralized scheme is proposed with the requirement that the communication topology form a fixed bidirectional ring. In contrast, the current framework via consensus strategies allows random packet loss for each communication link as well as dynamically changing communication topologies. In addition, in the current framework, there is no need to identify two adjacent neighbors in order to form a ring topology as in [15] since at each time each vehicle simply communicates with any available local neighbors.

3. Consensus Seeking for Virtual Structure Instantiations

Convergence to a common value is called *information consensus* or *agreement* in the literature. In this paper, the set of vehicles is said to achieve consensus if for all $\xi_i(0)$, $\xi_i(t) \to \xi_j(t)$ as $t \to \infty$. One common feature of most research in information consensus is that the common value on which consensus is reached is assumed to be inherently constant. However, information control problems, where the team moves with a velocity, the virtual structure state is dynamically evolving in time, which implies an exogenous input is required in the consensus strategies. In addition, the virtual structure state may be driven by noises due to unreliable information exchange between neighboring vehicles.
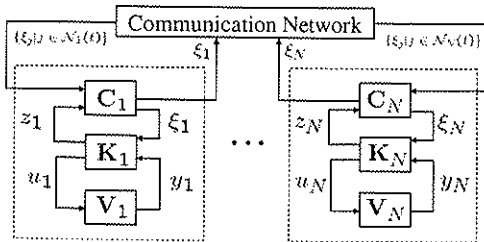

Fig. 2. A decentralized coordination framework.

In this section, we will consider a consensus strategy for the cases where the virtual structure state is driven by a common input or inputs with bounded inconsistency under fixed and switching interaction topologies. As mentioned in Section 2.2, the consensus strategy is implemented in consensus module C_i in Fig. 2. We provide boundedness analyses for the inconsistency of different instantiations of the virtual structure state when there exists inconsistent inputs using an input-to-state stability (ISS) analysis.

3.1. Definitions and Background

Let $\mathcal{A} = \{\mathcal{A}_i | i \in \mathcal{I}\}$ be a set of n vehicles, where $\mathcal{I} \triangleq \{1, 2, \ldots, n\}$. A directed graph $\mathcal{G}$ will be used to model the interaction topology among these vehicles. In $\mathcal{G}$, the ith node represents the ith vehicle $\mathcal{A}_i$ and a directed edge from $\mathcal{A}_i$ to $\mathcal{A}_j$, denoted as $(\mathcal{A}_i, \mathcal{A}_j)$, represents a unidirectional information exchange link from $\mathcal{A}_i$ to $\mathcal{A}_j$, that is, vehicle j can obtain information from vehicle i.

A directed path in $\mathcal{G}$ is a sequence of edges $(\mathcal{A}_{k_1}, \mathcal{A}_{k_2})$, $(\mathcal{A}_{k_2}, \mathcal{A}_{k_3})$, $(\mathcal{A}_{k_3}, \mathcal{A}_{k_4})$, ..., where $k_j \in \mathcal{I}$. Directed graph $\mathcal{G}$ is *strongly connected* if there is a directed path from $\mathcal{A}_i$ to $\mathcal{A}_j$ and $\mathcal{A}_j$ to $\mathcal{A}_i$ between every pair of distinct nodes $\mathcal{A}_i$ and $\mathcal{A}_j$, $\forall i, j \in \mathcal{I}$. A directed graph *has* or *contains a directed spanning tree* if there exists at least one node having a directed path to all other nodes. Fig. 3 shows a directed graph that has more than one possible directed spanning tree. The double arrows denote one possible directed spanning tree with $\mathcal{A}_1$ as the root. Directed spanning trees with $\mathcal{A}_2$ and $\mathcal{A}_3$ as the root are also possible.

Let $M_n(\mathbb{R})$ represent the set of all $n \times n$ real matrices. Given a matrix $\mathbf{A} = [a_{ij}] \in M_n(\mathbb{R})$, the directed graph of $\mathbf{A}$, denoted as $\Gamma(\mathbf{A})$, is the directed graph on n vertices V_i, $i \in \mathcal{I}$, such that there is a directed edge in $\Gamma(\mathbf{A})$ from V_j to V_i if and only if $a_{ij} \neq 0$ (c.f. [5]).

A matrix $\mathbf{A} = [a_{ij}] \in M_n(\mathbb{R})$ is *nonnegative*, denoted as $\mathbf{A} \geq 0$, if all its entries are nonnegative. Furthermore, if all its row sums are $+1$, $\mathbf{A}$ is said to be a *(row) stochastic matrix* [5].

3.2. Consensus Strategies

Let $\mathbb{N}$ and $\mathbb{N}^+$ denote, respectively, the set of non-negative integers and positive integers. Given T_s as the sampling period, a discrete-time consensus strategy with an exogenous input is given by

$$\xi_i[k+1] = \frac{1}{\sum_{j=1}^{n} \alpha_{ij}[k] g_{ij}[k]} \sum_{j=1}^{n} \alpha_{ij}[k] g_{ij}[k] \xi_j[k] + \mathbf{v}_i[k],$$

(2)

where $k \in \mathbb{N}$ is the discrete-time index, $\xi_i[k]$ is the ith instantiation of the virtual structure state at time $t = kT_s$, $i, j \in \mathcal{I}$, $\mathbf{v}_i[k]$ denotes the input at time $t = kT_s$, $\alpha_{ij}[k] > 0$ is uniformly lower and upper bounded, $g_{ii}[k] \triangleq 1$, and $g_{ij}[k]$, $\forall j \neq i$, is 1 if information flows from $\mathcal{A}_j$ to $\mathcal{A}_i$ and 0 otherwise. Eq. (2) is implemented in consensus module C_i in Fig. 2 to guarantee the consensus of the instantiations of the virtual structure state. Note that both α_{ij} and g_{ij} may be time-varying. Assume that $\xi_i \in \mathbb{R}^m$. Eq. (2) can be written in matrix form as

$$\xi[k+1] = (\mathbf{D}[k] \otimes \mathbf{I}_m) \xi[k] + \mathbf{v}[k],$$

(3)

where $\xi = [\xi_1^T, \ldots, \xi_n^T]^T$, $\mathbf{v} = [\mathbf{v}_1^T, \ldots, \mathbf{v}_n^T]^T$, $\otimes$ denotes the Kronecker product, $\mathbf{I}_m$ denotes the $m \times m$ identity matrix, and $\mathbf{D}[k] = [d_{ij}[k]]$, $(i, j) \in \mathcal{I}$, with $d_{ij}[k] = \frac{\alpha_{ij}[k] g_{ij}[k]}{\sum_{j=1}^{n} \alpha_{ij}[k] g_{ij}[k]}$. Note that $\mathbf{D}$ is a stochastic matrix.

When the virtual structure state is inherently constant and there does not exist information exchange noise, we simply let $\mathbf{v}_i = 0$. In contrast, when the virtual structure state is evolving with time, $\mathbf{v}_i$ in (2) can represent an exogenous feedforward signal to $\mathcal{A}_i$ denoted as $\mathbf{u}_i$ (i.e., the evolution velocity of the formation), a disturbance to $\mathcal{A}_i$ due to information exchange noise denoted as ω_i, a nonlinear dynamic evolution law (e.g., a group feedback term introduced to the ith instantiation of the virtual structure state) denoted as $\mathbf{f}(\xi_i, \mathbf{u}_i)$, or a combination of the three.

Two methods will be used to describe consensus errors in this paper. One is to define the consensus error variables for each vehicle based on how far each instantiation of the virtual structure state is from the consensus equilibrium $\bar{\xi}$ (i.e., the final common value to which all ξ_i converge). In this case, the consensus error variable for vehicle i can be denoted as

$$\tilde{\xi}_i = \xi_i - \bar{\xi}.$$

(4)

The other is to define the consensus error variables based on inconsistency between different instantiations of the virtual structure state. In this case, the consensus error variable between vehicle i and vehicle j can be denoted as

$$\hat{\xi}_{ij} = \xi_i - \xi_j.$$

(5)

Fig. 3. A directed graph that has more than one possible directed spanning tree, but is not strongly connected. One possible directed spanning tree is denoted with double arrows.

For a fixed interaction topology, the consensus equilibrium $\bar{\xi}$ can be defined explicitly as shown in Section 3.3. As a result, we can explicitly show the consensus equilibrium to which all ξ_i converge. We will use (4) to describe the consensus error variables in this case. Explicit bounds on the consensus errors will be derived. For switching interaction topologies, $\bar{\xi}$ will depend upon how the interaction topologies switch with time and cannot be defined explicitly. Instead, we will use (5) to describe the consensus error variables. Boundedness analyses of the consensus errors will be provided.

3.3. Analysis for a Fixed Interaction Topology

In this subsection, we will focus on the consensus-seeking problem with a fixed interaction topology and constant weighting factors, that is, matrix $\mathbf{D}$ is constant. In the following, we will focus on the case where $\xi_i \in \mathbb{R}$ and $v_i \in \mathbb{R}$ for simplicity. All results remain valid in the case of $\xi_i \in \mathbb{R}^m$ and $v_i \in \mathbb{R}^m$ since each component of ξ_i (respectively, v_i) is decoupled and the proof can be shown for each component of ξ_i (respectively, v_i).

Lemma 3.1: *Let* $\mathbf{D} = [d_{ij}] \in M_n(\mathbb{R})$ *be given (3). If directed graph $\mathcal{G}$ has a directed spanning tree, then one is the unique eigenvalue of $\mathbf{D}$ with maximum modulus and* $\lim_{m \to \infty} \mathbf{D}^m \to \mathbf{1}\mu^T$, *where* $m \in \mathbb{N}^+$, $\mathbf{1}$ *is an $n \times 1$ vector with all entries equal to one and* $\mu = [\mu_1, \ldots, \mu_n]^T \geq 0$ *satisfies* $\mathbf{D}^T \mu = \mu^1$ *and* $\mathbf{1}^T \mu = 1$.

Proof: Follows from Corollary 3.5 and Lemma 3.7 in [17]. ∎

Lemma 3.2: *If directed graph $\mathcal{G}$ has a directed spanning tree, then $\rho(\mathbf{D} - \mathbf{1}\mu^T) < 1$, where $\mathbf{D}$ and μ are defined in Lemma 3.1, and $\rho(\cdot)$ denotes the spectral radius of a matrix.*

Proof: Following part (f) of Lemma 8.2.7 in [5], we know that every nonzero eigenvalue of $\mathbf{D} - \mathbf{1}\mu^T$ is also an eigenvalue of $\mathbf{D}$. Since one is an eigenvalue of $\mathbf{D}$ with algebraic multiplicity one, we know that one is not an eigenvalue of $\mathbf{D} - \mathbf{1}\mu^T$ following part (g) of Lemma 8.2.7 in [5]. Also every eigenvalue of $\mathbf{D}$ other than one has modulus less than one. Combining above gives $\rho(\mathbf{D} - \mathbf{1}\mu^T) < 1$. ∎

Note that the solution to (3) is given by

$$\xi[k] = \mathbf{D}^k \xi[0] + \sum_{i=1}^{k} \mathbf{D}^{i-1} v[k-i], \quad \forall k \in \mathbb{N}^+. \quad (6)$$

[1]That is, μ is a left eigenvector of $\mathbf{D}$ associated with the eigenvalue one.

Consider the update scheme

$$\vartheta[k+1] = \vartheta[k] + \mu^T v[k], \quad (7)$$

where $\vartheta \in \mathbb{R}$, $\vartheta[0] = \mu^T \xi[0]$, and μ is defined in Lemma 3.1. The solution to (7) is given by $\vartheta[k] = \vartheta[0] + \sum_{i=1}^{k} \mu^T v[k-i]$, $\forall k \in \mathbb{N}^+$. Here ϑ denotes the consensus equilibrium $\bar{\xi}$ (i.e., $\bar{\xi} = \vartheta$). In the special case where $v_i[k] \equiv 0$ in (2), $\bar{\xi} = \vartheta = \sum_{i=1}^{n} \mu_i \xi_i(0)$, where μ_i is i^{th} component of μ, if $\mathcal{G}$ has a directed spanning tree. Defining $\tilde{\xi}_i = \xi_i - \bar{\xi}$ according to (4), gives

$$\tilde{\xi}[k] = (\mathbf{D}^k - \mathbf{1}\mu^T)\xi[0] + \sum_{i=1}^{k}(\mathbf{D}^{i-1} - \mathbf{1}\mu^T)v[k-i], \quad (8)$$

where $\tilde{\xi} = [\tilde{\xi}_1, \ldots, \tilde{\xi}_n]^T$. Note that $\xi_i[k] - \xi_j[k] = \tilde{\xi}_i[k] - \tilde{\xi}_j[k]$. If $|\tilde{\xi}_i[k] - \tilde{\xi}_j[k]| \to 0$, $\forall (i, j) \in \mathcal{I}$ as $k \to \infty$, then consensus is reached using discrete-time scheme (2).

Theorem 3.1. *Given discrete-time scheme (2), assume that directed graph $\mathcal{G}$ has a directed spanning tree. If $v_1[k] = \cdots = v_n[k] = v_*[k]$, $\forall k \in \mathbb{N}$, then $\xi_i[k] \to \mu^T \xi[0] + \sum_{j=1}^{k} v_*[k-j]$ and $\tilde{\xi}_i[k] \to 0$, $\forall i \in \mathcal{I}$, as $k \to \infty$. If $\|v[k] - \mathbf{1}(\min_\ell v_\ell[k])\|$ is uniformly bounded, so is $\|\tilde{\xi}[k]\|$.*

Proof: For the first statement of the lemma, we know that $\mathbf{D}^k \to \mathbf{1}\mu^T$ as $k \to \infty$ and $\mu^T \mathbf{1} = 1$ from Lemma 3.1. Note that $v[k] = \mathbf{1}v_*[k]$. Also note that $\mathbf{D}^i \mathbf{1} = \mathbf{1}$, $\forall i \in \mathbb{N}$, since $\mathbf{1}$ is an eigenvector of $\mathbf{D}$ and therefore an eigenvector of $\mathbf{D}^i$, $\forall i \in \mathbb{N}$, associated with eigenvalue one. From (6), we can see $\lim_{k \to \infty} \xi[k] \to \mathbf{1}\mu^T \xi(0) + \sum_{j=1}^{k} \mathbf{D}^{i-1} \mathbf{1}v_*[k-j] = \mathbf{1}\mu^T \xi(0) + \mathbf{1}\sum_{j=1}^{k} v_*[k-j]$. Therefore, consensus can be achieved with the consensus equilibrium $\bar{\xi} = \mu^T \xi(0) + \sum_{j=1}^{k} v_*[k-j]$ if each instantiation of the virtual structure state is driven by a common input $v_*[k]$.

For the second statement of the lemma, noting that $(\mathbf{D}^{i-1} - \mathbf{1}\mu^T)\mathbf{1} = 0$, we get

$$\|\tilde{\xi}[k]\| = \left\| (\mathbf{D}^k - \mathbf{1}\mu^T)\xi[0] + \sum_{i=1}^{k}(\mathbf{D}^{i-1} - \mathbf{1}\mu^T) \right.$$
$$\left. \times (v[k-i] - \mathbf{1}(\min_\ell v_\ell[k-i])) \right\|$$
$$\leq \|\mathbf{D}^k - \mathbf{1}\mu^T\| \|\xi[0]\|$$
$$+ \left(\sup_{i=0,\ldots,k-1} \left\| v[i] - \mathbf{1}(\min_\ell v_\ell[i]) \right\| \right)$$
$$\times \sum_{i=1}^{k} \|\mathbf{D}^{i-1} - \mathbf{1}\mu^T\|.$$

Noting that $\mathbf{D}^i - \mathbf{1}\mu^T = (\mathbf{D} - \mathbf{1}\mu^T)^i$ for $i \in \mathbb{N}^+$ following part (e) in Lemma 8.2.7 in [5]. Also note that $\rho(\mathbf{D} - \mathbf{1}\mu^T) < 1$ following Lemma 3.2. Therefore, there exists a matrix norm $\||| \cdot |||$ such that $\|||\mathbf{D} - \mathbf{1}\mu^T||| \leq r < 1$ from Lemma 5.6.10 in [5], which implies that $\|||\mathbf{D}^i - \mathbf{1}\mu^T||| = \||| (\mathbf{D} - \mathbf{1}\mu^T)^i ||| \leq r^i < 1$. Since every vector norm is equivalent on a finite Euclidean space, there exists a positive constant d_M such that $\|\cdot\| \leq d_M \||| \cdot |||$. As a result, we see that $\|\mathbf{D}^k - \mathbf{1}\mu^T\|$ is bounded, $\forall k \in \mathbb{N}^+$. Following the same reason as above, it is straightforward to see that

$$\sum_{i=1}^{k} \|\mathbf{D}^{i-1} - \mathbf{1}\mu^T\|$$

$$= \|\mathbf{I}_n - \mathbf{1}\mu^T\| + \sum_{i=2}^{k} \|\mathbf{D}^{i-1} - \mathbf{1}\mu^T\|$$

$$\leq \|\mathbf{I}_n - \mathbf{1}\mu^T\| + d_M \sum_{i=2}^{k} \||| (\mathbf{D} - \mathbf{1}\mu^T)^{i-1} |||$$

$$\leq \|\mathbf{I}_n - \mathbf{1}\mu^T\| + d_M r/(1-r).$$

Combining the above arguments, we see that if $\|\mathbf{v}[k] - \mathbf{1}(\min_\ell v_\ell[k])\|$ is uniformly bounded, so is $\|\tilde{\xi}[k]\|$. ∎

Corollary 3.2: *Let* $\mathbf{b} = [b_1, \ldots, b_n]^T$, *where* $b_i \in \mathbb{R}$, $\forall i \in \mathcal{I}$, *are arbitrary constants. Given discrete-time scheme (2), if directed graph* $\mathcal{G}$ *has a directed spanning tree and* $\mathbf{v}[k] = \mathbf{b}$, $\forall k \in \mathbb{N}$, *then* $\|\tilde{\xi}[k]\|$ *is uniformly bounded and* $\tilde{\xi}[k] \rightarrow (\mathbf{I}_n - \mathbf{1}\mu^T)\mathbf{b} + (\mathbf{D} - \mathbf{1}\mu^T) \times [\mathbf{I}_n - (\mathbf{D} - \mathbf{1}\mu^T)]^{-1}$ *as* $k \rightarrow \infty$, *that is, the inconsistency of different instantiations of the virtual structure state approaches a constant value.*

Proof: The uniform boundedness of $\tilde{\xi}[k]$ follows Theorem 3.1 since $\mathbf{v}[k]$ is a constant vector $\forall k \in \mathbb{N}$. When $\mathbf{v}[k] = \mathbf{b}$, (8) can be rewritten as

$$\tilde{\xi}[k] = (\mathbf{D}^k - \mathbf{1}\mu^T)\xi(0) + (\mathbf{I}_n - \mathbf{1}\mu^T)\mathbf{b}$$

$$+ \sum_{i=1}^{k-1} (\mathbf{D}^i - \mathbf{1}\mu^T)\mathbf{b}.$$

From Lemma 8.6.1 in [5], we know that $\sum_{i=1}^{k-1}(\mathbf{D}^i - \mathbf{1}\mu^T) = (\mathbf{D} - \mathbf{1}\mu^T)(\mathbf{I}_n - \mathbf{D}^{k-1} + \mathbf{1}\mu^T)[\mathbf{I}_n - (\mathbf{D} - \mathbf{1}\mu^T)]^{-1}$. The above limit as $k \rightarrow \infty$ then directly follows the fact that $\mathbf{D}^k \rightarrow \mathbf{1}\mu^T$ as $k \rightarrow \infty$. ∎

Note that although each $\xi_i[k]$, $i \in \mathcal{I}$, may become unbounded as $k \rightarrow \infty$ when driven by an input, their inconsistency is guaranteed to be bounded by the above analyses. Also note that if $|v_i[k]|$ is uniformly bounded for each $i \in \mathcal{I}$, the condition that $\|\mathbf{v}[k] - \mathbf{1}(\min_\ell v_\ell[k])\|$ is uniformly bounded is trivially satisfied. If the virtual structure state evolves
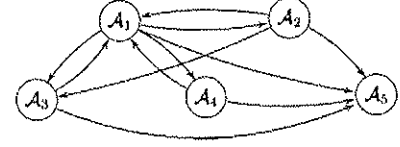

Fig. 4. Communication topology for five vehicles. An edge from $\mathcal{A}_j$ to $\mathcal{A}_i$ denotes that $g_{ij} = 1$.
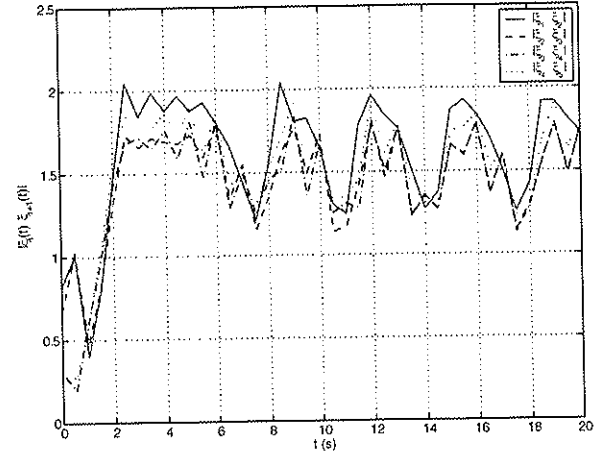

Fig. 5. Consensus with $v_i[k] = \sin(2\xi_i[k])$, $i = 1, \ldots, 5$.

according to some nonlinear dynamics $f(k, \xi_i[k], u[k])$, where u is the common exogenous input, then we can let $v_i[k] = f(k, \xi_i[k], u[k])$ in (2). In this case, consensus is not guaranteed to be achieved in general although a similar analysis to Theorem 3.1 guarantees that $\|\tilde{\xi}[k]\|$ is uniformly bounded if $|f(k, \xi_i[k], u[k]) - \min_\ell f(k, \xi_\ell[k], u[k])|$ is uniformly bounded, $\forall i \in \mathcal{I}$. Consider an example for five vehicles with a communication topology given by Fig. 4. We assume that $v_i[k] = \sin(2\xi_i[k])$ and $\alpha_{ij} = 1$, where $(i, j) = 1, \ldots, 5$. Fig. 5 shows $|\xi_i - \xi_{i+1}|$, $i = 1, \ldots, 4$. We can see that consensus is not achieved but the differences between ξ_i are uniformly bounded since $|\sin(2\xi_i[k])|$ is uniformly bounded.

3.4. Analysis for Switching Interaction Topologies

In the case of switching interaction topologies, matrix $\mathbf{D}$ is time-varying. Unlike the case of a fixed interaction topology, we generally cannot explicitly define the consensus equilibrium $\tilde{\xi}$ for the case of switching interaction topologies. Instead, we will provide qualitative boundedness analyses for the inconsistency between different instantiations of the virtual structure state. Define $\hat{\xi}_{ij} = \xi_i - \xi_j$ according to (5). Define $\hat{v}_{ij} = v_i - v_j$. Also define $\hat{\xi}$ and $\hat{v}$ as column vectors composed of each $\hat{\xi}_{ij}$ and $\hat{v}_{ij}$ respectively when $i \neq j$.

Lemma 3.3: *[17,19] Given discrete-time scheme (2), ξ_i achieves consensus under directed switching interaction topologies $\mathcal{G}[k]$ if there exist infinitely many, consecutive, uniformly bounded, time intervals such that the union of the directed interaction topologies across each such interval has a directed spanning tree.*

Theorem 3.3: *Given discrete-time scheme (2), assume that the conditions in Lemma 3.3 are satisfied. If $v_1[k] = \ldots = v_n[k] = v_*[k]$, then $|\hat{\xi}_{ij}[k]| \to 0$ as $k \to \infty$, $\forall i \neq j$. If $\|\hat{v}[k]\|$ is uniformly bounded, so is $|\hat{\xi}_{ij}[k]|$, $\forall i \neq j$.*

Proof: The first statement of the lemma follows a similar argument to that of Theorem 3.1.

For the second statement, following the structure of (2), we have the equation

$$\hat{\xi}[k+1] = \mathbf{D}[k]\hat{\xi}[k] + \hat{v}[k], \tag{9}$$

where $\mathbf{D}[k]$ is a time-varying matrix. When $v[k] = 0$, the solution to (2) is given by $\xi[k] = \Phi[k]\xi[0]$, where $\Phi[k] = \mathbf{D}[k-1]\mathbf{D}[k-2]\ldots\mathbf{D}[0]$. Noting that each $\mathbf{D}[j]$ is a stochastic matrix and the product of stochastic matrices is also a stochastic matrix, we know that $\Phi[k]$ is a stochastic matrix. As a result, we know that $\xi_i[k] = \sum_{j=1}^{n} \beta_{ij}[k]\xi_j[0]$, where $\beta_{ij}[k] \geq 0$ and $\sum_{j=1}^{n} \beta_{ij}[k] = 1$. Therefore, we know that $\max_j \xi_j[k] \leq \max_j \xi_j[0]$ and $\min_j \xi_j[k] \geq \min_j \xi_j[0]$, which in turn implies that $\max_{i,j} |\hat{\xi}_{ij}[k]| \leq \max_{i,j} |\hat{\xi}_{ij}[0]|$. Therefore, we see that $\|\hat{\xi}[k]\|_{\infty} \leq \|\hat{\xi}[0]\|_{\infty}$, $\forall k \in \mathbb{N}^+$, which implies that (9) is uniformly stable when $v[k] = 0$. From Lemma 3.3, we know that $\hat{\xi}[k] \to 0$ when $v[k] = 0$. Combining the above arguments, we know that (9) is uniformly asymptotically stable when $v[k] = 0$, which in turn implies that (9) is uniformly exponentially stable (see [20]). Following [20], we show that if $\|\hat{v}[k]\|$ is uniformly bounded, so is $\|\hat{\xi}[k]\|$. Therefore, we know that $|\hat{\xi}_{ij}[k]|$, $\forall i \neq j$ is uniformly bounded if $\|\hat{v}[k]\|$ is uniformly bounded. ∎

Note that if $|v_i[k]|$ is uniformly bounded for each $i \in \mathcal{I}$, the condition that $\|\hat{v}[k]\|$ is uniformly bounded is trivially satisfied. Also note that periodic packet losses are naturally considered in the case of directed switching interaction topologies.

4. Application to Formation Control

In this section, we apply the decentralized coordination framework to a multi-vehicle formation control scenario where 25 mobile robots need to preserve a formation shape when performing formation maneuvers.

The kinematic equations of a holonomic mobile robot are

$$\dot{\mathbf{z}}_i = \mathbf{u}_i, \quad i = 1, \ldots, 25 \tag{10}$$

where $\mathbf{z}_i = [x_i, y_i]^T$ represents the position of the ith robot, and $\mathbf{u}_i = [u_{xi}, u_{yi}]^T$ represents the control input. Note that (10) can also denote the kinematics for a nonholonomic mobile robot after feedback linearization for a fixed point off the center of the wheel axis [13]. While feedback linearization will result in loss of robot orientation information in the case of nonholonomic mobile robots, the virtual structure approach is sufficiently general to tackle the case without feedback linearization by employing tracking control laws that account for nonholonomic constraints. Note that although we use very simple robot kinematics, the decentralized coordination framework is applicable to formation control of vehicles with complicated dynamics.

For our tests, the 25-robot team is required to preserve a desired formation shape as shown in Fig. 6, where squares represent the desired positions of each robot and the two perpendicular arrows located at the geometric center of the formation denote formation frame C_F. In Fig. 6 one robot is located at the origin of the formation frame while the others are uniformly distributed along circles centered at the origin with a radius of 20 m and 40 m, respectively.

Define the virtual structure state as $\xi(t) = [x_F(t), y_F(t), \theta_F(t)]^T$, where $(x_F(t), y_F(t))$ and $\theta_F(t)$ denote, respectively, the position and orientation of formation frame C_F. Given $\xi(t)$, the desired trajectory for each robot can in turn be defined as

$$\mathbf{z}_i^d(t) = \begin{bmatrix} x_F(t) \\ y_F(t) \end{bmatrix} + \begin{bmatrix} \cos(\theta_F(t)) & -\sin(\theta_F(t)) \\ \sin(\theta_F(t)) & \cos(\theta_F(t)) \end{bmatrix} \begin{bmatrix} x_{iF}^d(t) \\ y_{iF}^d(t) \end{bmatrix},$$
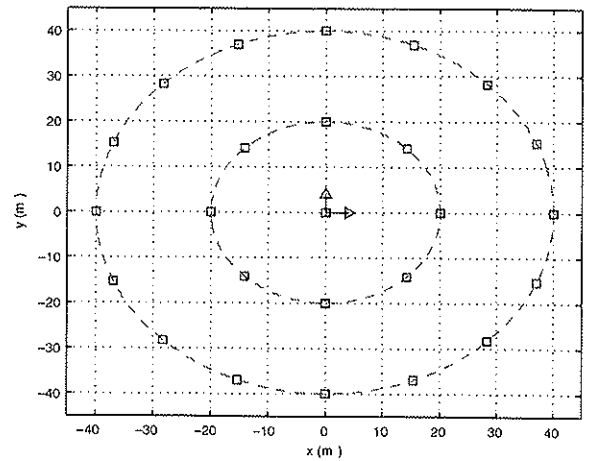

Fig. 6. 25-robot virtual structure with central formation frame C_F.

where $(x_{iF}^d(t), y_{iF}^d(t))$ is the specified desired deviation of each robot from the formation center. To further simplify the problem, we parameterize the virtual structure state by $s \in \mathbb{R}$, which is a function of time. As a result, the virtual structure state can be defined as $\xi(s(t)) = [x_F(s(t)), y_F(s(t)), \theta_F(s(t))]^T$. In this case, we can certainly instantiate ξ on each vehicle, denoted as ξ_i, and apply consensus strategy (2) to guarantee that each instantiation comes into consensus. However, noting that parameter s represents the information needed by each robot to coordinate its motion with the group, we can also drive s_i (i.e., each instantiation of s) into consensus via intervehicle communications. By doing so, the amount of information that needs to be communicated between vehicles is reduced from $\xi_i \in \mathbb{R}^3$ to $s_i \in \mathbb{R}$.

By applying (2) to s_i, we get

$$s_i[k+1] = \frac{1}{\sum_{j=1}^n \alpha_{ij}[k]g_{ij}[k]} \sum_{j=1}^n \alpha_{ij}[k]g_{ij}[k]$$
$$\times (s_j[k] + w_{ij}[k]) + \lambda_i,$$

where $\alpha_{ij}[k]$ is chosen as arbitrary positive constants, $w_{ij}[k]$ denotes the communication noise associated with the communication channel from robot j to robot i, λ_i is the input, and $g_{ij}[k]$ is 1 if information flows from robot j to robot i and 0 otherwise. Note that $\lambda_i + \frac{1}{\sum_{j=1}^n \alpha_{ij}[k]g_{ij}[k]} \sum_{j=1}^n \alpha_{ij}[k]g_{ij}[k]w_{ij}[k]$ corresponds to v_i in (2). In the simulation, we choose the sample period as $T_s = 0.5$ sec. Then each robot i can track its desired states specified by its parameter instantiation s_i based on a simple tracking law

$$\mathbf{u}_i = \dot{\mathbf{z}}_i^d(s_i) - \gamma_i(\mathbf{z}_i - \mathbf{z}_i^d(s_i)),$$

where $\gamma_i > 0$. Euler approximation is used to compute $\dot{\mathbf{z}}_i^d$ in the simulation.

We simulate the case where formation frame C_F follows a trajectory of a circle with radius 200 m. In the simulation, we let $x_F(s) = 200\cos(\frac{2\pi}{S}s)$, $y_F(s) = 200\sin(\frac{2\pi}{S}s)$, and $\theta_F(s) = \frac{2\pi}{S}s$, where S specifies the period of the desired trajectory for the formation frame in terms of parameter s. That is, when s evolves from 0 to S^2, the trajectory of the formation frame completes one cycle. We assume that each robot has a communication range of 30 m. Taking into account random communication packet losses, we assume that the ith robot can obtain information from the jth robot but not vice versa at a certain time. That is, the communication topology is generally bidirectional but may be sporadically unidirectional over one or more time steps. Specifically, we assume that there are 20% communication packet losses for any existing

[2] Not necessarily $t \in [0, S]$ since s is a function of t.

communication link. We also assume that each robot has limited control authority such that $|u_{xi}| \le 1$, $|u_{yi}| \le 1$. In the following, we assume that there is no collision between robots.

Table 1 shows parameter values used in three test cases. In Cases 1 and 2, each s_i is driven by a common exogenous input in the presence of communication noise. In Case 3, each s_i is driven by inputs with bounded inconsistency, particularly a nonlinear signal representing group feedback information, in the presence of communication noise.

Fig. 7 shows formation maneuvers of the 25 robots with each instantiation of parameter s driven by the same input $\lambda_i = \frac{T_s}{4}$ at $t = 0$, 400, 800, and 1200 sec, respectively, in Case 1. The green circle represents the desired trajectory of the formation center, square vertices denote the actual locations of each robot, and star vertices denote the desired locations of each robot. Fig. 8 shows consensus of s_i with random communication noise in Case 1. We can see that the differences between the instantiations are bounded. Fig. 9 shows the tracking errors and formation keeping errors with random communication noise in Case 1. Here $\text{dist}(\mathbf{a}, \mathbf{b})$ is defined as $\|\mathbf{a} - \mathbf{b}\|$. Due to the fact that the formation center evolves at a relatively low speed ($S = 500$ sec), the formation is preserved well even if there exists communication noise and each robot has control limitation.

Table 1. Parameter values used in Case $i, i = 1, 2, 3$.

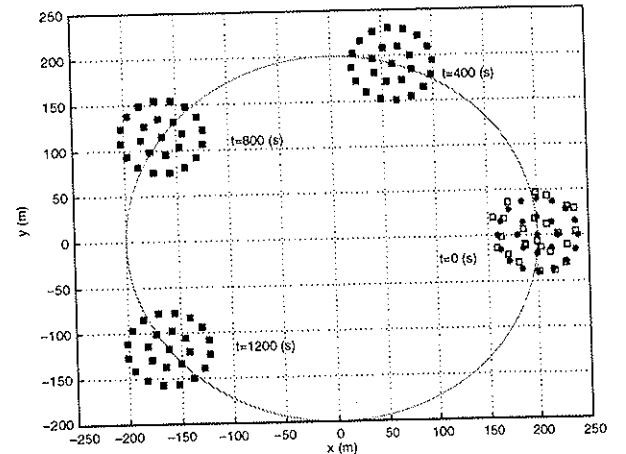| Case Number	Parameter Values
Case 1	$S = 500$ sec, $\lambda_i = \frac{T_s}{4}$
Case 2	$S = 300$ sec, $\lambda_i = \frac{T_s}{4}$
Case 3	$S = 300$ sec, $\lambda_i = \frac{T_s}{4}\eta(\|z_i - z_i^d\|)$

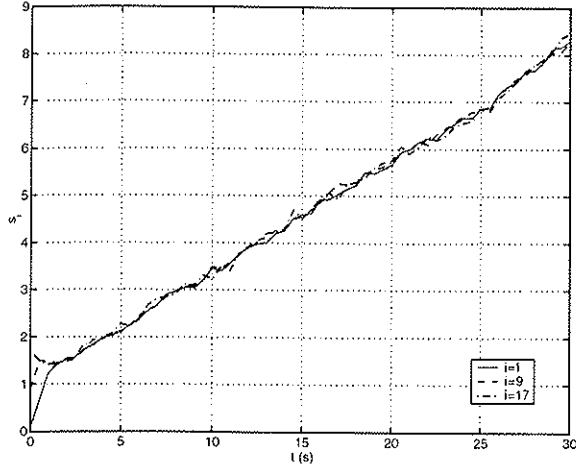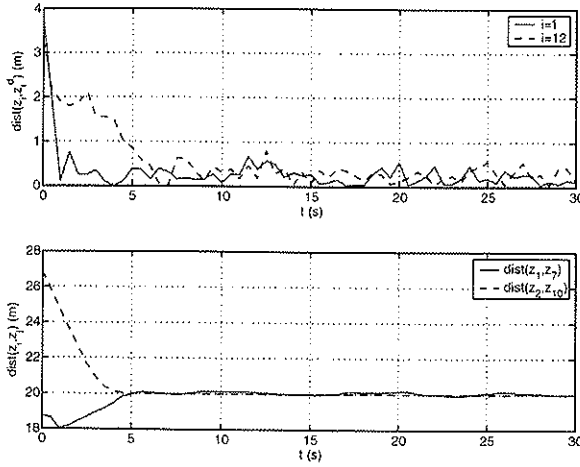**Fig. 7.** Virtual structure maintenance for Case 1.

Fig. 8. Consensus of s_i with random communication noise in Case 1.

Fig. 9. Tracking errors and formation keeping errors with random communication noise in Case 1.

Fig. 10. Consensus of s_i with random communication noise in Case 2.

Fig. 11. Tracking errors and formation keeping errors with random communication noise in Case 2.

In Case 2, each instantiation of parameter s is also driven by $\lambda_i = \frac{T_s}{4}$. However, in this case the formation center evo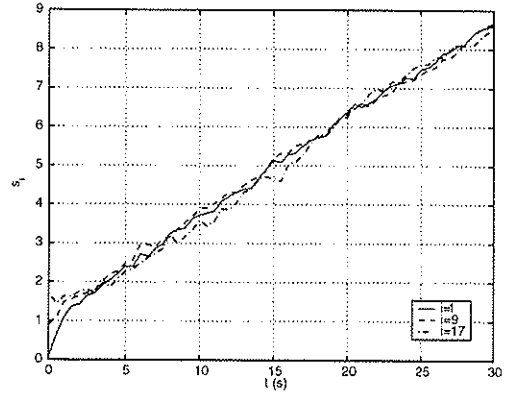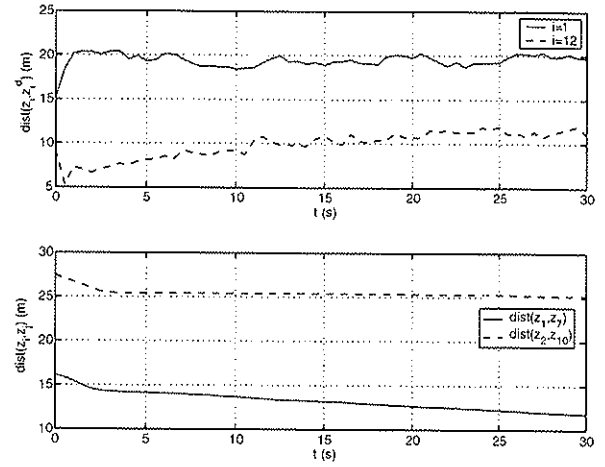lves at a higher speed ($S = 300$ sec). Fig. 10 shows consensus of s_i with random communication noise in Case 2. Fig. 11 shows the tracking errors and formation keeping errors with random communication noise in Case 2. Compared to Case 1, formation is not preserved well due to the limited control authority of each robot to track trajectories evolving relatively fast.

In Case 3, we replace the common exogenous input with group feedback from each vehicle to its instantiation of parameter s by defining $\lambda_i = \frac{T_s}{4}\eta(\|\mathbf{z}_i - \mathbf{z}_i^d\|)$, where $\eta(\cdot)$ is defined in such a way that $\eta(\|\mathbf{z}_i - \mathbf{z}_i^d\|) = 1$ if $\|\mathbf{z}_i - \mathbf{z}_i^d\| \leq \epsilon$ and $0 < \eta(\|\mathbf{z}_i - \mathbf{z}_i^d\|) < 1$ decreases as $\|\mathbf{z}_i - \mathbf{z}_i^d\|$ increases. As a result, if the tracking error for the ith robot is below a certain bound ϵ, the ith instantiation of parameter s evolves at its nominal speed. If the tracking error for the ith

robot exceeds the bound, the ith instantiation of parameter s evolves more slowly as the tracking error increases. In this paper, we simply define function $\eta(\cdot)$ as

$$
\eta(x) = \begin{cases} 1, & x \leq \epsilon \\ \dfrac{1}{1 + k(x - \epsilon)}, & x > \epsilon, \end{cases}
$$

where $\epsilon = 0.2$ and $k = 100$. Fig. 12 shows consensus of s_i with random communication noise in Case 3. Fig. 13 shows the tracking errors and formation keeping errors with random communication noise in Case 3. In this case, formation is preserved well even if S is also chosen to be 300 sec as in Case 2. By comparing Figs. 10 and 12, we can see that each s_i in Case 3 evolves more slowly than those in Case 2 due to the effect of group feedback. Note that there exists inconsistency between $\lambda_i = \frac{T_s}{4}\eta(\|z_i - z_i^d\|)$ due to the inconsistency between $z_i - z_i^d$. However, noting that $|\lambda_i|$ is bounded, we see that the inconsistency between
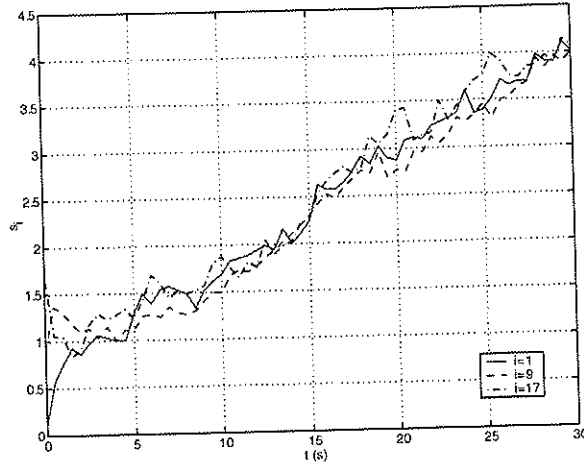
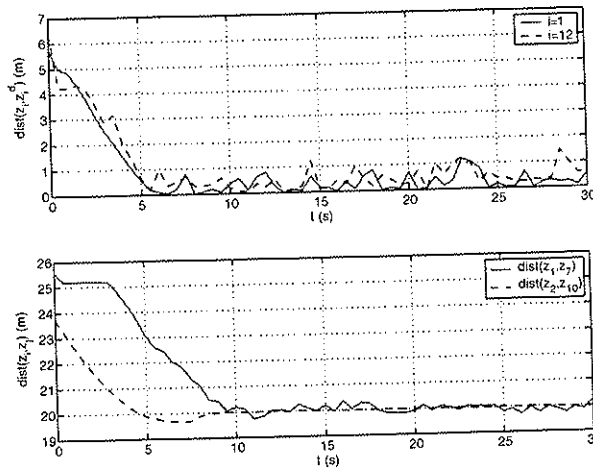**Fig. 12.** Consensus of s_i with random communication noise in Case 3.

Fig. 13. Tracking errors and formation keeping errors with random communication noise in Case 3.

λ_i is bounded since $|\lambda_i - \lambda_j| \leq |\lambda_i + \lambda_j|$. Therefore, according to Theorem 3.3, the inconsistency between s_i is bounded as shown in Fig. 12.

5. Conclusion

This paper has addressed the problem of decentralization of virtual structures in multi-vehicle systems via consensus strategies. Using discrete-time consensus schemes, we have shown conditions under which consensus can be achieved for each instantiation of the virtual structure state driven by a common input and performed boundedness analyses for the inconsistency between instantiations when there are inconsistent inputs under both fixed and switching interaction topologies. An application to robot formation maneuvering is presented to show the effectiveness of our results.

References

1. Beard RW, Lawton JR, Hadaegh FY. A coordination architecture for spacecraft formation control. *IEEE Trans Control Syst Technol*, 2001; 9(6): 777–790
2. Belta C, Kumar V. Abstraction and control for groups of robots, *IEEE Trans Robot Autom*, 2004; 20(5): 865–875
3. Egerstedt M, Hu X, Stotsky A. Control of mobile platforms using a virtual vehicle approach. *IEEE Trans Autom Control*, 2001; 46(11): 1777–1782
4. Fax JA, Murray RM. Information flow and cooperative control of vehicle formations. *IEEE Trans Autom Control*, 2004; 49(9): 1465–1476
5. Horn RA, Johnson CR. *Matrix Analysis*. Cambridge University Press, Cambridge, UK, 1985
6. Jadbabaie A, Lin J, Morse AS. Coordination of groups of mobile autonomous agents using nearest neighbor rules. *IEEE Trans Autom Control*, 2003; 48(6): 988–1001
7. Kang W, Xi N, Sparks A. Formation control of autonomous agents in 3D workspace. In *Proceedings of the IEEE International Conference on Robotics and Automation*, San Francisco, CA, 2000, pp 1755–1760
8. Leonard NE, Fiorelli E. Virtual leaders, artificial potentials and coordinated control of groups. In *Proceedings of the IEEE Conference on Decision and Control*, Orlando, Fl, 2001, pp. 2968–2973
9. Lewis MA, Tan K.-H. High precision formation control of mobile robots using virtual structures. *Auton Robots*, 1997; 4: 387–403
10. Moreau L. Stability of multi-agent systems with time-dependent communication links. *IEEE Trans Autom Control*, 2005; 50(2): 169–182
11. Ogren P, Egerstedt M, Hu X. A control Lyapunov function approach to multiagent coordination. *IEEE Trans Robot Autom*, 2002; 18(5): 847–851
12. Olfati-Saber R, Murray RM. Consensus problems in networks of agents with switching topology and time-delays. *IEEE Trans Autom Control*, 2004; 49(9): 1520–1533
13. Pomet J, Thuilot B, Bastin G, Campion G. A hybrid strategy for the feedback stabilization of nonholonomic mobile robots. In Proceedings of the IEEE International Conference on Robotics and Automation, Nice, France, 1992, pp. 129–134
14. Ren W. Decentralization of coordination variables in multi-vehicle systems. In Proceedings of the IEEE Conference on Networking, Sensing, and Control, Ft. Lauderdale, FL, 2006, pp. 550–555
15. Ren W, Beard RW. Decentralized scheme for spacecraft formation flying via the virtual structure approach. *J Guid Control Dyn*, 2004; 27(1): 73–82
16. Ren W, Beard RW. Formation feedback control for multiple spacecraft via virtual structures. *IEE Proc Control Theory Appl*, 2004; 151(3): 357–368

17. Ren W, Beard RW. Consensus seeking in multiagent systems under dynamically changing interaction topologies. *IEEE Trans Autom Control*, 2005; 50(5): 655–661

18. Ren W, Beard RW, Atkins EM. Information consensus in multivehicle cooperative control: Collective group behavior through local interaction. *IEEE Control Syst Mag*, 2007; 27(2): 71–82

19. Ren W, Beard RW, Kingston DB. Multi-agent Kalman consensus with relative uncertainty. In Proceedings of the American Control Conference, Portland, OR, 2005, pp. 1865–1870

20. Rugh WJ. *Linear System Theory, 2nd edn*. Prentice Hall, Englewood Cliffs, NJ, 1996

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