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Brief paper

Distributed containment control for Lagrangian networks with parametric uncertainties under a directed graph[☆]

Jie Mei[a,c], Wei Ren[b,1], Guangfu Ma[c]

[a] School of Mechanical Engineering & Automation, Harbin Institute of Technology Shenzhen Graduate School, Guangdong, 518055, PR China

[b] Department of Electrical Engineering, University of California, Riverside, CA, 92521, USA

[c] Department of Control Science & Engineering, Harbin Institute of Technology, Heilongjiang, 150001, PR China

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ABSTRACT

In this paper, we study the distributed containment control problem for networked Lagrangian systems with multiple dynamic leaders in the presence of parametric uncertainties under a directed graph that characterizes the interaction among the leaders and the followers. We propose a distributed adaptive control algorithm combined with distributed sliding-mode estimators. A necessary and sufficient condition on the directed graph is presented such that all followers converge to the dynamic convex hull spanned by the dynamic leaders asymptotically. As a byproduct, we show a necessary and sufficient condition on leaderless consensus for networked Lagrangian systems under a directed graph. Numerical simulation results are given to show the effectiveness of the proposed control algorithms.

1. Introduction

Recently, distributed coordination of multi-agent systems has gained much attention due to its broad applications, including consensus, flocking, and formation control. Many existing works in distributed coordination focus on the consensus problem when there is no leader. We refer the readers to Olfati-Saber, Fax, and Murray (2007) and Ren, Beard, and Atkins (2007) and references therein for more details. In reality, the presence of a single leader or multiple leaders can broaden the applications as a group objective can be encapsulated by the leader or the leaders. In the case where there exists one leader, Hong, Hu, and Gao (2006) studies the coordinated tracking problem with an active leader under the assumption that the leader's acceleration is known by all followers. In Cao and Ren (2012), the distributed coordinated tracking and swarm tracking problems are studied in the absence of velocity or acceleration measurements. Distributed sliding-mode estimators are proposed in Cao, Ren, and Meng (2010) to solve the finite-time formation tracking problem. In the case where there exist multiple leaders, Ji, Ferrari-Trecate, Egerstedt, and Buffa (2008) proposes a distributed containment control algorithm for agents with single-integrator dynamics such that a group of followers is driven to the convex hull spanned by multiple leaders under an undirected graph. The work of Ji et al. (2008) is extended in Cao, Stuart, Ren, and Meng (2011) to the case of a directed interaction graph and double-integrator dynamics and in Lou and Hong (2010) to the case of random switching topologies. Note that the above results focus on linear systems with single-integrator or double-integrator dynamics.

A class of mechanical systems including autonomous vehicles, robotic manipulators, and walking robots are Lagrangian systems. Therefore, distributed coordination of networked Lagrangian systems has many applications. Unfortunately, the results for single- and double-integrator dynamics cannot be directly applied to Lagrangian systems due to their inherent nonlinearity, especially when there exist parametric uncertainties. Recent work on coordination of networked Lagrangian systems focuses on the leaderless case (Chopra, Stipanovic, & Spong, 2008; Ren, 2009), the case with a single leader (Cheah, Hou, & Slotine, 2009; Chung & Slotine, 2009; Mei, Ren, & Ma, 2011; Spong & Chopra, 2007; Sun, Zhao, & Feng, 2007), and the case with multiple leaders (Dimarogonas, Tsiotras, & Kyriakopoulos, 2009; Meng, Ren, & You, 2010). In the leaderless case, a controller based on potential functions is

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E-mail addresses: meij.hit@gmail.com (J. Mei), ren@ee.ucr.edu (W. Ren), magf@hit.edu.cn (G. Ma).

[1] Tel.: +1 951 827 6204; fax: +1 951 827 2425.

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proposed in Chopra et al. (2008) for networked Lagrangian systems to achieve leaderless flocking. In Ren (2009), three distributed leaderless consensus algorithms are proposed for networked Lagrangian systems under an undirected graph. In the case of a single leader, output synchronization of networked Lagrangian systems is studied in Spong and Chopra (2007) under a passivity-based framework. Both fixed and switching graphs as well as communication delays are considered. Based on nonlinear contraction analysis, Chung and Slotine (2009) analyzes the stability of cooperative tracking control laws for multiple robotic manipulators. In Sun et al. (2007), a model-independent cross-coupled controller is proposed for position synchronization of multi-axis motions. In Mei et al. (2011), the distributed coordinated tracking problem for networked Lagrangian systems is solved in the presence of a dynamic leader, where the leader is a neighbor of only a subset of the followers and the followers have only local interaction. A region-based shape control scheme that utilizes potential functions is designed in Cheah et al. (2009) for a swarm of robots such that the robots move as a group inside a desired region while maintaining a minimum distance among themselves. In the case of multiple leaders, Dimarogonas et al. (2009) studies the distributed attitude containment control problem for multiple rigid bodies with multiple stationary leaders under an undirected graph. In Meng et al. (2010), the distributed finite-time containment control problem is studied for networked Lagrangian systems under the assumption that the interaction graph associated with the followers is undirected.

In this paper, we study the distributed containment control problem for networked Lagrangian systems with multiple dynamic leaders in the presence of parametric uncertainties under a directed graph that characterizes the interaction among the leaders and the followers by expanding on our preliminary work reported in Mei, Ren, and Ma (2011). The objective is that a team of followers modeled by Euler–Lagrange equations converge to the convex hull spanned by multiple dynamic leaders. The problem has many applications such as securing a group of followers in the area spanned by the leaders so that they can be away from dangerous sources outside the area. The case where there exists a single leader can be viewed as a special case. We propose a distributed adaptive control algorithm combined with distributed sliding-mode estimators. A necessary and sufficient condition on the directed graph is presented such that all followers converge to the dynamic convex hull spanned by the dynamic leaders asymptotically. As a byproduct, we show a necessary and sufficient condition on leaderless consensus for networked Lagrangian systems under a directed graph.

Comparison with existing work in the literature: In contrast to the containment control algorithms for first- and second-order linear dynamics (Cao et al., 2011; Ji et al., 2008; Lou & Hong, 2010), we study the nonlinear Lagrangian systems in the presence of parametric uncertainties. In contrast to the leaderless case or the case with a single leader for networked Lagrangian systems (Cheah et al., 2009; Chung & Slotine, 2009; Mei et al., 2011; Spong & Chopra, 2007; Sun et al., 2007), we consider the containment control problem with multiple leaders. In contrast to the rigid body attitude containment control problem in Dimarogonas et al. (2009) and the finite-time containment control problem for networked Lagrangian systems in Meng et al. (2010), we deal with the containment control problem for networked Lagrangian systems in the presence of parametric uncertainties under a directed graph.

Notations: Let $\mathbf{1}_m$ and $\mathbf{0}_m$ denote, respectively, the $m \times 1$ column vector of all ones and all zeros. Let $\mathbf{0}_{m\times n}$ denote the $m \times n$ matrix with all zeros and I_m denote the $m \times m$ identity matrix. For a point x and a set M, let $d(x, M) \triangleq \inf_{y\in M} \|x - y\|$ denote the distance between x and M. Throughout the paper, we use $\|\cdot\|$ to denote the Euclidean norm.

2. Background

2.1. Euler–Lagrange system

Suppose that there exist m followers, labeled as agents 1 to m, and $n - m$ ($n > m$) leaders labeled as agents $m + 1$ to n, in a team. The m followers are represented by Euler–Lagrange equations of the form

$$M_i(q_i)\ddot{q}_i + C_i(q_i, \dot{q}_i)\dot{q}_i + g_i(q_i) = \tau_i, \quad i = 1, \ldots, m, \tag{1}$$

where $q_i \in \mathbb{R}^p$ is the vector of generalized coordinates, $M_i(q_i) \in \mathbb{R}^{p\times p}$ is the symmetric positive-definite inertia matrix, $C_i(q_i, \dot{q}_i)\dot{q}_i \in \mathbb{R}^p$ is the vector of Coriolis and centrifugal torques, $g_i(q_i)$ is the vector of gravitational torque, and $\tau_i \in \mathbb{R}^p$ is the vector of control torque on the i th agent. We assume that the leaders' motions are independent of those of the followers.

Throughout the subsequent analysis we assume that the following assumptions hold (Kelly, Santibanez, & Loria, 2005; Spong, Hutchinson, & Vidyasagar, 2006):

(A1) Parameter Boundedness: For any i, there exist positive constants $k_{\underline{m}}$, $k_{\overline{m}}$, k_C, and k_{g_i} such that $0 < k_{\underline{m}} I_p \le M_i(q_i) \le k_{\overline{m}} I_p$, $\|C_i(x, y)\| \le k_C\|y\|$ for all vectors $x, y \in \mathbb{R}^p$, and $\|g_i(q_i)\| \le k_{g_i}$.
(A2) Skew symmetric property: $\dot{M}_i(q_i) - 2C_i(q_i, \dot{q}_i)$ is skew symmetric.
(A3) Linearity in the dynamic parameters: $M_i(q_i)x + C_i(q_i, \dot{q}_i)y + g_i(q_i) = Y_i(q_i, \dot{q}_i, x, y)\Theta_i$ for all vectors $x, y \in \mathbb{R}^p$, where $Y_i(q_i, \dot{q}_i, x, y)$ is the regressor and Θ_i is the constant parameter vector associated with the ith agent.

Remark 2.1. Assumptions (A1)–(A3) are three general properties for Euler–Lagrange systems. Examples include robot manipulators in joint space with unknown but constant masses, inertias and distances of the centers of mass of the links (Kelly et al., 2005; Spong et al., 2006), attitude dynamics of rigid bodies with unknown but constant inertias (Slotine & Li, 1991), and car-like robots with unknown mass and damping constants (Cheah et al., 2009), to name a few.

2.2. Graph theory

We use a directed graph to describe the network topology between the n agents. Let $\mathcal{G} \triangleq (\mathcal{V}, \mathcal{E})$ be a directed graph with the node set $\mathcal{V} \triangleq \{1, \ldots, n\}$ and the edge set $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$. An edge $(i, j) \in \mathcal{E}$ denotes that agent j can obtain information from agent i, but not vice versa. Here, node i is the parent node while node j is the child node. Equivalently, node i is a neighbor of node j. A directed path from node i to node j is a sequence of edges of the form $(i_1, i_2), (i_2, i_3), \ldots$, in a directed graph. A directed tree is a directed graph, where every node has exactly one parent except for one node, called the root, and the root has directed paths to every other node. A directed spanning tree of a directed graph is a direct tree that contains all nodes of the directed graph. A directed graph has a spanning tree if there exists a directed spanning tree as a subset of the directed graph.

The adjacency matrix $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{n\times n}$ associated with $\mathcal{G}$ is defined as $a_{ij} > 0$ if $(j, i) \in \mathcal{E}$, and $a_{ij} = 0$ otherwise. In this paper, self-edges are not allowed, *i.e.*, $a_{ii} = 0$. The (nonsymmetric) Laplacian matrix $\mathcal{L}_A = [l_{ij}] \in \mathbb{R}^{n\times n}$ associated with $\mathcal{A}$ and hence $\mathcal{G}$ is defined as $l_{ii} = \sum_{j=1,j\neq i}^{n} a_{ij}$ and $l_{ij} = -a_{ij}$, $i \neq j$.

Lemma 2.1 (*Ren & Beard, 2008*)**.** *Let $\mathcal{G}$ be a directed graph of order n and $\mathcal{L}_A \in \mathbb{R}^{n\times n}$ be the associated (nonsymmetric) Laplacian matrix. The following three statements are equivalent:*

(1) *The matrix $\mathcal{L}_A$ has a single zero eigenvalue and all other eigenvalues have positive real parts;*
(2) *$\mathcal{G}$ has a directed spanning tree;*

(3) *Given a system* $\dot{z} \triangleq -\mathcal{L}_A z$*, where* $z = [z_1, \ldots, z_n]^T$*, consensus is reached exponentially. In particular, for all* $i = 1, \ldots, n$*, and all* $z_i(0)$, $z_i(t) \to \sum_{i=1}^n p_i z_i(0)$ *exponentially as* $t \to \infty$*, where* $\mathbf{p} \triangleq [p_1, \ldots, p_n]^T$ *is a nonnegative left eigenvector of* $\mathcal{L}_A$ *associated with the zero eigenvalue satisfying* $\sum_{i=1}^n p_i = 1$.

For the n agents with $m (m < n)$ followers and $n - m$ leaders, we use $\mathcal{V}_F \triangleq \{1, \ldots, m\}$ and $\mathcal{V}_L \triangleq \{m+1, \ldots, n\}$ to denote, respectively, the follower set and the leader set. Let q_F and q_L be the column stack vectors of, respectively, $q_i, \forall i \in \mathcal{V}_F$, and $q_i, \forall i \in \mathcal{V}_L$. In this paper, we assume that the directed graph $\mathcal{G}$ satisfies the following assumption.

Assumption 2.2. For each of the m followers, there exists at least one leader that has a directed path to the follower.

2.3. Mathematic background

Definition 2.3 (Rockafellar, 1972)**.** Let $\mathcal{C}$ be a set in a real vector space $\mathcal{S} \subseteq \mathbb{R}^n$. The set $\mathcal{C}$ is convex if, for any x and y in $\mathcal{C}$, the point $(1-t)x + ty \in \mathcal{C}$ for any $t \in [0, 1]$. The *convex hull* for a set of points $X \triangleq \{x_1, \ldots, x_n\}$ in $\mathcal{S}$ is the minimal convex set containing all points in X. We use $\mathrm{Co}(X)$ to denote the convex hull of X. In particular, $\mathrm{Co}(X) \triangleq \{\sum_{i=1}^n \alpha_i x_i | x_i \in X, \alpha_i \geq 0, \sum_{i=1}^n \alpha_i = 1\}$.

Definition 2.4 (Berman & Plemmons, 1979)**.** Let $Z_n \subset \mathbb{R}^{n\times n}$ denote the set of all square matrices of dimension n with nonpositive off-diagonal entries. A matrix $A \in \mathbb{R}^{n\times n}$ is said to be a *nonsingular M-matrix* if $A \in Z_n$ and all eigenvalues of A have positive real parts.

Lemma 2.2 (Berman & Plemmons, 1979)**.** *A matrix* $A \in Z_n$ *is a nonsingular M-matrix if and only if* A^{-1} *exists and each entry of* A^{-1} *is nonnegative.*

We assume that the leaders have no neighbors. Therefore, the (nonsymmetric) Laplacian matrix $\mathcal{L}_A$ associated with $\mathcal{A}$ hence $\mathcal{G}$ can be written as

$$\mathcal{L}_A = \begin{bmatrix} L_1 & L_2 \\ \mathbf{0}_{(n-m)\times m} & \mathbf{0}_{(n-m)\times(n-m)} \end{bmatrix}, \tag{2}$$

where $L_1 \in \mathbb{R}^{m\times m}$ and $L_2 \in \mathbb{R}^{m\times(n-m)}$.

Lemma 2.3. *The matrix* L_1 *defined as in (2) is a nonsingular M-matrix if and only if Assumption 2.2 holds. In addition, if Assumption 2.2 holds, then each entry of* $-L_1^{-1}L_2$ *is nonnegative and all row sums of* $-L_1^{-1}L_2$ *equal to one.*

Proof. See Appendix A. □

Lemma 2.4 (Khalil, 2002)**.** *Consider the system*

$$\dot{x} = f(t, x, u), \tag{3}$$

where $f(t, x, u)$ *is continuously differentiable and globally Lipschitz in* (x, u)*, uniformly in* t*. If the unforced system* $\dot{x} = f(t, x, 0)$ *has a globally exponentially stable equilibrium point at the origin* $x = 0$*, then the system (3) is input-to-state stable.*

3. Distributed containment control with multiple dynamic leaders

In this section, we deal with the distributed containment control problem where the leaders have varying vectors of generalized coordinate derivatives. Suppose that the leaders' vectors of generalized coordinate derivatives and their first-order and second-order derivatives are all bounded. We will design a distributed control algorithm for (1) such that all followers converge to the convex hull spanned by the dynamic leaders.

Before moving on, we introduce the following auxiliary variables

$$\hat{\dot{q}}_{ri} \triangleq \hat{v}_i - \alpha \sum_{j\in\mathcal{V}_L\bigcup\mathcal{V}_F} a_{ij}(q_i - q_j), \tag{4}$$

$$\hat{\ddot{q}}_{ri} \triangleq \hat{a}_i - \alpha \sum_{j\in\mathcal{V}_L\bigcup\mathcal{V}_F} a_{ij}(\dot{q}_i - \dot{q}_j), \tag{5}$$

$$\hat{s}_i \triangleq \dot{q}_i - \hat{\dot{q}}_{ri} = \dot{q}_i - \hat{v}_i + \alpha \sum_{j\in\mathcal{V}_L\bigcup\mathcal{V}_F} a_{ij}(q_i - q_j), \quad i \in \mathcal{V}_F, \tag{6}$$

where α is a positive constant, a_{ij} is the (i, j)th entry of the adjacency matrix $\mathcal{A}$ associated with $\mathcal{G}$, and $\hat{v}_i$ (respectively $\hat{a}_i$) is the ith follower's estimate of its desired vector of generalized coordinate derivatives (respectively, accelerations) in the convex hull spanned by those of the dynamic leaders that will be designed later. We then propose the following distributed algorithm combined with distributed sliding-mode estimators

$$\tau_i = -K_i\hat{s}_i + Y_i(q_i, \dot{q}_i, \hat{\dot{q}}_{ri}, \hat{\ddot{q}}_{ri})\widehat{\Theta}_i, \tag{7a}$$

$$\dot{\hat{v}}_i = -\beta_1 \mathrm{sgn}\left[\sum_{j\in\mathcal{V}_F} a_{ij}(\hat{v}_i - \hat{v}_j) + \sum_{j\in\mathcal{V}_L} a_{ij}(\hat{v}_i - \dot{q}_j)\right] \tag{7b}$$

$$\dot{\hat{a}}_i = -\beta_2 \mathrm{sgn}\left[\sum_{j\in\mathcal{V}_F} a_{ij}(\hat{a}_i - \hat{a}_j) + \sum_{j\in\mathcal{V}_L} a_{ij}(\hat{a}_i - \ddot{q}_j)\right], \tag{7c}$$

$$\dot{\widehat{\Theta}}_i = -\Lambda_i Y_i^T(q_i, \dot{q}_i, \hat{\dot{q}}_{ri}, \hat{\ddot{q}}_{ri})\hat{s}_i, \quad i \in \mathcal{V}_F, \tag{7d}$$

where K_i and Λ_i are symmetric positive-definite matrices, β_1 and β_2 are positive constants, $\mathrm{sgn}(\cdot)$ is the signum function defined componentwise, $\widehat{\Theta}_i$ is the estimate of Θ_i, and $Y_i(q_i, \dot{q}_i, \hat{\dot{q}}_{ri}, \hat{\ddot{q}}_{ri})$ is defined as in (8).

Remark 3.1. The distributed discontinuous sliding-model estimators (7b) and (7c) are inspired by the finite-time coordinated tracking algorithm proposed in Cao et al. (2010). As shown in Lemma 3.1, $\hat{v}_i$ and $\hat{a}_i$, $i \in \mathcal{V}_F$, converge to some certain limits located in the convex hull spanned by the vectors of, respectively, the generalized coordinate derivatives and the generalized coordinate accelerations of the leaders in finite time. Actually, after some finite time, $\hat{v}_i$ and $\hat{a}_i$ become the final desired vectors of, respectively, the generalized coordinate derivatives and the generalized coordinate accelerations of the ith follower.

Remark 3.2. The auxiliary variable $\hat{s}_i$ is inspired by the sliding variable introduced in Slotine and Li (1991). The control algorithm (7a) is designed to drive the variable $\hat{s}_i$ to zero. Then on the sliding surface $\hat{s}_i = 0$, one can conclude that the followers converge to the convex hull spanned by the dynamic leaders asymptotically.

Lemma 3.1. *Suppose that Assumption 2.2 holds. Let* $q_d \triangleq [q_{d1}^T, \ldots, q_{dm}^T]^T = -(L_1^{-1}L_2 \otimes I_p)q_L$*, where* $q_{di} \in \mathbb{R}^p$.[2] *If* $\beta_1 > \|\ddot{q}_d\|$*, then* $\|\hat{v}_i(t) - \dot{q}_{di}(t)\| \to 0, \forall i \in \mathcal{V}_F$*, in finite time. Similarly, if* $\beta_2 > \|\dddot{q}_d\|$*, then* $\|\hat{a}_i(t) - \ddot{q}_{di}(t)\| \to 0, \forall i \in \mathcal{V}_F$*, in finite time.*

Proof. See Appendix B. □

Theorem 3.3. *Suppose that the leaders have varying vectors of generalized coordinate derivatives,* $\beta_1 > \|\ddot{q}_d\|$*, and* $\beta_2 > \|\dddot{q}_d\|$*, where* q_d *is defined in Lemma 3.1. Using (7) for (1),* $d\{q_i(t), Co[q_L(t)]\} \to 0$ *as* $t \to \infty, \forall i \in \mathcal{V}_F$*, for arbitrary initial conditions in the presence of parametric uncertainties if and only if Assumption 2.2 holds. Specifically,* $\|q_F(t) + (L_1^{-1}L_2 \otimes I_p)q_L(t)\| \to 0$ *as* $t \to \infty$.

[2] When Assumption 2.2 holds, it follows from Lemma 2.3 that L_1^{-1} exists. Therefore, q_d is well defined.

Proof (*Sufficiency*)**.** First, we show that for bounded initial values $q_i(0)$ and $\dot{q}_i(0)$, using the control algorithm (7) for (1), the states $q_i(t)$ and $\dot{q}_i(t)$, $\forall i \in \mathcal{V}_F$, will remain bounded in finite time. From (7b) and (7c), we can get that $\hat{v}_i(t)$ and $\hat{a}_i(t)$, $\forall i \in \mathcal{V}_F$, are bounded in finite time for bounded initial values $\hat{v}_i(0)$ and $\hat{a}_i(0)$. For bounded states q_i and $\dot{q}_i$, $\forall i \in \mathcal{V}_F$, we can get that $\hat{s}_i$, $\hat{\dot{q}}_{ri}$ and $\hat{\ddot{q}}_{ri}$, $\forall i \in \mathcal{V}_F$, are bounded. From Assumption (A3), it follows that

$$M_i(q_i)\hat{\ddot{q}}_{ri} + C_i(q_i, \dot{q}_i)\hat{\dot{q}}_{ri} + g_i(q_i) = Y_i(q_i, \dot{q}_i, \hat{\ddot{q}}_{ri}, \hat{\dot{q}}_{ri})\Theta_i, \quad i \in \mathcal{V}_F. \tag{8}$$

From Assumption (A1), we can get that $Y_i(q_i, \dot{q}_i, \hat{\ddot{q}}_{ri}, \hat{\dot{q}}_{ri})$ is bounded for bounded states q_i and $\dot{q}_i$, $\forall i \in \mathcal{V}_F$, and therefore, $\widehat{\Theta}_i(t)$ is bounded for bounded initial value $\widehat{\Theta}_i(0)$. Thus, we can get from (7a) that τ_i is bounded. Finally, from (1), for bounded q_i, $\dot{q}_i$, and τ_i, under Assumption (A1), we can get that $\ddot{q}_i$ is also bounded. Thus, we can conclude that for bounded initial values $q_i(0)$ and $\dot{q}_i(0)$, $q_i(t)$ and $\dot{q}_i(t)$, $\forall i \in \mathcal{V}_F$, remain bounded in finite time. Let

$$\dot{q}_{ri} \triangleq \dot{q}_{di} - \alpha \sum_{j \in \mathcal{V}_L \bigcup \mathcal{V}_F} (q_i - q_j), \tag{9}$$

and

$$s_i \triangleq \dot{q}_i - \dot{q}_{ri} = \dot{q}_i - \dot{q}_{di} + \alpha \sum_{j \in \mathcal{V}_L \bigcup \mathcal{V}_F} (q_i - q_j), \quad i \in \mathcal{V}_F, \tag{10}$$

where q_{di} is defined in Lemma 3.1. Under the condition of the theorem, using the sliding-mode estimators (7b) and (7c), we can get from Lemma 3.1 that $\hat{v}_i(t) \equiv \dot{q}_{di}(t)$ and $\hat{a}_i(t) \equiv \ddot{q}_{di}(t)$ when $t \geq \max\{T_1, T_2\} \triangleq T_0$. Therefore, $\hat{\dot{q}}_{ri}(t) \equiv \dot{q}_{ri}(t)$, $\hat{\ddot{q}}_{ri}(t) \equiv \ddot{q}_{ri}(t)$, and $\hat{s}_i(t) \equiv s_i(t)$, $\forall i \in \mathcal{V}_F$, when $t \geq T_0$. Let $\widetilde{\Theta}_i \triangleq \Theta_i - \widehat{\Theta}_i$. Also let s_F, $\dot{q}_r$, $\hat{\dot{q}}_r$, $\hat{\ddot{q}}_r$, $\widetilde{\Theta}$, Θ, and $\widehat{\Theta}$ be, respectively, the column stack vectors of s_i, $\dot{q}_{ri}$, $\hat{\dot{q}}_{ri}$, $\hat{\ddot{q}}_{ri}$, $\widetilde{\Theta}_i$, Θ_i, and $\widehat{\Theta}_i$, $\forall i \in \mathcal{V}_F$.

Hence, using (7) and (8), when $t \geq T_0$, the closed-loop system (1) can be written in a vector form as

$$M(q_F)\dot{s}_F = -C(q_F, \dot{q}_F)s_F - K_F s_F - Y(q_F, \dot{q}_F, \hat{\ddot{q}}_r, \hat{\dot{q}}_r)\widetilde{\Theta}, \tag{11}$$

where $M(q_F)$, $C(q_F, \dot{q}_F)$, $Y(q_F, \dot{q}_F, \hat{\ddot{q}}_r, \hat{\dot{q}}_r)$, and K_F are, respectively, the block diagonal matrices of $M_i(q_i)$, $C_i(q_i, \dot{q}_i)$, $Y_i(q_i, \dot{q}_i, \hat{\ddot{q}}_{ri}, \hat{\dot{q}}_{ri})$, and K_i, $\forall i \in \mathcal{V}_F$.

When $t \geq T_0$, consider the following Lyapunov function candidate

$$V(t) = \frac{1}{2}s_F^T M(q_F)s_F + \frac{1}{2}\widetilde{\Theta}^T \Xi \widetilde{\Theta}, \tag{12}$$

where Ξ is the block diagonal matrix of Λ_i^{-1}, $\forall i \in \mathcal{V}_F$. Taking the derivative of V along (11) gives that

$$\begin{aligned}\dot{V}(t) &= s_F^T M(q_F)\dot{s}_F + \frac{1}{2}s_F^T \dot{M}(q_F)s_F + \widetilde{\Theta}^T \Xi \dot{\widetilde{\Theta}} \\ &= -s_F^T K_F s_F,\end{aligned} \tag{13}$$

where we have used Assumption (A2) and (7d) to obtain (13). Because K_F is symmetric positive definite, we can get $\dot{V}(t) \leq 0$, which means that s_F and $\widetilde{\Theta}$ are bounded when $t \geq T_0$. Also note that $\dot{q}_d$ is bounded. Combining with the fact that for bounded initial values $q_i(0)$, $\dot{q}_i(0)$, $\widehat{\Theta}_i(0)$, $q_i(t)$, $\dot{q}_i(t)$, and $\widehat{\Theta}_i(t)$, $\forall i \in \mathcal{V}_F$, remain bounded in finite time, we can conclude that $s_F(t)$ and $\widetilde{\Theta}(t)$ are bounded for all $t \geq 0$. If Assumption 2.2 holds, it follows from Lemma 2.3 that L_1 is a nonsingular M-matrix, which implies that L_1^{-1} exists. Note that (10) can be written in a vector form as

$$\dot{\bar{q}}_F = -\alpha(L_1 \otimes I_p)\bar{q}_F + s_F, \tag{14}$$

where

$$\bar{q}_F \triangleq q_F + (L_1^{-1}L_2 \otimes I_p)q_L. \tag{15}$$

Because L_1 is a nonsingular M-matrix, it follows from Definition 2.4 that all eigenvalues of L_1 have positive real parts. It thus follows that when $s_F = \mathbf{0}_{mp}$, (14) is globally exponentially stable at the origin $\bar{q}_F = \mathbf{0}_{mp}$. We can conclude from Lemma 2.4 that (14) is input-to-state stable with respect to the input s_F and the state $\bar{q}_F$.

Because s_F is bounded, so is $\bar{q}_F$. It follows from (14) that $\dot{\bar{q}}_F$ is bounded. Because $\dot{q}_d$ is bounded, we can get from (9) that $\dot{q}_{ri}$, $\forall i \in \mathcal{V}_F$, is bounded. Differentiating (9), we can get that $\ddot{q}_{ri}$, $\forall i \in \mathcal{V}_F$, is bounded because $\ddot{q}_d$ is bounded. It also follows from (10) that $\dot{q}_F$ is bounded. Note that $\hat{\dot{q}}_r$ and $\hat{\ddot{q}}_r$ are bounded when $t \leq T_0$ and $\hat{\dot{q}}_r \equiv \dot{q}_r$ and $\hat{\ddot{q}}_r \equiv \ddot{q}_r$ when $t \geq T_0$. We can conclude that $\hat{\dot{q}}_r$ and $\hat{\ddot{q}}_r$ are bounded for all $t \geq 0$. Note from Assumption (A1) that $\|C_i(q_i, \dot{q}_i)\hat{\dot{q}}_{ri}\| \leq k_C\|\dot{q}_i\|\,\|\hat{\dot{q}}_{ri}\|$ and $\|g_i(q_i)\| \leq k_{g_i}$, $\forall i \in \mathcal{V}_F$. Therefore, both $\|C_i(q_i, \dot{q}_i)\hat{\dot{q}}_{ri}\|$ and $\|g_i(q_i)\|$ are bounded. Note that in (8), $M_i(q_i)$, $\hat{\ddot{q}}_{ri}$, $C_i(q_i, \dot{q}_i)\hat{\dot{q}}_{ri}$ and $g_i(q_i)$, $\forall i \in \mathcal{V}_F$, are all bounded. We conclude from (8) that $Y_i(q_i, \dot{q}_i, \hat{\ddot{q}}_{ri}, \hat{\dot{q}}_{ri})$ is bounded. Note again from Assumption (A1) that $\|C_i(q_i, \dot{q}_i)s_i\| \leq k_C\|\dot{q}_i\|\|s_i\|$, $\forall i \in \mathcal{V}_F$. From (11), we can get that $\dot{s}_F$ is bounded. By differentiating (13), we can see that $\ddot{V}(t)$ is bounded. Therefore, $\dot{V}(t)$ is uniformly continuous in time. From Barbalat's Lemma (Khalil, 2002), we can conclude that $\dot{V}(t) \to 0$ as $t \to \infty$, *i.e.*, $s_F(t) \to \mathbf{0}_{mp}$ as $t \to \infty$. Because (14) is input-to-state stable with respect to the input s_F and the state $\bar{q}_F$, we have that $\bar{q}_F(t) \to \mathbf{0}_{mp}$ as $t \to \infty$. As a result, it follows that $q_F(t) \to -(L_1^{-1}L_2 \otimes I_p)q_L$ and $\dot{q}_F \to \mathbf{0}_{mp}$ as $t \to \infty$. If Assumption 2.2 holds, it follows from Lemma 2.3 that each entry of $-L_1^{-1}L_2$ is nonnegative and each row of $-L_1^{-1}L_2$ has a sum equal to one. We then get from Definition 2.3 that $-(L_1^{-1}L_2 \otimes I_p)q_L$ is within the convex hull spanned by the dynamic leaders. It thus follows that $d[q_i(t), \mathrm{Co}(q_L)] \to 0$ as $t \to \infty$. This concludes the sufficiency part.

(*Necessity*) We prove the necessity part by contradiction. If Assumption 2.2 does not hold, there exists a subset of the followers who cannot receive any information from the leaders directly or indirectly. That is, the motions of these followers are independent of the states of the leaders. Therefore, these followers cannot always converge to the convex hull spanned by the dynamic leaders for arbitrary initial conditions. □

Remark 3.4. Note that a single leader is a special case of multiple leaders. Therefore, the distributed coordinated tracking problem with a single leader for networked Lagrangian systems is a special case of the distributed containment control problem. Thus the results in the current paper can be used to deal with the coordinated tracking problem and hence extend the work in Chung and Slotine (2009), Mei et al. (2011) and Spong and Chopra (2007) to a directed graph.

It is worth to mention that the distributed adaptive control algorithm (7) can deal with the general case where the leaders have varying vectors of generalized coordinate derivatives but rely on two discontinuous sliding-mode estimators. We next show that if the leaders have constant vectors of generalized coordinate derivatives, only one distributed continuous estimator is required for each follower. Define the following auxiliary variables

$$\dot{q}_{ri} \triangleq \hat{v}_i - \alpha \sum_{j \in \mathcal{V}_L \bigcup \mathcal{V}_F} a_{ij}(q_i - q_j), \tag{16}$$

$$s_i \triangleq \dot{q}_i - \dot{q}_{ri} = \dot{q}_i - \hat{v}_i + \alpha \sum_{j \in \mathcal{V}_L \bigcup \mathcal{V}_F} a_{ij}(q_i - q_j), \quad i \in \mathcal{V}_F, \tag{17}$$

where α is a positive constant, a_{ij} is defined as in (4), and $\hat{v}_i$ is the ith follower's estimate of its desired vector of generalized coordinate derivatives in the convex hull spanned by those of the leaders that will be designed later. In this case, we propose the following control algorithm for (1) in the presence of parametric uncertainties

$$\tau_i = -K_i s_i + Y_i(q_i, \ddot{q}_{ri}, \dot{q}_i, \dot{q}_{ri})\widehat{\Theta}_i, \tag{18a}$$

$$\dot{\hat{v}}_i = -\beta\left[\sum_{j\in\mathcal{V}_F} a_{ij}(\hat{v}_i - \hat{v}_j) + \sum_{j\in\mathcal{V}_L} a_{ij}(\hat{v}_i - \dot{q}_j)\right], \tag{18b}$$

$$\dot{\widehat{\Theta}}_i = -\Lambda_i Y_i^T(q_i, \ddot{q}_{ri}, \dot{q}_i, \dot{q}_{ri})s_i, \quad i \in \mathcal{V}_F, \tag{18c}$$

where β is a positive constant, and K_i, Λ_i, and $\widehat{\Theta}_i$ are defined as in (7). We next state the main result of containment control with multiple dynamic leaders that have constant vectors of generalized coordinate derivatives.

Corollary 3.5. *Suppose that the leaders have constant vectors of generalized coordinate derivatives. Using (18) for (1), $d\{q_i(t), Co[q_L(t)]\} \to 0, \forall i \in \mathcal{V}_F$, as $t \to \infty$ for arbitrary initial conditions in the presence of parametric uncertainties if and only if Assumption 2.2 holds. Specifically, $\|q_F(t) + (L_1^{-1}L_2 \otimes I_p)q_L(t)\| \to 0$ as $t \to \infty$.*

Furthermore, we show that if the leaders are stationary, no estimators are required. In this case, $\dot{q}_d$ defined in Lemma 3.1 equals to zero. Thus, (9) and (10) reduce to

$$\dot{q}_{ri} \triangleq -\alpha \sum_{\mathcal{V}_L \bigcup \mathcal{V}_F} a_{ij}(q_i - q_j), \tag{19}$$

$$s_i \triangleq \dot{q}_i + \alpha \sum_{\mathcal{V}_L \bigcup \mathcal{V}_F} a_{ij}(q_i - q_j), \quad i \in \mathcal{V}_F. \tag{20}$$

We propose the following distributed adaptive control algorithm for (1) in the presence of parametric uncertainties

$$\tau_i = -K_i s_i + Y_i(q_i, \dot{q}_i, \ddot{q}_{ri}, \dot{q}_{ri})\widehat{\Theta}_i, \tag{21a}$$

$$\dot{\widehat{\Theta}}_i = -\Lambda_i Y_i^T(q_i, \dot{q}_i, \ddot{q}_{ri}, \dot{q}_{ri})s_i, \quad i \in \mathcal{V}_F, \tag{21b}$$

where K_i, Λ_i, and $\widehat{\Theta}_i$ are defined as in (7).

Corollary 3.6. *Suppose that all leaders are stationary. Using (21) for (1), $d[q_i(t), Co(q_L)] \to 0$ and $\dot{q}_i \to \mathbf{0}_p$ as $t \to \infty, \forall i \in \mathcal{V}_F$, for arbitrary initial conditions in the presence of parametric uncertainties if and only if Assumption 2.2 holds. Specifically, $q_F(t) \to -(L_1^{-1}L_2 \otimes I_p)q_L$ as $t \to \infty$, that is, the final vectors of generalized coordinates of the followers are given by $-(L_1^{-1}L_2 \otimes I_p)q_L$.*

For the leaderless consensus problem for networked Lagrangian systems, we have the following result.

Corollary 3.7. *Suppose that $\mathcal{V}_L = \emptyset$.[3] Using (21) for (1), $\|q_i(t) - q_j(t)\| \to 0$ and $\dot{q}_i(t) \to \mathbf{0}_p$ as $t \to \infty$ for arbitrary initial conditions in the presence of parametric uncertainties if and only if the directed graph $\mathcal{G}$ associated with the n agents has a directed spanning tree.*

Proof (*Sufficiency*). Because $\mathcal{V}_L = \emptyset$, (20) can be written in a vector form as

$$\dot{q} = -\alpha(\mathcal{L}_A \otimes I_p)q + s, \tag{22}$$

where $\mathcal{L}_A \in \mathbb{R}^{n\times n}$ is the (nonsymmetric) Laplacian matrix associated with $\mathcal{G}$, and q and s are column stack vectors of q_i and s_i, $i = 1, \ldots, n$. Following the same steps as in the proof of Theorem 3.3, we can get that $s(t) \to \mathbf{0}_{np}$ as $t \to \infty$. For the linear system $\dot{q} = -\alpha(\mathcal{L}_A \otimes I_p)q$, if $\mathcal{G}$ has a directed spanning tree, then it follows from Lemma 2.1 that consensus is reached exponentially. Thus, there exists $\bar{q} = \sum_{i=1}^n p_i q_i(0)$, where p_i is defined in Lemma 2.1, such that $\mathbf{1}_n \otimes \bar{q}$ is a globally exponentially stable equilibrium point of $\dot{q} = -\alpha(\mathcal{L}_A \otimes I_p)q$. We can conclude from Lemma 2.4 that the system (22) is input-to-state stable with the input s and the state $q - \mathbf{1}_n \otimes \bar{q}$. Note that $s(t) \to \mathbf{0}_{np}$ as $t \to \infty$. We can conclude that $\|q_i(t) - q_j(t)\| \to 0$ and $\dot{q}_i(t) \to \mathbf{0}_p$ as $t \to \infty$.

(*Necessity*) The proof of the necessity part is the same as Theorem 3.3 and is omitted here. □

3 In this case, there does not exist a leader. Therefore, (21) becomes a leaderless consensus algorithm accounting for parametric uncertainties.

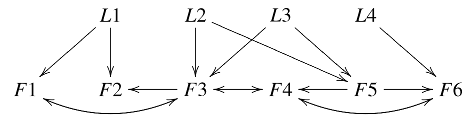

Fig. 1. The directed graph that characterizes the interaction among the four leaders and the six followers, where L_i, $i = 1, \ldots, 4$, denotes the ith leader and F_i, $i = 1, \ldots, 6$, denotes the ith follower.

Remark 3.8. In Corollary 3.7, we have shown that the adaptive control algorithm (21) can be used to deal with the leaderless consensus problem for networked Lagrangian system. Thus, Corollary 3.7 extends the first algorithm in Ren (2009) to a directed graph in the presence of parameter uncertainties.

Remark 3.9. In the current paper, we assume that the interaction graph among the followers and the leaders is fixed. As a result, the followers converge to some certain limits located in the convex hull spanned by the leaders. But the case where the interaction graph is time varying occurs commonly in real-world applications. In future work, it will be interesting to consider the distributed containment control problem for multiple nonlinear Lagrangian systems in the presence of parametric uncertainties under time-varying interaction graphs and introduce the strict containment-preserving mechanisms.

4. Simulation results

In this section, numerical simulations are performed to show the effectiveness of the proposed control algorithm. We consider the containment control problem for ten agents with four leaders and six followers. The dynamic equation of each follower is modeled by Cheah et al. (2009)

$$m_i\ddot{q}_i + \beta_i\dot{q}_i = \tau_i, \quad i = 1, \ldots, 6,$$

where $q_i \in \mathbb{R}^2$, and m_i and β_i represent, respectively, the mass and damping constants of the i follower, which are assumed to be constant but unknown.

Fig. 1 shows the directed graph that characterizes the interaction among the leaders and the followers. In our simulation, we choose $a_{ij} = 1$, $i = 1, \ldots, 6$, $j = 1, \ldots, 10$, if agent j is a neighbor of agent i, and $a_{ij} = 0$ otherwise. For simplicity, the dynamics of the followers are assumed to be the same, and we let $m_i = 1$, and $\beta_i = 0.5$, $i = 1, \ldots, 6$. Also let the initial positions of the six followers be, respectively, $[-4, 4]^T$, $[0, 6]^T$, $[4, 4]^T$, $[-4, -4]^T$, $[0, 6]^T$, and $[4, -4]^T$, and the initial velocities of the six followers be, respectively, $[5, -1]^T$, $[1, -3]^T$, $[-12, -8]^T$, $[10, 3]^T$, $[0, 6]^T$, and $[-7, 0]^T$. In the following, we call $\bar{q}_F = q_F + (L_1^{-1}L_2 \otimes I_2)q_L$ *the containment error vector* and let $\bar{q}_F = [\bar{q}_{F1}^T, \ldots, \bar{q}_{F6}^T]^T$, where $\bar{q}_{Fi} \in \mathbb{R}^2$. Let $\bar{q}_{Fi}^{(1)}$ and $\bar{q}_{Fi}^{(2)}$ denote, respectively, the first and the second component of $\bar{q}_{Fi}$, $\forall i = 1, \ldots, 6$.

Let the initial positions of the fours leaders be, respectively, $[-2, 2]^T$, $[2, 2]^T$, $[-2, -2]^T$, and $[2, -2]^T$, the initial velocities be identical, $[2, 4]^T$, and the accelerations be identical, $[0, -4\sin(t)]^T$. Here for better visualization of the plot, we have chosen identical accelerations and initial velocities for the leaders. However, the

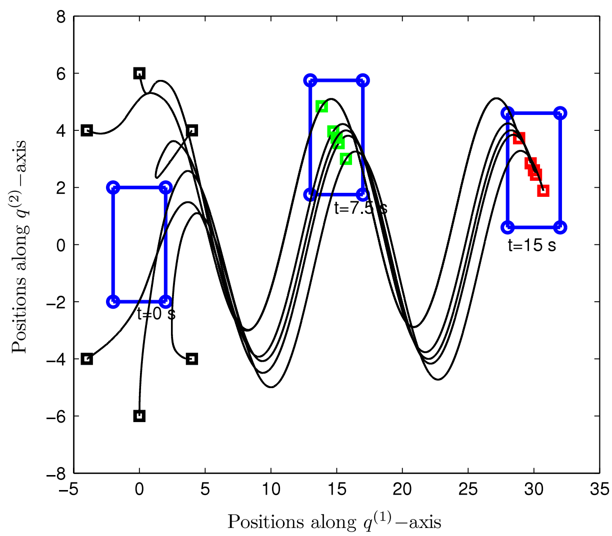

Fig. 2. Trajectories of the followers with four dynamic leaders that move with varying velocities using (7). The blue circles denote the four leaders, and the blue rectangle is the convex hull spanned by the leaders. The black, green, and red squares denote, respectively, the positions of the followers at, respectively, $t = 0$ s, $t = 7.5$ s, and $t = 15$ s, while the black lines are the trajectories of the followers. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

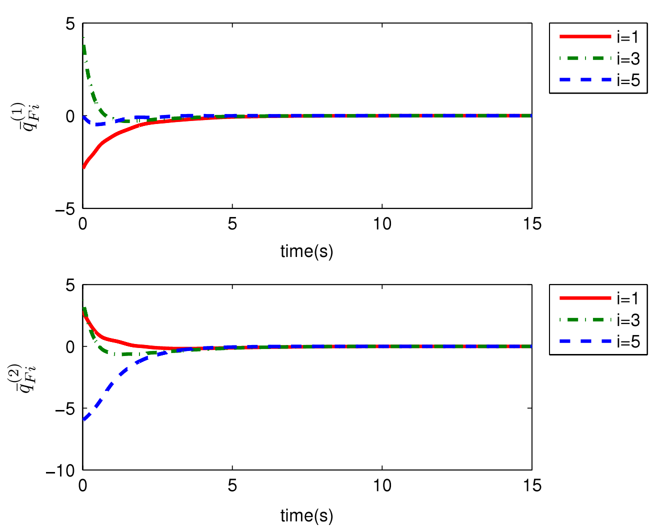

Fig. 3. The containment errors of followers 1, 3, and 5 with four dynamic leaders that move with varying velocities using (7).

leaders can have different accelerations and initial velocities, which means that the shape of the convex hull can be changing over time. The control parameters are chosen as $\alpha = 0.5, K_i = 0.8I_2$, and $\Lambda_i = 5I_2, \forall i = 1, \ldots, 6$. The gains for the distributed sliding-mode estimators are chosen as $\beta_1 = \beta_2 = 4$. Fig. 2 shows the trajectories of the six followers when the leaders move with varying velocities using (7). Fig. 3 shows the containment errors of followers 1, 2, and 3 using (7). We can see that the six followers converge to the dynamic convex hull spanned by the four dynamic leaders and the containment errors converge to zero asymptotically.

5. Conclusions

The distributed containment control problem for networked Lagrangian systems with multiple dynamic leaders in the presence of parametric uncertainties has been studied under a directed graph that characterizes the interaction among the leaders and the followers. We have proposed a distributed adaptive control algorithm combined with distributed sliding-mode estimators and a necessary and sufficient condition on the directed graph such that all followers converge to the dynamic convex hull spanned by the dynamic leaders asymptotically. As a byproduct, we have presented a necessary and sufficient condition on a leaderless consensus algorithm for networked Lagrangian systems under a directed graph.

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Appendix A. Proof of Lemma 2.3

Consider the following new (nonsymmetric) Laplacian matrix given by

$$\overline{\mathcal{L}} = \begin{bmatrix} L_1 & L_2\mathbf{1}_{n-m} \\ \mathbf{0}_m^T & 0 \end{bmatrix}.$$

Treat the set of leaders as a new leader, labeled as agent 0. Note that all entries of the matrix L_2 are nonpositive. Thus, for $i \in \mathcal{V}_F$, if there exists an entry of L_2, denoted as $-a_{ij}, j \in \mathcal{V}_L$, such that $-a_{ij} < 0$ (equivalently, there exists a leader in $\mathcal{V}_L$ that is a neighbor of follower i), the ith entry of the vector $L_2\mathbf{1}_{(n-m)}$ is negative (equivalently, the new leader 0 is a neighbor of follower i); on the other hand, for $i \in \mathcal{V}_F$, if the ith entry of the vector $L_2\mathbf{1}_{(n-m)}$ is negative, there must exist an entry of L_2, denoted as $-a_{ij}, j \in \mathcal{V}_L$, such that $-a_{ij} < 0$. We can conclude that Assumption 2.2 holds if and only if the graph associated with $\overline{\mathcal{L}}$, denoted as $\overline{\mathcal{G}}$, has a directed spanning tree. From Lemma 2.1, $\overline{\mathcal{G}}$ has a directed spanning tree if and only if $\overline{\mathcal{L}}$ has exactly one zero eigenvalue, which is equivalent to the fact that L_1 is nonsingular.

Note that $L_1 \in Z_m$ is diagonally dominant with positive diagonal entries. From the Gershgorin disc theorem (Horn & Johnson, 1985), it follows that the eigenvalues of L_1 have nonnegative real parts. Also note that L_1 is nonsingular,which means that all eigenvalues of L_1 have positive real parts. It thus follows from Definition 2.4 that L_1 is a nonsingular M-matrix.

In addition, note from (2) that $L_1\mathbf{1}_m + L_2\mathbf{1}_{(n-m)} = \mathbf{0}_m$, *i.e.*, $-L_1^{-1}L_2\mathbf{1}_{n-m} = \mathbf{1}_m$. Because L_1 is a nonsingular M-matrix, we can conclude from Lemma 2.2 that each entry of L_1^{-1} is nonnegative. Also note that each entry of L_2 is nonpositive, it thus follows that $-L_1^{-1}L_2$ is a matrix with nonnegative entries and each row of $-L_1^{-1}L_2$ has a sum equal to 1.

Appendix B. Proof of Lemma 3.1

Let $\hat{v}$ and $\hat{a}$ be, respectively, the column stack vectors of $\hat{v}_i$ and $\hat{a}_i, \forall i \in \mathcal{V}_F$. Note that (7b) can be written in a vector form as

$$\dot{\hat{v}} = -\beta_1 \mathrm{sgn}\big[(L_1 \otimes I_p)(\hat{v} - \dot{q}_d)\big]. \tag{23}$$

Let $\bar{v}_i \triangleq \hat{v}_i - \dot{q}_{di}$ and $a_{i0} \triangleq \sum_{j\in\mathcal{V}_L} a_{ij}$. Then (23) can be written as

$$\dot{\bar{v}}_i = -\beta_1 \mathrm{sgn}\left[\sum_{j=0}^{m} a_{ij}(\bar{v}_i - \bar{v}_j)\right] - \ddot{q}_{di}, \quad i \in \mathcal{V}_F, \tag{24}$$

where $\bar{v}_0 \triangleq \mathbf{0}_p$. If Assumption 2.2 holds, from the proof of Theorem 3.1 in Cao et al. (2010), we can get that if $\beta_1 > \|\ddot{q}_{di}\|_\infty$, $\bar{v}_i(t) \to \mathbf{0}_p$

in finite time, *i.e.*, $\|\hat{v}_i(t) - q_{di}(t)\| \to 0, \forall i \in \mathcal{V}_F$, in finite time. Note that we have $\beta_1 > \|\ddot{q}_d\| \geq \|\ddot{q}_{di}\|_\infty$. The settling time T_1 is upper bounded by $\frac{\max_{i\in\mathcal{V}_F}\|\bar{v}_i(0)\|_\infty}{\beta_1-\sup_{t\geq 0}\|\ddot{q}_d\|}$. Similarly, we can show that if $\beta_2 > \|\dddot{q}_d\|$, $\|\hat{a}_i(t) - \ddot{q}_{di}(t)\| \to 0, \forall i \in \mathcal{V}_F$, in finite time, and the settling time T_2 is upper bounded by $\frac{\max_{i\in\mathcal{V}_F}\|\bar{a}_i(0)\|_\infty}{\beta_2-\sup_{t\geq 0}\|\dddot{q}_d\|}$, where $\bar{a}_i \triangleq \hat{a}_i - \ddot{q}_{di}$.

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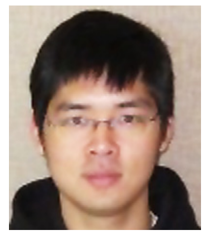

Jie Mei received the B.S. degree in information and computational science from Jilin University, China, in 2007, and the Ph.D. degree in Control Science and Engineering from Harbin Institute of Technology, China, in 2011. Since February 2012, he has been with the School of Mechanical Engineering and Automation, Harbin Institute of Technology Shenzhen Graduate School. From 2009 to 2011, he was an exchange Ph.D. student supported by China Scholarship Council with the Department of Electrical and Computer Engineering at Utah State University, Logan. His research interest focuses on coordination of distributed multi-agent systems.

Wei Ren received the B.S. degree in electrical engineering from Hohai University, China, in 1997, the M.S. degree in mechatronics from Tongji University, China, in 2000, and the Ph.D. degree in electrical engineering from Brigham Young University, Provo, UT, in 2004. From October 2004 to July 2005, he was a Postdoctoral Research Associate with the Department of Aerospace Engineering, University of Maryland, College Park. He was an assistant professor (August 2005–June 2010) and an associate professor (July 2010–June 2011) with the Department of Electrical and Computer Engineering, Utah State University, Logan. Since July 2011, he has been with the Department of Electrical Engineering, University of California, Riverside, where he is currently an Associate Professor. He is an author of two books Distributed Coordination of Multi-agent Networks (Springer-Verlag, 2011) and Distributed Consensus in Multi-vehicle Cooperative Control (Springer-Verlag, 2008). Dr. Ren was a recipient of the National Science Foundation CAREER Award in 2008. He is currently an Associate Editor for Automatica and Systems and Control Letters and an Associate Editor on the IEEE Control Systems Society Conference Editorial Board.

Guangfu Ma is currently a Professor in the Department of Control Science and Engineering, Harbin Institute of Technology. He received his Ph.D. and M.S. in electrical engineering from Harbin Institute of Technology in 1993 and 1987, respectively. In 1992, he became an Associate Professor and in 1997, Professor at Harbin Institute of Technology, where he currently teaches and performs research in the fields of nonlinear control, satellite attitude control and formation flying.