Systems & Control Letters 60 (2011) 365–372

Contents lists available at ScienceDirect

Systems & Control Letters

journal homepage: www.elsevier.com/locate/sysconle

Collective rotating motions of second-order multi-agent systems in three-dimensional space

Peng Lin [a], Kaiyu Qin [a], Zhongkui Li [b], Wei Ren [c,*]

[a] Institute of Astronautics and Aeronautics, University of Electronic Science and Technology of China, Chengdu, 610054, China
[b] School of Automation, Beijing Institute of Technology, Beijing, 100081, China
[c] Department of Electrical and Computer Engineering, Utah State University, Logan, UT 84322, USA

ARTICLE INFO

Article history:
Received 4 October 2010
Received in revised form
3 March 2011
Accepted 9 March 2011
Available online 19 April 2011

Keywords:
Rotating motion
Formation control
Multi-agent system

ABSTRACT

This paper addresses collective rotating motions of second-order multi-agent systems in three-dimensional space (3D). Two distributed control protocols are proposed and sufficient conditions are derived under which all agents rotate around a common point with a specified formation structure. Simulation results are provided to illustrate the effectiveness of the theoretical results.

1. Introduction

Distributed control of multi-agent systems has attracted great attention in many fields such as biology, physics, robotics and control engineering [1–22]. For example, in [4], Ren introduced several second-order consensus algorithms for state consensus of systems by taking into account actuator saturation and limited available information. Also, in [5], Hong et al. addressed a coordination problem for second-order multi-agent systems in the presence of an active leader. Moreover, in [14], Lafferriere et al. investigated a method for decentralized stabilization of vehicle formations, while, in [16], Tanner et al. gave a set of control laws to enable second-order agents to generate stable flocking motions.

A class of collective circular motions widely exist in nature. Examples include flocks of birds flying along a circular orbit and the motion of celestial bodies, which can be applied to formation flight of satellites, circular mobile sensor networks and so on. However, currently, rare results are derived to generate such motions. Only recently, motivated by the applications of autonomous underwater vehicles (AUVs) in oceanographic sampling, a novel rotating formation control problem was solved in [18] to make all agents circle around a common point with some special structures at a unit speed. In [19], 3D circular motion coordination was studied in the presence of a time-invariant flow field. Here all agents finally move on a unit circle. Also, in [20], collective circular motions were addressed by introducing a cyclic pursuit policy to make one vehicle pursuit another vehicle along the line of sight rotated by a common offset angle. Subsequently, the results of [20] were extended in [21] by introducing a rotation matrix to an existing second-order consensus protocol. However, in [18–21], the radii of the circles and the desired formation shape cannot be arbitrarily set. While the circular motions with the same radius or *a priori* unknown radii might be appropriate for some applications, there exist other applications such as persistent surveillance, where it is desirable to have different desired radii of circular orbits for different agents. To specify a desired circle center, a possible approach is to introduce a virtual leader. However, in real applications and a distributed control context, it might not be realistic to assume that the virtual leader's state is known by all agents. Using the algorithms in [18,19], it is not clear how to introduce a virtual leader whose state is known by only a subset of the agents to define a desired circle center. Moreover, in [20,21] a certain control parameter, namely, the rotation angle, must be exactly equal to a certain value to generate circular motions. As a result, the algorithms are not robust in this case. Motivated by the works of [18–21], a collective rotating formation control problem was investigated in [22] for second-order multi-agent systems in 2D. Control protocols were proposed to make all agents surround a common point with a desired formation structure. However, the approach adopted in [22] is based on the complex system theory and cannot be directly applied to the 3D situation.

In this paper, we extend the work of [22] to address the collective rotating formation control problem for second-order

* Corresponding author. Tel.: +1 435 797 2831; fax: +1 435 797 3054.
E-mail address: wei.ren@usu.edu (W. Ren).

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doi:10.1016/j.sysconle.2011.03.002

multi-agent systems in a plane in 3D. We are interested in collection motions where all agents finally converge to a desired plane in 3D and move on desired circular orbits. This kind of collective motion is relevant for many practical applications. Due to the existence of the rotating mode, the desired relative position between every two agents is time varying and thus the approach used in the common formation control problems by introducing desired relative separation vectors to the corresponding consensus problems is invalid for the rotating formation control problem. To solve the rotating formation control problem, we propose two protocols and employ a Lyapunov-based approach to give sufficient conditions under which all agents finally move around a common point with a specific formation structure. When there exists a virtual leader that specifies the desired center and radii of the orbits, it is only required that one agent knows the desired circle center and radius. Compared with [18–21], the protocols designed in this paper guarantees that the desired rotating formation can be arbitrarily set except for some singular points. Also, the protocols are robust to control parameter changes. Therefore, the results in this paper complements some of the existing results [18–22].

The following notations will be used throughout this paper. $\mathbb{R}^m$ and $\mathbb{C}^m$ denote the set of all m dimensional real and complex column vectors; I_m denotes the m dimensional unit matrix; $\otimes$ denotes the Kronecker product; $\mathbf{1}$ represents $[1, \dots, 1]^T$ with compatible dimensions (sometimes, we use $\mathbf{1}_n$ to denote $\mathbf{1}$ with a dimension n); 0 denotes the zero vector or zero matrix with appropriate dimensions; $\mathcal{I} = \{1, 2, \dots, n\}$; $\arg(\cdot)$ denotes the argument of a vector; i_j denotes the imaginary unit; $\|\cdot\|$ denotes the 2-norm; $*$ denotes the conjugate transpose; $\mathrm{diag}\{\cdots\}$ denotes a diagonal matrix; $\oplus$ denotes the direct sum.

2. Preliminaries

2.1. Graph theory

Let $\mathcal{G}(\mathcal{V}, \mathcal{E}, \mathcal{A})$ be an undirected graph of order n, where $\mathcal{V} = \{s_1, \dots, s_n\}$ is the set of nodes, $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the set of edges, and $\mathcal{A} = [a_{ij}]$ is the weighted adjacency matrix. The node indices belong to a finite index set $\mathcal{I}$. An edge of $\mathcal{G}$ is denoted by $e_{ij} = (s_i, s_j)$. The weighted adjacency matrix is defined as $a_{ii} = 0$ and $a_{ij} \geq 0$, where $a_{ij} = a_{ji} > 0$ if and only if $e_{ij} \in \mathcal{E}$. Since the graph considered is undirected, it means that once $e_{ij} \in \mathcal{E}$, then $e_{ji} \in \mathcal{E}$. Thus, $\mathcal{A}$ is a symmetric nonnegative matrix. The set of neighbors of node s_i is denoted by $N_i = \{s_j \in \mathcal{V} : (s_i, s_j) \in \mathcal{E}\}$. The in-degree and out-degree of node s_i are defined as $d_{in}(s_i) = \sum_{j=1}^{n} a_{ji}$ and $d_o(s_i) = \sum_{j=1}^{n} a_{ij}$, respectively. Then, the Laplacian corresponding to the undirected graph $\mathcal{G}$ is defined as $L = [l_{ij}]$, where $l_{ii} = d_o(s_i)$ and $l_{ij} = -a_{ij},\ i \neq j$. Obviously, the Laplacian of any undirected graph is symmetric. A path is a sequence of ordered edges of the form $(s_{i_1}, s_{i_2}), (s_{i_2}, s_{i_3}), \dots$, where $i_j \in \mathcal{I}$ and $s_{i_j} \in \mathcal{V}$. If there is a path from every node to every other node, the graph is said to be connected. If the undirected graph $\mathcal{G}$ is connected, then its Laplacian L has a zero eigenvalue with an associated eigenvector $\mathbf{1}$ and all its other $n-1$ eigenvalues are all positive [23].

2.2. Transformation of coordinates

In this subsection, we introduce some concepts and results of transformation of coordinates (referring to [24]). Consider two rectangular coordinate systems with a common origin, denoted by $S_a(x, y, z)$ and $S_b(\xi, \eta, \zeta)$. Let $\mathrm{i}_x, \mathrm{i}_y, \mathrm{i}_z \in \mathbb{R}^3$ and $\mathrm{i}_\xi, \mathrm{i}_\eta, \mathrm{i}_\zeta \in \mathbb{R}^3$ be two sets of orthogonal unit vectors parallel to their respective coordinate axes. Consider an arbitrary vector p expressed in terms of the components along the x, y, z axes and the ξ, η, ζ axes. Then we have two equivalent representations: $p = x\mathrm{i}_x + y\mathrm{i}_y + z\mathrm{i}_z = \xi\mathrm{i}_\xi + \eta\mathrm{i}_\eta + \zeta\mathrm{i}_\zeta$. To obtain any coordinate representation in one system in terms of that of another system, we simply take the scalar product of the above identity with the corresponding unit vector. In this manner, we obtain two sets of three linear equations which can be written in a vector form as

$$[x, y, z]^T = R[\xi, \eta, \zeta]^T \quad \text{or} \quad [\xi, \eta, \zeta]^T = R^T[x, y, z]^T, \tag{1}$$

where $R \in \mathbb{R}^{3\times 3}$. The matrix R is called the rotation matrix of S_b with respect to S_a and it has the property that $RR^T = R^T R = I_3$.

Let $k \in \mathbb{R}^3$ be a unit vector whose coordinate representation in S_a is $k = [k_x, k_y, k_z]^T$, and S_c be a new coordinate system obtained by rotating S_a about the axis k with an angle θ. Then the rotation matrix of S_c with respect to S_a is [24] $R_k(\theta) = (\cos\theta)I_3 + (1 - \cos\theta)kk^T + (\sin\theta)S(k)$, where $S(k) \triangleq \begin{bmatrix} 0 & -k_z & k_y \\ k_z & 0 & -k_x \\ -k_y & k_x & 0 \end{bmatrix}$.

3. Problem statement

Consider a multi-agent system consisting of n agents. Each agent is regarded as a node in an undirected graph $\mathcal{G}$. Each edge $(s_j, s_i) \in \mathcal{E}$ corresponds to an available information channel between agents s_i and s_j. Moreover, each agent updates its current state based upon the information received from its neighbors. Suppose that each agent has the dynamics as follows:

$$\dot{r}_i = v_i, \qquad \dot{v}_i = u_i, \tag{2}$$

where $r_i = [r_{i1}, r_{i2}, r_{i3}]^T \in \mathbb{R}^3$ and $v_i = [v_{i1}, v_{i2}, v_{i3}]^T \in \mathbb{R}^3$ are the position and velocity of agent s_i and $u_i(t) \in \mathbb{R}^3$ is the control input (or protocol). It should be noted that all vectors are represented in an inertial rectangular coordinate system, denoted by S_o, throughout this paper unless otherwise stated.

In practice, groups of agents often need to move around a common point while maintaining some specific formation structure, e.g., satellite formation flying and the motion of celestial bodies. In such multi-agent systems, each agent can only use its neighbors' information to cooperate with other agents, and the desired relative position between every two agents is time varying which is different from the common formation control problems as discussed in [14].

In this paper, our main objective is to design rules to make all agents surround a common point with a desired formation structure on a plane whose normal is a specified unit vector $\mathrm{i}_\rho \in \mathbb{R}^3$.

Note that the agents may finally move in the clockwise direction or counterclockwise direction. Without loss of generality, we assume that all agents finally move in the counterclockwise direction. Moreover, for convenience of discussion, we introduce a new rectangular coordinate system S_n such that the third coordinate axis of S_n is parallel to the unit vector i_ρ and S_n and S_o share the common origin. The rotation matrix from S_o to S_n is denoted as $R_{no} \in \mathbb{R}^{3\times 3}$. Here, we do not define the other two axes of S_n explicitly and thus the rotation matrix R_{no} can take different values, which does not affect the analysis and results in this paper.

Definition 3.1. A rotating formation in $\mathbb{R}^3$ is a constant vector

$$\begin{aligned} F(h(\rho, \theta), \mathrm{i}_\rho) &= [h_1^T, \dots, h_n^T, \mathrm{i}_\rho^T]^T \\ &= [(\rho_1\cos\theta_1, \rho_1\sin\theta_1), \dots, (\rho_n\cos\theta_n, \rho_n\sin\theta_n), \mathrm{i}_\rho^T]^T \\ &\in \mathbb{R}^{2n+3}, \end{aligned}$$

where $h_i = (\rho_i\cos\theta_i, \rho_i\sin\theta_i)^T, \theta_i \in [0, 2\pi)$ and $\rho_i \in \mathbb{R}$ is a positive constant for any $i \in \mathcal{I}$. The multi-agent system (2) converges to the formation $F(h, \theta, \mathrm{i}_\rho)$ if

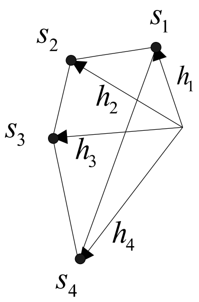

Fig. 1. One example of a formation structure with four agents.

$$\lim_{t\to+\infty} \mathrm{i}_\rho^T v_i(t) = 0, \tag{3}$$

$$\lim_{t\to+\infty} \left[\dot{v}_i(t) - \omega R_{no}^T R_{l_0}\left(\frac{\pi}{2}\right) R_{no} v_i(t)\right] = 0, \tag{4}$$

$$\lim_{t\to+\infty} \left[\left(r_i(t) + \omega^{-1} R_{no}^T R_{l_0}\left(\frac{\pi}{2}\right) R_{no} v_i(t)\right) - \left(r_k(t) + \omega^{-1} R_{no}^T R_{l_0}\left(\frac{\pi}{2}\right) R_{no} v_k(t)\right)\right] = 0, \tag{5}$$

$$\lim_{t\to+\infty} \left[\frac{1}{\rho_i} R_{no}^T R_{l_0}(-\theta_i) R_{no} v_i(t) - \frac{1}{\rho_k} R_{no}^T R_{l_0}(-\theta_k) R_{no} v_k(t)\right] = 0, \tag{6}$$

$$\lim_{t\to+\infty} \left[\left\|\frac{v_i(t)}{\omega}\right\| - \|h_i\|\right] = 0, \tag{7}$$

for any $i, k \in \mathcal{I}$ and a positive constant ω, where $R_{l_0}(\theta_i) = \begin{bmatrix} \cos\theta_i & -\sin\theta_i & 0 \\ \sin\theta_i & \cos\theta_i & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is the rotation matrix of a rotation about the axis $l_0 = [0, 0, 1]^T$ with an angle θ_i.[1] In particular, if only the conditions (3)–(6) are satisfied, it is said that a quasi-rotating formation is achieved.

In Definition 3.1, condition (3) means that each agent finally travels on a plane perpendicular to the unit vector i_ρ. Note that the term $R_{no} v_i(t)$ is the coordinate representation of $v_i(t)$ in S_n. The terms $R_{no}^T R\left(\frac{\pi}{2}\right) R_{no} v_i(t)$ and $R_{no}^T R(-\theta_i) R_{no} v_i(t)$ are the coordinate representations of v_i rotated about the axis i_ρ with an angle $\theta = \frac{\pi}{2}$ and $\theta = -\theta_i$, respectively, in S_o. Then, condition (4) means that the acceleration of each agent tends to contain only the component of the centripetal acceleration $\omega R_{no}^T R\left(\frac{\pi}{2}\right) R_{no} v_i(t)$. By mechanical knowledge, it can be obtained that conditions (3) and (4) guarantee that each agent finally moves on a circle with the angular velocity ω on a plane perpendicular to the vector i_ρ. Also, $r_i(t) + \omega^{-1} R_{no}^T R\left(\frac{\pi}{2}\right) R_{no} v_i(t)$ denotes the circle center surrounded by agent s_i at time t. Then, condition (5) means that the circle centers surrounded by all agents tend to be the same as $t \to +\infty$. Condition (6) means that $\lim_{t\to+\infty}\left(\left\|\frac{v_i}{\rho_i}\right\| - \left\|\frac{v_k}{\rho_k}\right\|\right) = 0$ and $\lim_{t\to+\infty}[(\arg(v_i) - \arg(v_k)) - (\theta_i - \theta_k)] = 0$ for any $i, k \in \mathcal{I}$, which makes all agents finally form a formation that has the same shape but perhaps a different size from the desired formation h. Condition (7) guarantees that the final formation has the same size as the desired formation h. In summary, under conditions (3)–(7), all agents finally surround a common point in the counterclockwise direction with an angular velocity ω on a plane perpendicular to the vector i_ρ while maintaining a specific formation h. For simplicity of the following analysis, we assume $\omega = 1$.

Fig. 1 shows a specific formation structure with four agents. Fig. 2 shows a rotating formation with four agents, where the agents maintain a specific formation structure as shown in Fig. 1 while surrounding a common point with a constant angular velocity ω on a plane perpendicular to the vector i_ρ. Fig. 3 shows one example of two agents, agents s_i and s_k, in a desired rotating formation, where the vectors from the circle center to the positions of agents s_i and s_k are $-\omega^{-1} R_{no}^T R\left(\frac{\pi}{2}\right) R_{no} v_i(t)$ and $-\omega^{-1} R_{no}^T R\left(\frac{\pi}{2}\right) R_{no} v_k(t)$, which correspond to h_i and h_k, respectively.

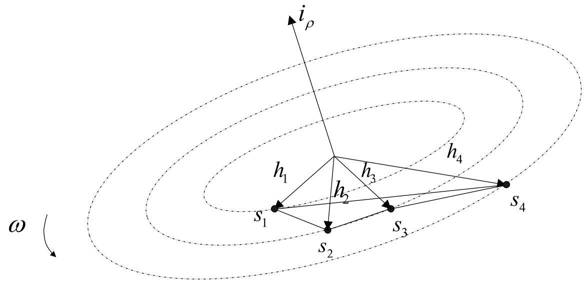

Fig. 2. A rotating formation with a desired formation structure $h = [h_1^T, h_2^T, h_3^T, h_4^T]^T$ as shown in Fig. 1.

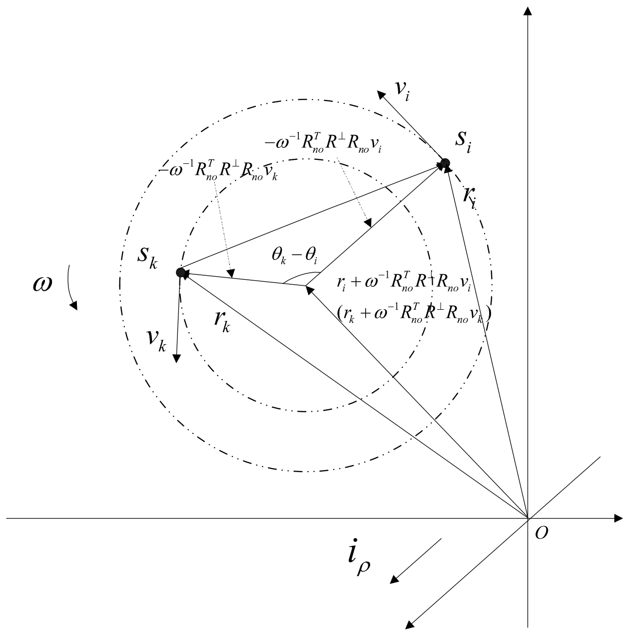

Fig. 3. One example of two agents in a desired rotating formation.

4. Rotating formation control of second-order multi-agent systems in 3D

Due to the existence of the rotating mode, the desired relative position between every two agents is time varying and thus the approach used in the common formation control problems by introducing desired relative separation vectors to the corresponding consensus problems is invalid for the rotating formation control problem.

4.1. Quasi-rotating formation

In this subsection, we study quasi-rotating formation control to make all agents finally surround a common point on a plane perpendicular to the vector i_ρ while maintaining a formation structure that has the same shape but perhaps a different size from the desired formation structure h. The protocol is given as

$$u_i = u_{i1} + u_{i2}, \tag{8}$$

[1] To simplify the notation, we replace all "$R_{l_0}(\cdot)$" with "$R(\cdot)$" in the remainder of this paper.

$$c_i = r_i + R_{no}^T R\left(\frac{\pi}{2}\right) R_{no} v_i, \qquad \xi = [v_1^T, \ldots, v_n^T, c_1^T, \ldots, c_n^T]^T \quad \text{and}$$

$$\Gamma = \begin{bmatrix} I_n \otimes (R_{no}^T R^{\perp} R_{no}) & 0_{3n\times 3n} \\ 0_{3n\times 3n} & 0_{3n\times 3n} \end{bmatrix} - \begin{bmatrix} F[(H^{-1}LH^{-1}) \otimes I_3]F^T & L \otimes I_3 \\ F\left[(H^{-1}LH^{-1}) \otimes \left(R_{no}^T R\left(\frac{\pi}{2}\right) R_{no}\right)\right] F^T & L \otimes \left(R_{no}^T R\left(\frac{\pi}{2}\right) R_{no}\right) \end{bmatrix} \in \mathbb{R}^{6n}$$

where $H = \text{diag}\{\rho_1, \rho_2, \ldots, \rho_n\}$, $F = \text{diag}\{R_{no}^T R(\theta_1) R_{no}, \ldots, R_{no}^T R(\theta_n) R_{no}\}$ and L is the Laplacian of the graph $\mathcal{G}$.

Box I.

where $u_{i1} = R_{no}^T R^{\perp} R_{no} v_i$ and $u_{i2} = -\sum_{s_k\in N_i} a_{ij}[\rho_i^{-2} v_i - \rho_i^{-1}\rho_k^{-1} R_{no}^T R(\theta_i - \theta_k) R_{no} v_k] - \sum_{s_k\in N_i} a_{ij}[(r_i - r_k) + R_{no}^T R(\frac{\pi}{2}) R_{no}(v_i - v_k)]$ for any $i \in \mathcal{I}$, where a_{ij} is the (i,j)th entry of the adjacency matrix $\mathcal{A}$, N_i is the neighbor set of agent s_i and $R^{\perp} = \text{diag}\left\{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, -1\right\}$. In protocol (8), u_{i1} is a local velocity feedback term that plays a role in making each agent surround a point on a plane perpendicular to the vector i_ρ, while u_{i2} is a distributed state information feedback term that plays a role in making all agents converge to a specific structure.

We define the equations as in Box I, Here, it should be noted that $R(-\theta_i) = R(\theta_i)^T = R(\theta_i)^{-1}$, $R(\theta_i - \theta_k) = R(\theta_i)R(-\theta_k)$ and $R(\theta_i)R(\theta_k) = R(\theta_k)R(\theta_i)$. Then using protocol (8) for (2), the closed-loop system can be written in a vector form as

$$\dot{\xi} = \Gamma \xi. \tag{9}$$

Let $\bar{\xi} = (I_n \otimes R_{no})\xi = [\bar{v}_1^T, \ldots, \bar{v}_n^T, \bar{c}_1^T, \ldots, \bar{c}_n^T]^T \in \mathbb{R}^{6n}$, where $\bar{v}_i = [\bar{v}_{i1}^T, \bar{v}_{i2}^T, \bar{v}_{i3}^T]^T \in \mathbb{R}^3$ and $\bar{c}_i = [\bar{c}_{i1}^T, \bar{c}_{i2}^T, \bar{c}_{i3}^T]^T \in \mathbb{R}^3$ for all $i \in \mathcal{I}$. Then system (9) can be written as

$$\dot{\bar{\xi}} = \bar{\Gamma}\bar{\xi}, \tag{10}$$

where $\bar{\Gamma} = \begin{bmatrix} I_n \otimes R^{\perp} - \bar{F}[(H^{-1}LH^{-1}) \otimes I_3]\bar{F}^T & -L \otimes I_3 \\ -\bar{F}\left[(H^{-1}LH^{-1}) \otimes R\left(\frac{\pi}{2}\right)\right]\bar{F}^T & -L \otimes R\left(\frac{\pi}{2}\right) \end{bmatrix}$ and

$$\bar{F} = \text{diag}\{R(\theta_1), R(\theta_2), \ldots, R(\theta_n)\}.$$

Further, let

$$\varphi_1 = [\bar{v}_{11}, \bar{v}_{12}, \ldots, \bar{v}_{n1}, \bar{v}_{n2}, \bar{c}_{11}, \bar{c}_{12}, \ldots, \bar{c}_{n1}, \bar{c}_{n2}]^T \in \mathbb{R}^{4n}$$

and

$$\varphi_2 = [\bar{v}_{13}, \bar{v}_{23}, \ldots, \bar{v}_{n3}, \bar{c}_{13}, \bar{c}_{23}, \ldots, \bar{c}_{n3}]^T \in \mathbb{R}^{2n}.$$

Also, let $\bar{R}(\theta_i) = \begin{bmatrix} \cos\theta_i & -\sin\theta_i \\ \sin\theta_i & \cos\theta_i \end{bmatrix}$ and $\bar{\bar{F}} = \text{diag}\{\bar{R}(\theta_1), \ldots, \bar{R}(\theta_n)\}$. Here, it should be also noted that $\bar{R}(\theta_i)$ also satisfies that $\bar{R}(-\theta_i) = \bar{R}(\theta_i)^T = \bar{R}(\theta_i)^{-1}$, $\bar{R}(\theta_i - \theta_k) = \bar{R}(\theta_i)\bar{R}(-\theta_k)$ and $\bar{R}(\theta_i)\bar{R}(\theta_k) = \bar{R}(\theta_k)\bar{R}(\theta_i)$. Then system (9) is equivalent to the following two systems:

$$\dot{\varphi}_1 = \Phi_1\varphi_1, \tag{11}$$

and

$$\dot{\varphi}_2 = \Phi_2\varphi_2, \tag{12}$$

where

$$\Phi_1 = \begin{bmatrix} I_n \otimes \bar{R}\left(\frac{\pi}{2}\right) - \bar{\bar{F}}[(H^{-1}LH^{-1}) \otimes I_2]\bar{\bar{F}}^T & -L \otimes I_2 \\ -\bar{\bar{F}}\left[(H^{-1}LH^{-1}) \otimes \bar{R}\left(\frac{\pi}{2}\right)\right]\bar{\bar{F}}^T & -L \otimes \bar{R}\left(\frac{\pi}{2}\right) \end{bmatrix}$$

and

$$\Phi_2 = \begin{bmatrix} -I_n - H^{-1}LH^{-1} & -L \\ -H^{-1}LH^{-1} & -L \end{bmatrix}.$$

Remark 4.1. In fact, system (10) or (11)–(12) is the representation of system (9) in the rectangular coordinate system S_n. Specifically, $\bar{v}_i$, $i = 1, 2, \ldots, n$, are the coordinate representations of velocities of all agents in S_n whereas $\bar{c}_i$, $i = 1, 2, \ldots, n$, are the coordinate representations of the circle centers of all agents in S_n. Then it is easy to see that system (9) satisfies conditions (3)–(6) if and only if system (10) or system (11)–(12) satisfies that $\lim_{t\to+\infty}\left(\frac{1}{\rho_i}\bar{R}(-\theta_i)[\bar{v}_{i1}, \bar{v}_{i2}]^T - \frac{1}{\rho_k}\bar{R}(-\theta_k)[\bar{v}_{k1}, \bar{v}_{k2}]^T\right) = 0$, $\lim_{t\to+\infty}\bar{v}_{i3} = 0$, $\lim_{t\to+\infty}(\bar{c}_i - \bar{c}_k) = 0$ and $\lim_{t\to+\infty}([\dot{\bar{v}}_{i1}, \dot{\bar{v}}_{i2}]^T - \bar{R}(\frac{\pi}{2})[\bar{v}_{i1}, \bar{v}_{i2}]^T) = 0$ for any $i, k \in \mathcal{I}$.

Remark 4.2. It should be noted that the approach used in [22] cannot be applied directly to analyze the stability of the system (11) because the multi-agent system in [22] is described in a complex number field, which has a lower dimension and is thus much easier to analyze than system (11).

Define

$$\text{sgn}(x) = \begin{cases} 1, & \text{if } x = 0 \\ -1, & \text{if } x = \pi \\ 0, & \text{otherwise} \end{cases}$$

and

$$M = \text{diag}\{1, \text{sgn}(|\theta_2 - \theta_1|), \text{sgn}(|\theta_3 - \theta_1|), \ldots, \text{sgn}(|\theta_n - \theta_1|)\}.$$

Assumption 1. $\mathbf{1}_n^T HM\mathbf{1}_n \neq 0$ when $|\theta_i - \theta_k| = 0$ or $|\theta_i - \theta_k| = \pi$ for all $i, k \in \mathcal{I}$.

Remark 4.3. When $\mathbf{1}_n^T HM\mathbf{1}_n = 0$ and $|\theta_i - \theta_k| = 0$ or $|\theta_i - \theta_k| = \pi$ hold for all $i, k \in \mathcal{I}$, then it is easy to see that $\sum_{i=1}^n h_i = 0$ and all agents in the desired formation are in a line. This might make the matrix Φ_1 have other imaginary eigenvalues that are not equal to $\pm \mathrm{i}_j$ and 0. For example, consider a multi-agent system consisting of two agents with $[\rho_1, \theta_1, \rho_2, \theta_2] = [1, 0, 1, \pi]$ and $L = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$. By simple calculations, it can be obtained that Φ_1 has imaginary eigenvalues at $\pm 2\mathrm{i}_j$ except 0 and $\pm \mathrm{i}_j$, which renders the desired quasi-rotating formation to be unreachable. Therefore, Assumption 1 is made to eliminate these unreachable formation structures.

Lemma 4.1. *Suppose that the graph $\mathcal{G}$ is connected. Under Assumption 1, the following statements hold.*

(a) *Φ_1 has a zero eigenvalue of multiplicity 2 with associated eigenvectors $[0_{2n}^T, \mathbf{1}_n^T \otimes [1, 0]]^T$ and $[0_{2n}^T, \mathbf{1}_n^T \otimes [0, 1]]^T$.*
(b) *Φ_1 has two simple eigenvalues at i_j and $-\mathrm{i}_j$ with associated eigenvectors $[\beta_1^T, 0_{2n}^T]^T$ and $[\beta_2^T, 0_{2n}^T]^T$, where $\beta_1 = \bar{\bar{F}}[(\mathbf{1}_n^T H) \otimes [\mathrm{i}_j, 1]]^T$ and $\beta_2 = \bar{\bar{F}}[(\mathbf{1}_n^T H) \otimes [-\mathrm{i}_j, 1]]^T$.*
(c) *All its other $4n - 4$ eigenvalues have negative real parts.*

Proof. Let λ_a be an eigenvalue of Φ_1 and $[z_1^T, z_2^T]^T$ be its corresponding eigenvector, where $z_1, z_2 \in \mathbb{C}^{2n}$. Then we have that

$$\left(I_n \otimes \bar{R}\left(\frac{\pi}{2}\right)\right) z_1 - \bar{\bar{F}}[(H^{-1}LH^{-1}) \otimes I_2]\bar{\bar{F}}^T z_1 - (L \otimes I_2) z_2 = \lambda_a z_1, \tag{13}$$

$$-\bar{\bar{F}}\left[(H^{-1}LH^{-1}) \otimes \bar{R}\left(\frac{\pi}{2}\right)\right]\bar{\bar{F}}^T z_1 - \left(L \otimes \bar{R}\left(\frac{\pi}{2}\right)\right) z_2 = \lambda_a z_2. \tag{14}$$

For statement (a), it can be checked that Φ_1 has a zero eigenvalue with two associated linearly independent eigenvectors $[z_1^T, z_2^T] = [0_{2n}^T, (\mathbf{1}_n^T \otimes ([1,0]^T))^T]^T$ and $[z_1^T, z_2^T] = [0_{2n}^T, (\mathbf{1}_n^T \otimes ([0,1]^T))^T]^T$. To prove that the multiplicity of the zero eigenvalue is 2, we only need to prove that Φ_1 has no generalized eigenvector of a grade higher than 1 associated with the zero eigenvalue. Suppose that there is a grade 2 generalized eigenvector associated with the zero eigenvalue. That is, there is a nonzero $4n \times 1$ column vector, denoted by $[\tilde{z}_1^T, \tilde{z}_2^T]^T$ where $\tilde{z}_1, \tilde{z}_2 \in \mathbb{R}^{2n}$, such that $(\Phi_1 - 0)[\tilde{z}_1^T, \tilde{z}_2^T]^T = [z_1^T, z_2^T]^T$. After simple calculations, it can be obtained that $\tilde{z}_1 = z_2$ and thus $(I_n \otimes \bar{R}(\frac{\pi}{2}))z_2 - \bar{\bar{F}}[(H^{-1}LH^{-1}) \otimes I_2]\bar{\bar{F}}^T z_2 - (L \otimes I_2)\tilde{z}_2 = 0$, because $z_1 = 0_n$ and $z_2 = \mathbf{1}_n \otimes ([1,0]^T)$ or $\mathbf{1}_n \otimes ([0,1]^T)$. On the other hand, since $\bar{R}(\frac{\pi}{2}) + \bar{R}(\frac{\pi}{2})^T = 0$, then $\bar{\bar{F}}[(H^{-1}LH^{-1}) \otimes \bar{R}(\frac{\pi}{2})]\bar{\bar{F}}^T + [\bar{\bar{F}}[(H^{-1}LH^{-1}) \otimes \bar{R}(\frac{\pi}{2})]\bar{\bar{F}}^T]^T = 0$. Therefore, $-z_2^*(I_n \otimes \bar{R}(\frac{\pi}{2}))[(I_n \otimes \bar{R}(\frac{\pi}{2}))z_2 - \bar{\bar{F}}[(H^{-1}LH^{-1}) \otimes I_2]\bar{\bar{F}}^T z_2 - (L \otimes I_2)\tilde{z}_2] + [-z_2^*(I_n \otimes \bar{R}(\frac{\pi}{2}))[(I_n \otimes \bar{R}(\frac{\pi}{2}))z_2 - \bar{\bar{F}}[(H^{-1}LH^{-1}) \otimes I_2]\bar{\bar{F}}^T z_2 - (L \otimes I_2)\tilde{z}_2]]^* = 2n + z_2^*\bar{\bar{F}}[(H^{-1}LH^{-1}) \otimes \bar{R}(\frac{\pi}{2})]\bar{\bar{F}}^T z_2 + [z_2^*\bar{\bar{F}}[(H^{-1}LH^{-1}) \otimes \bar{R}(\frac{\pi}{2})]\bar{\bar{F}}^T z_2]^* = 2n \neq 0$, which is a contradiction. Therefore, the multiplicity of the zero eigenvalue is 2. Statement (a) is proved. $\square$

For statements (b) and (c), pre-multiplying both sides of (14) by $I_n \otimes \bar{R}\left(\frac{\pi}{2}\right)$ yields that

$$\begin{aligned} &-\left(I_n \otimes \bar{R}\left(\frac{\pi}{2}\right)\right)\bar{\bar{F}}\left[(H^{-1}LH^{-1}) \otimes \bar{R}\left(\frac{\pi}{2}\right)\right]\bar{\bar{F}}^T z_1 \\ &\quad -\left(I_n \otimes \bar{R}\left(\frac{\pi}{2}\right)\right)\left(L \otimes \bar{R}\left(\frac{\pi}{2}\right)\right)z_2 \\ &= \bar{\bar{F}}[(H^{-1}LH^{-1}) \otimes I_2]\bar{\bar{F}}^T z_1 + (L \otimes I_2)z_2 \\ &= \lambda_a\left(I_n \otimes \bar{R}\left(\frac{\pi}{2}\right)\right)z_2. \end{aligned} \tag{15}$$

Substituting (15) into (13), we have that

$$\lambda_a\left(I_n \otimes \bar{R}\left(\frac{\pi}{2}\right)\right)z_2 = \left(I_n \otimes \bar{R}\left(\frac{\pi}{2}\right) - \lambda_a I_{2n}\right)z_1. \tag{16}$$

Combining (15) and (16), we have that

$$K_1 z_1 - K_2 z_1 - K_3 z_1 = 0, \tag{17}$$

where $K_1 = I_n \otimes \bar{R}\left(\frac{\pi}{2}\right) - \lambda_a I_{2n}$, $K_2 = \bar{\bar{F}}[(H^{-1}LH^{-1}) \otimes I_2]\bar{\bar{F}}^T$ and $K_3 = \frac{1}{\lambda_a}(L \otimes I_2)\left(I_n \otimes \bar{R}\left(\frac{\pi}{2}\right)^{-1}\right)\left(I_n \otimes \bar{R}\left(\frac{\pi}{2}\right) - \lambda_a I_{2n}\right)$. Consider a system given by

$$\dot{x} = (K_1 - K_2 - K_3)x, \tag{18}$$

where $x \in \mathbb{C}^n$. Construct a Lyapunov function for system (18) as $V = x^*x$. Calculating $\dot{V}$, we have that

$$\begin{aligned} \dot{V} &= x^*\left[K_1 + K_1^* - 2K_2 + (L \otimes I_2)\left(I_n \otimes \bar{R}\left(\frac{\pi}{2}\right)^{-1}\right)\right. \\ &\quad \left. + (L \otimes I_2)\left(I_n \otimes \bar{R}\left(\frac{\pi}{2}\right)^{-1}\right)^* - \frac{\lambda_a + \lambda_a^*}{\|\lambda_a\|^2}(L \otimes I_2)\right]x \\ &= x^*\left[-(\lambda_a + \lambda_a^*)I_{2n} - 2K_2 - \frac{\lambda_a + \lambda_a^*}{\|\lambda_a\|^2}(L \otimes I_2)\right]x. \end{aligned} \tag{19}$$

Because K_2 and $L \otimes I_2$ are both positive semi-definite, if the real part of λ_a is positive, then $\dot{V} \le 0$. This implies that all the eigenvalues of the matrix $K_1 - K_2 - K_3$ have negative real parts. Eq. (17) implies that z_1 should be a zero vector and hence z_2 should also be a zero vector from (16). Therefore, the eigenvector of Φ_1 associated with the eigenvalue λ_a is a zero vector, which is a contradiction. Thus, the real part of λ_a is nonpositive.

Suppose that $\lambda_a = b\mathrm{i}_j$ $(b \neq 0)$. Then the derivative of V becomes $\dot{V} = -2x^*K_2 x \le 0$. Note that $\mathrm{Rank}(K_2) = 2n - 2$ and $K_2\tilde{\beta}_1 = K_2\tilde{\beta}_2 = 0$, where $\tilde{\beta}_1 = \bar{\bar{F}}[(\mathbf{1}_n^T H) \otimes [1,0]]^T$ and $\tilde{\beta}_2 = \bar{\bar{F}}[(\mathbf{1}_n^T H) \otimes [0,1]]^T$. Then by LaSalle's Invariant Principle, it follows that the solution of (18) will converge to the kernel space of the matrix K_2 that is spanned by the vectors $\tilde{\beta}_1$ and $\tilde{\beta}_2$. Also, note that z_1 is an eigenvector of the matrix $K_1 - K_2 - K_3$ associated with the zero eigenvalue from (17). Therefore, the vector z_1 falls into the kernel space of the matrix K_2, i.e., $K_2 z_1 = 0$. Then z_1 can be represented by a linear combination of $\tilde{\beta}_1$ and $\tilde{\beta}_2$. That is, there exist two numbers $b_1, b_2 \in \mathbb{C}$ such that $b_1\tilde{\beta}_1 + b_2\tilde{\beta}_2 = [\tilde{\beta}_1, \tilde{\beta}_2][b_1, b_2]^T = z_1$.

We now prove that Φ_1 has two simple eigenvalues at i_j and $-\mathrm{i}_j$ with associated eigenvectors $[\beta_1^T, 0_{2n}^T]^T$ and $[\beta_2^T, 0_{2n}^T]^T$. From (16), we have $z_2 = \frac{1}{\lambda_a}z_1 - \left(I_n \otimes \bar{R}\left(\frac{\pi}{2}\right)\right)z_1$. Observing the form of $\tilde{\beta}_1$ and $\tilde{\beta}_2$, we easily see that $\bar{\bar{F}}\left[(H^{-1}LH^{-1}) \otimes \bar{R}\left(\frac{\pi}{2}\right)\right]\bar{\bar{F}}^T z_1 = 0$ and z_2 can be represented linearly by $\tilde{\beta}_1$ and $\tilde{\beta}_2$ since z_1 and $\left(I_n \otimes \bar{R}\left(\frac{\pi}{2}\right)\right)z_1$ can both be represented linearly by $\tilde{\beta}_1$ and $\tilde{\beta}_2$. It follows that $-\left(L \otimes \bar{R}\left(\frac{\pi}{2}\right)\right)z_2 = \lambda_a z_2$ from (14). Then as in the subsequent proof of statement (c), it can be proved that $z_2 = 0$ when $\lambda_a = b\mathrm{i}_j \neq 0$ by a contradiction approach. Thus by simple calculations, it can be checked that Φ_1 has two eigenvalues at $\pm\mathrm{i}_j$ with associated eigenvectors $[\beta_1^T, 0_{2n}^T]^T$ and $[\beta_2^T, 0_{2n}^T]^T$. We then prove that the multiplicities of the eigenvalues $\pm\mathrm{i}_j$ are both 1. Suppose that there is a grade 2 generalized eigenvector associated with the eigenvalue i_j. That is, there is a nonzero $4n \times 1$ column vector, denoted by $[\tilde{z}_1^T, \tilde{z}_2^T]^T$, such that

$$\begin{aligned} &\left(I_n \otimes \bar{R}\left(\frac{\pi}{2}\right)\right)\tilde{z}_1 - \bar{\bar{F}}[(H^{-1}LH^{-1}) \otimes I_2]\bar{\bar{F}}^T\tilde{z}_1 - \mathrm{i}_j\tilde{z}_1 \\ &\quad -(L \otimes I_2)\tilde{z}_2 = \beta_1, \end{aligned} \tag{20}$$

$$\begin{aligned} &-\bar{\bar{F}}\left[(H^{-1}LH^{-1}) \otimes \bar{R}\left(\frac{\pi}{2}\right)\right]\bar{\bar{F}}^T\tilde{z}_1 \\ &\quad -\left(L \otimes \bar{R}\left(\frac{\pi}{2}\right)\right)\tilde{z}_2 - \mathrm{i}_j\tilde{z}_2 = 0_{2n}, \end{aligned} \tag{21}$$

where $\tilde{z}_1, \tilde{z}_2 \in \mathbb{C}^{2n}$. Pre-multiplying both sides of (20) with $I_n \otimes \bar{R}\left(\frac{\pi}{2}\right)$, we have

$$\begin{aligned} &-\tilde{z}_1 - \bar{\bar{F}}\left[(H^{-1}LH^{-1}) \otimes \bar{R}\left(\frac{\pi}{2}\right)\right]\bar{\bar{F}}^T\tilde{z}_1 \\ &-\mathrm{i}_j\left(I_n \otimes \bar{R}\left(\frac{\pi}{2}\right)\right)\tilde{z}_1 - \left(L \otimes \bar{R}\left(\frac{\pi}{2}\right)\right)\tilde{z}_2 = \left(I_n \otimes \bar{R}\left(\frac{\pi}{2}\right)\right)\beta_1. \end{aligned} \tag{22}$$

Rewriting β_1, we have $\beta_1 = [\rho_1\mathrm{i}_j e^{\mathrm{i}_j\theta_1}, \rho_1 e^{\mathrm{i}_j\theta_1}, \ldots, \rho_n\mathrm{i}_j e^{\mathrm{i}_j\theta_n}, \rho_n e^{\mathrm{i}_j\theta_n}]^T$. Then combining (21) and (22) yields

$$\tilde{z}_2 = I_n \otimes \begin{bmatrix} -\mathrm{i}_j & -1 \\ 1 & -\mathrm{i}_j \end{bmatrix}\tilde{z}_1 + \beta_1. \tag{23}$$

Pre-multiplying both sides of (20) with β_1^*, we have

$$-\beta_1^*(L \otimes I_2)\tilde{z}_2 = \beta_1^*\beta_1,$$

i.e.,

$$\begin{aligned} &-\beta_1^*(L \otimes I_2)\left(I_n \otimes \begin{bmatrix} -\mathrm{i}_j & -1 \\ 1 & -\mathrm{i}_j \end{bmatrix}\tilde{z}_1 + \beta_1\right) \\ &\quad = -\beta_1^*(L \otimes I_2)\beta_1 = \beta_1^*\beta_1. \end{aligned}$$

Since L is symmetric and positive semi-definite, then $-\beta_1^*(L \otimes I_2)\beta_1 \le 0$. Note that $\beta_1^*\beta_1 > 0$. Then we have $-\beta_1^*(L \otimes I_2)\beta_1 \neq \beta_1^*\beta_1$. It is a contradiction. This proves that the multiplicity of the eigenvalue i_j is 1. Similarly, it can be proved that the multiplicity of the eigenvalue $-\mathrm{i}_j$ is 1. Statement (b) is proved. $\square$

Next, we prove statement (c). It is easy to see that $(K_1 - K_3)z_1 = \left(I_n \otimes \bar{R}\left(\frac{\pi}{2}\right) - \lambda_a I_{2n}\right)\left[I_{2n} + \frac{1}{\lambda_a}\left(L \otimes \bar{R}\left(\frac{\pi}{2}\right)\right)\right]z_1 = 0$. Since

$I_n \otimes \bar{R}\left(\frac{\pi}{2}\right) - \lambda_a I_{2n}$ is nonsingular for any $\lambda_a \neq \pm \mathrm{i}_j$, then it follows that $\left[I_{2n} + \frac{1}{\lambda_a}\left(L \otimes \bar{R}\left(\frac{\pi}{2}\right)\right)\right] z_1 = 0$, i.e., $\left(L \otimes \bar{R}\left(\frac{\pi}{2}\right)\right) z_1 = -\lambda_a z_1$. It thus follows that z_1 is also an eigenvector of $L \otimes \bar{R}\left(\frac{\pi}{2}\right)$ associated with a nonzero eigenvalue and has the form of $e \otimes ([\mathrm{i}_j, 1]^T)$ or $e \otimes ([\mathrm{i}_j, -1]^T)$, where $e = [e_1, \ldots, e_n]^T \in \mathbb{R}^n$ is an eigenvector of L associated with a nonzero eigenvalue. If $z_1 = e \otimes ([\mathrm{i}_j, 1]^T)$, then $z_1 = [\tilde{\beta}_1, \tilde{\beta}_2][b_1, b_2]^T$ can be decomposed into $\rho_m \bar{R}(\theta_m)[b_1, 0]^T + \rho_m \bar{R}(\theta_m)[0, b_2]^T = \rho_m \bar{R}(\theta_m)[b_1, b_2]^T = e_m[\mathrm{i}_j, 1]^T$ for all $m \in \mathcal{I}$. Clearly, all the included angles of each pair of $\rho_m[b_1, b_2]^T$ and $|e_m|[\mathrm{i}_j, 1]^T$ are equal for all $m \in \mathcal{I}$. Then it follows that $\theta_k = \theta_m$ or $|\theta_k - \theta_m| = \pi$ for all $k, m \in \mathcal{I}$. Moreover, if $z_1 = e \otimes ([\mathrm{i}_j, -1]^T)$, it can be similarly obtained that all θ_k should satisfy that $\theta_k = \theta_m$ or $|\theta_k - \theta_m| = \pi$ for all $k, m \in \mathcal{I}$. Then, $\tilde{\beta}_1$ and $\tilde{\beta}_2$ can be written as $\tilde{\beta}_1 = [(\mathbf{1}_n^T HM) \otimes ([1, 0]\bar{R}(\theta_1)^T)]^T$ and $\tilde{\beta}_2 = [(\mathbf{1}_n^T HM) \otimes ([0, 1]\bar{R}(\theta_1)^T)]^T$. Note that $L\mathbf{1}_n = 0$ and L is a symmetric matrix. Then $\mathbf{1}_n^T e = 0$ and hence $\mathbf{1}_n^T HM\mathbf{1}_n = 0$, which contradicts Assumption 1. Statement (c) is proved. □

Lemma 4.2. *If the graph $\mathcal{G}$ is connected, then $\lim_{t\to+\infty} \bar{v}_{i3}(t) = 0$ and $\lim_{t\to+\infty}[\bar{c}_{i3}(t) - \bar{c}_{k3}(t)] = 0$ for any $i, k \in \mathcal{I}$. That is, all agents finally converge to a plane perpendicular to the vector i_ρ.*

Proof. Consider system (12). Let λ_b be an eigenvalue of Φ_2 and $[y_1^T, y_2^T]^T$ be its corresponding eigenvector, where $y_1, y_2 \in \mathbb{R}^n$. Then we have

$$-y_1 - H^{-1}LH^{-1}y_1 - Ly_2 = \lambda_b y_1, \tag{24}$$

$$-H^{-1}LH^{-1}y_1 - Ly_2 = \lambda_b y_2. \tag{25}$$

Similar to the proof of Lemma 4.1, it can be proved that Φ_2 has a simple zero eigenvalue with an associated eigenvector $[0_n^T, \mathbf{1}_n^T]^T$ and all its other $2n - 1$ eigenvalues have negative real parts. Then it is easy to see that the solution of system (12) converges to the eigenvector space, $\mathrm{span}\{[0_n^T, \mathbf{1}_n^T]^T\}$, of Φ_2 associated with the zero eigenvalue. This implies that $\lim_{t\to+\infty} \bar{v}_{i3}(t) = 0$ and $\lim_{t\to+\infty}[\bar{c}_{i3}(t) - \bar{c}_{k3}(t)] = 0$ for any $i, k \in \mathcal{I}$. □

Remark 4.4. As Φ_2 has no imaginary eigenvalues except 0, Assumption 1 is not required in Lemma 4.2, which is different from Lemma 4.1.

Theorem 4.5. *Consider a network of second-order agents with a fixed topology. If the graph $\mathcal{G}$ is connected, the multi-agent system (2) with protocol (8) achieves a desired quasi-rotating formation $F(h(\rho, \theta), i_\rho)$ under Assumption 1.*

Proof. From Lemma 4.1, Φ_1 has a zero eigenvalue of multiplicity 2 with associated eigenvectors $[0_{2n}^T, \mathbf{1}_n^T \otimes [1, 0]]^T$ and $[0_{2n}^T, \mathbf{1}_n^T \otimes [0, 1]]^T$, two simple eigenvalues at i_j and $-\mathrm{i}_j$ with associated eigenvectors $[\beta_1^T, 0_{2n}^T]^T$ and $[\beta_2^T, 0_{2n}^T]^T$ and all its other $4n - 4$ eigenvalues have negative real parts. Also, the initial condition $\varphi_1(0)$ can be decomposed into $\varphi_1(0) = \mu_1[0_{2n}^T, \mathbf{1}_n^T \otimes [1, 0]]^T + \mu_2[0_{2n}^T, \mathbf{1}_n^T \otimes [0, 1]]^T + \mu_3[\beta_1^T, 0_{2n}^T]^T + \bar{\mu}_3[\beta_2^T, 0_{2n}^T]^T + \varphi_{10}$, where $\mu_1, \mu_2 \in \mathbb{R}, \varphi_{10} \in \mathbb{R}^{3n}$ and $\mu_3, \bar{\mu}_3 \in \mathbb{C}$ are conjugate complex numbers. Then, it is easy to see that

$$\lim_{t\to+\infty} [\varphi_1(t) - \mu_1[0_{2n}^T, \mathbf{1}_n^T \otimes [1, 0]]^T - \mu_2[0_{2n}^T, \mathbf{1}_n^T \otimes [0, 1]]^T - \mu_3 \mathrm{e}^{\mathrm{i}_j t}[\beta_1^T, 0_{2n}^T]^T - \bar{\mu}_3 \mathrm{e}^{-\mathrm{i}_j t}[\beta_2^T, 0_{2n}^T]^T] = 0,$$

i.e.,

$$\lim_{t\to+\infty} [\varphi_1(t) - \mu_1[0_{2n}^T, \mathbf{1}_n^T \otimes [1, 0]]^T - \mu_2[0_{2n}^T, \mathbf{1}_n^T \otimes [0, 1]]^T - [\bar{\bar{F}}[(\mathbf{1}_n^T H) \otimes \hat{\beta}_{12}^T]^T, 0_{2n}^T]^T] = 0,$$

where $\hat{\beta}_{12}^T = [\mathrm{i}_j(\mu_3 - \bar{\mu}_3)\cos t - (\mu_3 + \bar{\mu}_3)\sin t, (\mu_3 + \bar{\mu}_3)\cos t + \mathrm{i}_j(\mu_3 - \bar{\mu}_3)\sin t] \in \mathbb{R}^2$. This implies that $\lim_{t\to+\infty}(\bar{c}_{i1} - \bar{c}_{k1}) = 0$, $\lim_{t\to+\infty}(\bar{c}_{i2} - \bar{c}_{k2}) = 0$ and $\lim_{t\to+\infty}(\frac{1}{\rho_i}\bar{R}(-\theta_i) \times [\bar{v}_{i1}, \bar{v}_{i2}]^T - \frac{1}{\rho_k}\bar{R}(-\theta_k)[\bar{v}_{k1}, \bar{v}_{k2}]^T) = 0$ for any $i, k \in \mathcal{I}$. From Lemma 4.2, we have $\lim_{t\to+\infty} \bar{v}_{i3} = 0$ and $\lim_{t\to+\infty}(\bar{c}_{i3} - \bar{c}_{k3}) = 0$ for any $i, k \in \mathcal{I}$. Moreover, calculating $[\dot{\bar{v}}_{i1}, \dot{\bar{v}}_{i2}]^T$, we have $\lim_{t\to+\infty}\left([\dot{\bar{v}}_{i1}, \dot{\bar{v}}_{i2}]^T - \bar{R}\left(\frac{\pi}{2}\right)[\bar{v}_{i1}, \bar{v}_{i2}]^T\right) = 0$. Therefore, it follows from Remark 4.1 that the multi-agent system (2) with protocol (8) achieves a desired quasi-rotating formation $F(h(\rho, \theta), i_\rho)$. □

Remark 4.6. By analyzing the eigenvalue–eigenvector solution of the system as in [2, Theorem 3], the final motions of systems (11) and (12) can be accurately given in terms of the left and right eigenvectors of Φ_1 and Φ_2. Then through coordinate transformation, the final point, i.e., the common center of the circles, surrounded by the agents and the corresponding radii of the circles can also be calculated. We have omitted these expressions here due to their complexity.

4.2. Rotating formation

In this subsection, we will study the rotating formation control problem based on protocol (8) to make all agents finally surround a common point on a plane perpendicular to the vector i_ρ while maintaining the desired formation structure h. To achieve this, we introduce a virtual leader approach.

Suppose that one of the agents, denoted by s_{i_0}, has access to the desired velocity $v_0 = R_{no}^T[\rho_{i_0}\cos t, \rho_{i_0}\sin t, 0]^T \in \mathbb{R}^3$ and the desired circle center $c_0 \in \mathbb{R}^3$. The protocol of agent s_{i_0} is given as

$$u_{i_0} = u_{i_0 1} + u_{i_0 2}, \tag{26}$$

where $u_{i_0 1} = R_{no}^T R^\perp R_{no} v_{i_0}$ and

$$u_{i_0 2} = -\sum_{s_k \in N_{i_0}} a_{i_0 j}[\rho_{i_0}^{-2} v_{i_0} - \rho_{i_0}^{-1}\rho_k^{-1} R_{no}^T R(\theta_{i_0} - \theta_k) R_{no} v_k] - \sum_{s_k \in N_{i_0}} a_{i_0 j}\left[(r_{i_0} - r_k) + R_{no}^T R\left(\frac{\pi}{2}\right) R_{no}(v_{i_0} - v_k)\right] - (v_{i_0} - v_0) - \left(r_{i_0} + R_{no}^T R\left(\frac{\pi}{2}\right) R_{no} v_{i_0} - c_0\right).$$

The protocols of all other agents are given as

$$u_i = u_{i1} + u_{i2}, \quad i \in \mathcal{I} \setminus \{i_0\}, \tag{27}$$

where $u_{i1} = R_{no}^T R^\perp R_{no} v_i$ and

$$u_{i2} = -\sum_{s_k \in N_i} a_{ij}[\rho_i^{-2} v_i - \rho_i^{-1}\rho_k^{-1} R_{no}^T R(\theta_i - \theta_k) R_{no} v_k] - \sum_{s_k \in N_i} a_{ij}\left[(r_i - r_k) + R_{no}^T R\left(\frac{\pi}{2}\right) R_{no}(v_i - v_k)\right].$$

In (26), the term $-\left(r_{i_0} + R_{no}^T R\left(\frac{\pi}{2}\right) R_{no} v_{i_0} - c_0\right)$ plays a role in making the final point surrounded by all agents converge to c_0 while the term $-(v_{i_0} - v_0)$ plays a role in guaranteeing that the final formation has the desired size.

We define the equations as in Box II. It can be easily obtained that

$$\begin{aligned}\dot{v}_0 &= R_{no}^T R^\perp [\rho_{i_0}\cos t, \rho_{i_0}\sin t, 0]^T \\ &= R_{no}^T R^\perp R_{no} R_{no}^T [\rho_{i_0}\cos t, \rho_{i_0}\sin t, 0]^T \\ &= R_{no}^T R^\perp R_{no} v_0.\end{aligned}$$

Thus,

$$\begin{aligned}\dot{\tilde{v}}_i &= \dot{v}_i - \frac{\rho_i}{\rho_{i_0}} R_{no}^T R(\theta_i - \theta_{i_0}) R_{no} \dot{v}_0 \\ &= \dot{v}_i - \frac{\rho_i}{\rho_{i_0}} R_{no}^T R(\theta_i - \theta_{i_0}) R_{no} R_{no}^T R^\perp R_{no} v_0 \\ &= \dot{v}_i - R_{no}^T R^\perp R_{no} \times \frac{\rho_i}{\rho_{i_0}} R_{no}^T R(\theta_i - \theta_{i_0}) R_{no} v_0.\end{aligned}$$

$$\tilde{v}_i = v_i - \frac{\rho_i}{\rho_{i_0}} R_{no}^T R(\theta_i - \theta_{i_0}) R_{no} v_0, \qquad \tilde{c}_i = r_i + R_{no}^T R\left(\frac{\pi}{2}\right) R_{no} v_i - c_0,$$

$$\tilde{\xi} = [\tilde{v}_1, \ldots, \tilde{v}_n, \tilde{c}_1, \ldots, \tilde{c}_n]^T,$$

$$\tilde{\Gamma} = \begin{bmatrix} I_n \otimes (R_{no}^T R^{\perp} R_{no}) & 0_{3n\times 3n} \\ 0_{3n\times 3n} & 0_{3n\times 3n} \end{bmatrix} - \begin{bmatrix} F[(H^{-1}LH^{-1} + E_1) \otimes I_3]F^T & (L+E_1) \otimes I_3 \\ F\left[(H^{-1}LH^{-1} + E_1) \otimes \left(R_{no}^T R\left(\frac{\pi}{2}\right) R_{no}\right)\right] F^T & (L+E_1) \otimes \left(R_{no}^T R\left(\frac{\pi}{2}\right) R_{no}\right) \end{bmatrix} \in \mathbb{R}^{6n}$$

where E_1 is a diagonal matrix whose i_0th diagonal entry is 1 and all other entries are 0.

Box II.

Moreover, it is easy to see that

$$\begin{aligned} &\rho_i^{-2} v_i - \rho_i^{-1} \rho_k^{-1} R_{no}^T R(\theta_i - \theta_k) R_{no} v_k \\ &= \left(\rho_i^{-2} v_i - \frac{1}{\rho_i \rho_{i_0}} R_{no}^T R(\theta_i - \theta_{i_0}) R_{no} v_0\right) \\ &\quad - \left(\rho_i^{-1} \rho_k^{-1} R_{no}^T R(\theta_i - \theta_k) R_{no} v_k - \frac{1}{\rho_i \rho_{i_0}} R_{no}^T R(\theta_i - \theta_{i_0}) R_{no} v_0\right) \\ &= \rho_i^{-2} \tilde{v}_i - \rho_i^{-1} \rho_k^{-1} R_{no}^T R(\theta_i - \theta_k) R_{no} \tilde{v}_k \end{aligned}$$

and

$$\begin{aligned} &(r_i - r_k) + R_{no}^T R\left(\frac{\pi}{2}\right) R_{no}(v_i - v_k) \\ &= \left(r_i + R_{no}^T R\left(\frac{\pi}{2}\right) R_{no} v_i - c_0\right) - \left(r_k + R_{no}^T R\left(\frac{\pi}{2}\right) R_{no} v_k - c_0\right) \\ &= \tilde{c}_i - \tilde{c}_k. \end{aligned}$$

Then by simple calculations, using protocol (26)–(27) for (2), the closed-loop system can be written in a vector form as

$$\dot{\tilde{\xi}} = \tilde{\Gamma} \tilde{\xi}, \qquad (28)$$

where L is the Laplacian of the graph $\mathcal{G}$. It should be noted that $\dot{v}_0(t) = R_{no}^T R^{\perp} R_{no} v_0$. Clearly, if the graph $\mathcal{G}$ is connected, $L + E_1$ and $L + \rho_{i_0}^2 E_1$ are both symmetric positive definite matrices according to [5, Lemma 3] and hence $H^{-1}LH^{-1} + E_1$ is also a symmetric positive definite matrix. Then by a similar argument to that of the proofs of Lemma 4.1 and Theorem 4.5, the following lemma and theorem can be obtained.

Lemma 4.3. *If the graph $\mathcal{G}$ is connected, then all the eigenvalues of $\tilde{\Gamma}$ have negative real parts.*

Theorem 4.7. *Consider a network of second-order agents with a fixed topology. If the graph $\mathcal{G}$ is connected, the multi-agent system (2) with protocol (26)–(27) achieves the desired rotating formation $F(h(\rho, \theta), \mathrm{i}_\rho)$. Moreover, $\lim_{t\to+\infty}\left(v_i - \frac{\rho_i}{\rho_{i_0}} R_{no}^T R(\theta_i - \theta_{i_0}) R_{no} v_0\right) = 0$ and $\lim_{t\to+\infty}\left(r_i + R_{no}^T R\left(\frac{\pi}{2}\right) R_{no} v_i - c_0\right) = 0$ for all $i \in \mathcal{I}$.*

Remark 4.8. Theorem 4.7 shows that protocol (26)–(27)makes all agents finally surround a pre-specified point c_0 with a desired formation structure h. It should be noted that if we eliminate the term $-\left(r_{i_0} + R_{no}^T R\left(\frac{\pi}{2}\right) R_{no} v_{i_0} - c_0\right)$, then the multi-agent system (2) can also reach the desired rotating formation h but the common point surrounded by the agents might not be c_0.

Remark 4.9. In [21], to achieve circular motions, one control gain, namely, the rotation angle, must be exactly equal to a certain value to have the eigenvalues on the imaginary axis, which is not robust. In this paper, we do not impose such an assumption and the control parameters $\rho_i, \theta_i, i = 1, \ldots, n$, can be arbitrarily chosen in the real number field except for the points satisfying that $\mathbf{1}_n^T HM \mathbf{1}_n = 0$ when $|\theta_i - \theta_k| = 0$ or $|\theta_i - \theta_k| = \pi$ for all $i, k \in \mathcal{I}$.

Remark 4.10. In protocols (8) and (26)–(27), each agent need access its neighbors' position and velocity information, which might bring in communication time-delays. Further research could be directed towards considering the communication time-delays.

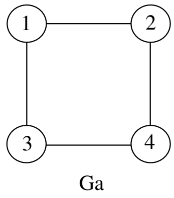

Fig. 4. The communication topology for the multi-agent system (2).

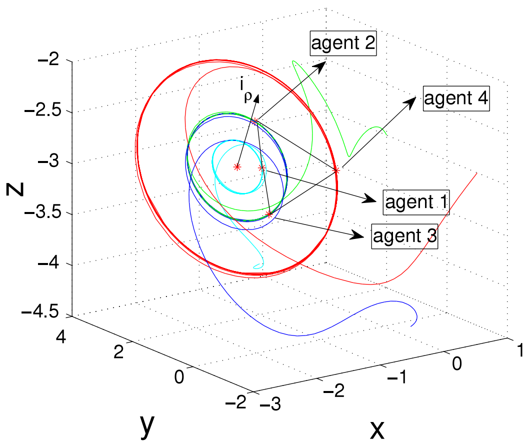

Fig. 5. Position trajectories of the multi-agent system using protocol (8).

5. Simulations

Numerical simulations will be given to illustrate the theoretical results obtained in the previous sections. The graph $\mathcal{G}_a$ in Fig. 4 is the communication topology for the multi-agent system (2), where the weight of each edge is 1. The initial condition of the multi-agent system is taken as $[r_1^T, v_1^T, r_2^T, v_2^T, r_3^T, v_3^T, r_4^T, v_4^T]^T = [-1, 2, -4, -3, -2, 0, 1, 2, -3, 0, 2, 1, -1, -3, -4, 0, -3, 1, 1, -1, -3, -3, 0, -2]^T$ and i_ρ is taken as $\left[\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right]^T$.

We first present simulation results to illustrate Theorem 4.5, we take $[\theta_1, \theta_2, \theta_3, \theta_4] = [0, \pi/3, -\pi/3, 0]$ and $[\rho_1, \rho_2, \rho_3, \rho_4] = [0.5, 1, 1, 2]$. Fig. 5 shows the position trajectories of all agents with protocol (8). It is obvious that all agents finally surround a common point with a formation structure that has the same shape but a different size from the desired formation structure on a plane perpendicular to the vector i_ρ. That is, the multi-agent system (2) reaches a desired quasi-rotating formation, which is consistent with Theorem 4.5.

Now, we give simulation results to illustrate Theorem 4.7. Suppose that the second agent s_2 has access to the desired velocity

$$v_0 = \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ -\frac{2}{\sqrt{6}} & 0 & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} \cos t \\ \sin t \\ 0 \end{bmatrix}$$

and the desired circle center $c_0 = [-1, -1, -1]^T$. Fig. 6 shows the position trajectories of all agents with protocol (26)–(27). It is clear that all agents finally

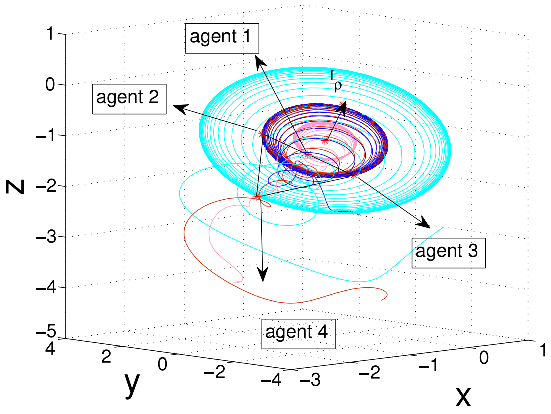

Fig. 6. Position trajectories of the multi-agent system using protocol (26)–(27).

surround a common point with the desired formation structure on a plane perpendicular to the vector i_ρ. That is, the multi-agent system reaches the desired rotating formation, which is consistent with Theorem 4.7.

6. Conclusions

In this paper, we have investigated the collective rotating formation control problem of second-order multi-agent systems in 3D. We proposed two distributed control protocols and employed a new Lyapunov-based approach to give conditions to make all agents surround a common point with a desired formation structure. Simulation results were provided to illustrate the theoretical results. Finally, it should be pointed out that the communication graph considered in this paper is undirected and fixed, and future research could be directed towards considering directed and switching communication graph.

Acknowledgements

This work was supported by the NSFC (60672029, 60334030, and 60774003) and the NSF CAREER Award (ECCS–0748287).

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