# Distributed Containment Control with Multiple Dynamic Leaders for Double-Integrator Dynamics Using Only Position Measurements

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***Abstract*—This note studies the distributed containment control problem for a group of autonomous vehicles modeled by double-integrator dynamics with multiple dynamic leaders. The objective is to drive the followers into the convex hull spanned by the dynamic leaders under the constraints that the velocities and the accelerations of both the leaders and the followers are not available, the leaders are neighbors of only a subset of the followers, and the followers have only local interaction. Two containment control algorithms via only position measurements of the agents are proposed. Theoretical analysis shows that the followers will move into the convex hull spanned by the dynamic leaders if the network topology among the followers is undirected, for each follower there exists at least one leader that has a directed path to the follower, and the parameters in the algorithm are properly chosen. Numerical results are provided to illustrate the theoretical results.**

***Index Terms*—Containment control, distributed control, double-integrator dynamics, multi-agent systems.**

## I. Introduction

The distributed multi-vehicle cooperative control has received increasing attention from researchers in different areas. This is due to its broad applications and its advantages such as low cost, high adaptivity, and easy maintenance, compared with its centralized counterpart. The leaderless consensus problem is a fundamental problem in distributed multi-vehicle cooperative control. The objective is to reach an agreement on certain quantities of interest among the vehicles through local interaction. Recently, significant progress has been made in the leaderless consensus problem (See [1]–[3] and references therein).

A more challenging problem in distributed multi-vehicle cooperative control is the coordinated tracking problem, where there exists a single or multiple dynamic leaders. In the single-leader case, the objective is to drive the states of the followers to approach the state of the dynamic leader. This problem and its variants were investigated in [4]–[7], [16]. In the multi-leader case, the objective is to drive the states of the followers into the convex hull spanned by those of the dynamic leaders, also called the containment control problem. The containment control problem has many applications in practice. For example, suppose that a group of robots are to move from one place to another, but only a subset of them has the ability to detect the hazardous obstacles. This subset of robots can be designed as leaders. The other robots can be designated as followers. For the followers, one way to reach the target area safely is to stay in the moving safe area formed by the leaders. In [8], a stop-and-go strategy was proposed for vehicles modeled by single-integrator kinematics under a fixed undirected network topology. In [9], the partial differential equation theory was exploited and a hybrid control schemes was proposed for the leaders. An extension to a switching directed network topology was given in [10], where the Lyapunov-based approach was used. In [11], the set input-to-state stability and the set integral input-to-state stability problems were considered for multi-agent systems with multiple leaders, where all the followers had nonlinear neighbor-based coordination rules. Note that [8]–[11] consider the multi-agent systems with single-integrator dynamics. In [12], the followers were assumed to have double-integrator dynamics, but the dynamics of the leaders were single integrators. In [13], both the leaders and the followers have double-integrator dynamics. However, the algorithms proposed in [13] require the velocity measurements to be available.

In reality, it is more difficult to obtain velocity and acceleration measurements than position measurements. We are hence motivated to design distributed containment control algorithms for autonomous vehicles with double-integrator dynamics in the presence of multiple dynamic leaders using only position measurements. The case where there exists a single dynamic leader can be treated as a special case of multiple dynamic leaders. When the absolute position measurements of the vehicles are available, we propose a distributed finite-time containment control algorithm. We show that the followers are driven into the convex hull spanned by the dynamic leaders in finite time if the network topology among the followers is undirected, for each follower there exists at least one leader that has directed path to the follower, and the parameters in the algorithm are properly chosen. When the absolute position measurements of the vehicles are not available, we propose a distributed adaptive containment control algorithm using the relative position measurements. We show that the followers will ultimately move into the convex hull spanned by the dynamic leaders under similar conditions to the case where the absolute position measurements are available.

The salient features of the algorithms proposed in this note are as follows. First, both algorithms can solve the distributed containment control problem with multiple dynamic leaders for vehicles with double-integrator dynamics while removing the requirement on the velocity measurements. Second, the first algorithm guarantees finite-time convergence without the requirement that the velocities of the leaders are identical. Third, in the second algorithm, the bound on the accelerations of the leaders is not required to be known. Fourth, the parameters in the second algorithm are not required to satisfy any condition related to the network topology. In contrast, existing algorithms for vehicles with double-integrator dynamics in [13] require the velocity measurements, can guarantee finite-time convergence only when the velocities of the leaders are identical, require the bound on the accelerations of the leaders to be known, and require parameters in the algorithm to satisfy a certain condition related to the network topology when the accelerations of the leaders are not identical. A preliminary version of the work has appeared in [15].

*Notations:* Define $\mathbf{1}_p \triangleq [1,\cdots,1]^T \in \mathbb{R}^p$. Given a vector $\nu = [\nu_1,\cdots,\nu_p]^T \in \mathbb{R}^p$ and $\alpha \in \mathbb{R}$, define $\mathrm{sig}(\nu)^\alpha \triangleq [\mathrm{sgn}(\nu_1)|\nu_1|^\alpha,\cdots,\mathrm{sgn}(\nu_p)|\nu_p|^\alpha]^T$ and $\mathrm{sgn}(\nu) \triangleq [\mathrm{sgn}(\nu_1),\cdots,\mathrm{sgn}(\nu_p)]^T$, where $\mathrm{sgn}(\cdot)$ is the signum function. We use $\mathrm{diag}(\nu_1,\cdots,\nu_p)$ to denote the diagonal matrix of all $\nu_1,\cdots,\nu_p$, $\nu_{(l)}$ to denote the $l$th element of $\nu$, and $I_p$ to denote the $p$ by $p$ identical matrix.

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## II. Background and Preliminaries

Consider a group of $n+s$ vehicles. We use a graph $\mathcal{G} \triangleq (\mathcal{V}, \mathcal{E})$ to denote the network topology among vehicles 1 to $n+s$, where $\mathcal{V} \triangleq \{1, \cdots, n+s\}$ is the node set and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the edge set. A directed edge $(j,i) \in \mathcal{E}$ if vehicle $i$ can access information from vehicle $j$ but not necessarily vice versa. An undirected edge $(j,i) \in \mathcal{E}$ if vehicle $i$ and vehicle $j$ can access information from each other. Here we assume that $(i,i) \notin \mathcal{E}$. The neighbor set $\mathcal{N}_i$ of vehicle $i$ is defined as $\mathcal{N}_i \triangleq \{j | (j,i) \in \mathcal{E}\}$. Suppose that vehicles 1 to $n$ have at least one neighbor and vehicles $n+1$ to $n+s$ have no neighbor. We call vehicles 1 to $n$ the *followers* and vehicles $n+1$ to $n+s$ the *leaders*. A graph is undirected if $(i,j) \in \mathcal{E}$ implies that $(j,i) \in \mathcal{E}$. We assume that the graph associated with the followers are undirected and further assume that $a_{ij} = a_{ji}$, $i,j = 1, \cdots, n$. A directed path is a sequence of directed edges of the form $(i_1, i_2), (i_2, i_3), \ldots$, where $i_j \in \mathcal{V}$. An undirected path is defined analogously. The adjacency matrix $\mathcal{A}_d \triangleq [a_{ij}] \in \mathbb{R}^{(n+s)\times(n+s)}$ is defined as $a_{ij} > 0$ if $(j,i) \in \mathcal{E}$ and $a_{ij} = 0$ otherwise. It is easy to see that $a_{ij} = 0, i = n+1, \cdots, n+s, j = 1, \cdots, n+s$ because the leaders have no neighbors. The Laplacian matrix $\mathcal{L} \triangleq [l_{ij}] \in \mathbb{R}^{(n+s)\times(n+s)}$ is defined as $l_{ii} = \sum_{j=1, j\neq i}^{n+s} a_{ij}$ and $l_{ij} = -a_{ij}, i \neq j$. Note that $\mathcal{L}$ can be rewritten as

$$\mathcal{L} = \begin{bmatrix} L_1 & L_2 \\ 0_{s\times n} & 0_{s\times s} \end{bmatrix}. \tag{1}$$

Suppose that all the vehicles have double-integrator dynamics given by

$$\dot{x}_i(t) = v_i(t), \quad \dot{v}_i(t) = u_i(t), \quad i = 1, \cdots, n+s \tag{2}$$

where $x_i(t)$, $v_i(t)$ and $u_i(t) \in \mathbb{R}^m$ are, respectively, the position, velocity and control input associated with the $i$th vehicle. Suppose that all leaders' control inputs have been *a priori* chosen as $u_i(t) = f_i(t)$, $i = n+1, \cdots, n+s$, where $f_i(t)$ specify the leaders' accelerations and hence the dynamic convex hull formed by the leaders. In this note, we focus on the controller design for the followers. We have the following definition.

*Definition 2.1:* Let $\mathcal{C} \subseteq \mathbb{R}^p$. The set $\mathcal{C}$ is said to be convex if for any $x$ and $y$ in $\mathcal{C}$, the point $(1-\alpha)x + \alpha y$ is in $\mathcal{C}$ for any $\alpha \in [0,1]$. The convex hull of a set of points $X = \{x_1, \cdots, x_q\}$ is the minimal convex set containing all points in $X$. We use $\mathrm{Co}(X)$ to denote the convex hull of $X$. Let $\Omega(t) \triangleq \mathrm{Co}\{x_{n+1}(t), \cdots, x_{n+s}(t)\}$ and $\Upsilon(t) \triangleq \mathrm{Co}\{v_{n+1}(t), \cdots, v_{n+s}(t)\}$.

The objective of the distributed containment control problem is to design $u_i(t)$ for all the followers such that the followers move into the convex hull spanned by the dynamic leaders, i.e., $\inf_{y(t)\in\Omega(t)} \|x_i(t) - y(t)\| \to 0$ and $\inf_{y(t)\in\Upsilon(t)} \|v_i(t) - y(t)\| \to 0$, $i = 1, \cdots, n$, as $t \to \infty$. Before moving on, we need the following assumptions and lemmas.

*Assumption 2.2:* $\|v_i(t)\|_\infty$, $\|f_i(t)\|_\infty$ and $\|\dot{f}_i(t)\|_\infty$, $i = n+1, \cdots, n+s$, are all bounded.

*Assumption 2.3:* For each follower, there exists at least one leader that has a directed path to the follower.

*Lemma 2.1:* [14] Under Assumption 2.3, $L_1$ defined in (1) is symmetric positive definite.

Note from Lemma 2.1 that $L_1$ is invertible. Let $x_L(t) \triangleq [x_{n+1}^T(t), \cdots, x_{n+s}^T(t)]^T$, $v_L(t) \triangleq [v_{n+1}^T(t), \cdots, v_{n+s}^T(t)]^T$, and $x_d(t) \triangleq [x_{d1}^T(t), \cdots, x_{dn}^T(t)]^T \triangleq -(L_1^{-1}L_2 \otimes I_m)x_L(t)$, where $x_{di}(t) \in \mathbb{R}^m$. It follows that $\ddot{x}_d(t) = -(L_1^{-1}L_2 \otimes I_m)v_L(t)$ and $\ddot{x}_d(t) = -(L_1^{-1}L_2 \otimes I_m)\dot{v}_L(t)$.

*Lemma 2.2:* Under Assumption 2.3, $\inf_{y(t)\in\Omega(t)} \|x_{di}(t) - y(t)\| = 0$ and $\inf_{y(t)\in\Upsilon(t)} \|\dot{x}_{di}(t) - y(t)\| = 0$, $i = 1, \cdots, n$, for all $t$.

*Proof:* Under Assumption 2.3, by Lemma 4 in [14] we know that each entry of $-L_1^{-1}L_2$ is nonnegative and each row sum of $-L_1^{-1}L_2$ is equal to one, which implies that $\inf_{y(t)\in\Omega(t)} \|x_{di}(t) - y(t)\| = 0$ and $\inf_{y(t)\in\Upsilon(t)} \|\dot{x}_{di}(t) - y(t)\| = 0$, $i = 1, \cdots, n$. ■

Note from Lemma 2.2 that $x_{di}(t)$ and $\dot{x}_{di}(t)$ belong to, respectively, the convex hull formed by the positions and velocities of the leaders. If for each follower $x_i(t)$ can be driven to $x_{di}(t)$ and $v_i(t)$ can be driven to $\dot{x}_{di}(t)$, then the containment control problem is solved. Therefore, $x_{di}(t)$ and $\dot{x}_{di}(t)$ can be regarded as, respectively, the desired position and velocity of the $i$th follower in the convex hull formed by the leaders. Because $\|v_i(t)\|_\infty$ and $\|f_i(t)\|_\infty$, $i = n+1, \cdots, n+s$, are bounded (see Assumption 2.2), it follows that $\|\dot{x}_d(t)\|_\infty$ and $\|\ddot{x}_d(t)\|_\infty$ are also bounded. We hence assume that $\|\dot{x}_d(t)\|_\infty \leq \eta_a$ and $\|\ddot{x}_d(t)\|_\infty \leq \eta_b$.

## III. Main Results

### A. Containment Control Using Absolute Position Measurements

In this section, we assume that the absolute position measurements of the vehicles are available but the velocity and acceleration measurements are not available. We propose the following containment control algorithm:

$$u_i(t) = -\alpha \mathrm{sgn}\left\{z_{1i}(t) + \beta \mathrm{sig}\left[x_i(t) - \hat{x}_i(t)\right]^{\frac{1}{2}}\right\} \tag{3a}$$

$$\dot{z}_{0i}(t) = z_{1i}(t) - k_1 \mathrm{sig}\left\{\bar{z}_{0i}(t)\right\}^{\frac{1}{2}} \tag{3b}$$

$$\begin{aligned}\dot{z}_{1i}(t) = &- k_2 \mathrm{sgn}\left\{\bar{z}_{0i}(t)\right\} \\ &- \alpha \mathrm{sgn}\left\{z_{1i}(t) + \beta \mathrm{sig}\left[x_i(t) - \hat{x}_i(t)\right]^{\frac{1}{2}}\right\}\end{aligned} \tag{3c}$$

$$\begin{aligned}\dot{\hat{x}}_i(t) = &- k_3 \mathrm{sgn}\left\{\sum_{j=1}^{n} a_{ij}\left[\hat{x}_i(t) - \hat{x}_j(t)\right] + \sum_{j=n+1}^{n+s} a_{ij}\left[\hat{x}_i(t) - x_j(t)\right]\right\}, \\ & i = 1, \cdots, n\end{aligned} \tag{3d}$$

where $\hat{x}_i(0) = 0$, $\bar{z}_{0i}(t) \triangleq z_{0i}(t) - [x_i(t) - \hat{x}_i(t)]$, for $i = 1, \cdots, n$, $k_1, k_2, k_3, \alpha$ and $\beta$ are positive constant scalars, and $a_{ij}$, $i = 1, \cdots, n$, $j = 1, \cdots, n+s$, is the $(i,j)$th entry of the adjacency matrix $\mathcal{A}_d$. Throughout this note, the solutions to the closed-loop systems are understood in the Filippov sense [19].

*Remark 3.1:* In (3d), $\hat{x}_i(t)$ is used to estimate $x_{di}(t)$, like $\hat{x}_{fi}(t)$ in [14]. As will be shown in Lemma 3.1, using (3d), $\hat{x}_i(t)$ will converge to $x_{di}(t)$ in finite time. Without loss of generality, let $\hat{x}_i(t) = x_{di}(t)$ for $t \geq T$. Therefore, when $t \geq T$, $\hat{x}_i(t)$ in (3a), (3b) and (3c) can be replaced with $x_{di}(t)$. Controller (3a)–(3c) is motivated by controller (8)–(10) in [18] with a little modification. In (3b) and (3c), $z_{0i}(t)$ and $z_{1i}(t)$ are adopted to estimate, respectively, $x_i(t) - x_{di}(t)$ and $v_i(t) - \dot{x}_{di}(t)$. If $z_{1i}(t)$ converges to $v_i(t) - \dot{x}_{di}(t)$ in finite time, $u_i(t)$ in (3a) can then drive $x_i(t)$ to $x_{di}(t)$ and $v_i(t)$ to $\dot{x}_{di}(t)$ in finite time.

Before moving on, we need the following lemmas.

*Lemma 3.1:* Using (3d), $\|\hat{x}_i(t) - x_{di}(t)\| \to 0$, $i = 1, \cdots, n$, in finite time if $k_3 > \eta_a$.

*Proof:* Let $\bar{\hat{x}}_i(t) = \hat{x}_i(t) - x_{di}(t)$, $i = 1, \cdots, n$, $\bar{\hat{x}}(t) = [\bar{\hat{x}}_1^T(t), \cdots, \bar{\hat{x}}_n^T(t)]^T$ and $\bar{V} = (1/2)\bar{\hat{x}}(t)(L_1 \otimes I_m)\bar{\hat{x}}(t)$. Following a similar proof to that of Theorem 2 in [14], we can get that $\|\hat{x}_i(t) - x_{di}(t)\| \to 0$, $i = 1, \cdots, n$, in finite time if $k_3 > \eta_a$. ■

*Lemma 3.2:* [17] Consider the system

$$\begin{aligned}\dot{x}_1(t) &= x_2(t) - k_1 \mathrm{sig}\left[x_1(t)\right]^{\frac{1}{2}}, \\ \dot{x}_2(t) &= -k_2 \mathrm{sgn}\left[x_1(t)\right] + \rho(t,x)\end{aligned}$$

where $x_1(t), x_2(t) \in \mathbb{R}$, $k_1, k_2$ are constant positive scalars and $\rho(t,x)$ is a bounded perturbation with $x \triangleq [x_1(t)\ x_2(t)]^T$. Suppose that there exists a symmetric positive-definite matrix $P$ such that the linear matrix inequality

$$A^T P + PA + \varepsilon^2 C^T C + PBB^T P < 0 \tag{4}$$

is satisfied, where $A \triangleq \begin{bmatrix} -1/2k_1 & 1/2 \\ -k_2 & 0 \end{bmatrix}$, $B \triangleq \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $C \triangleq [1\,0]$, and $\varepsilon$ is a positive constant scalar. Then $x_1(t)$ and $x_2(t)$ will converge to zero in finite time for all bounded perturbations satisfying $|\rho(t,x)| \le \varepsilon$.

*Lemma 3.3:* [18] Consider the system $\ddot{x}(t) = f(t,x) - \alpha K(t,x)\mathrm{sgn}\{\dot{x}(t) + \beta \mathrm{sig}[x(t)]^{1/2}\}$, where $x(t) \in \mathbb{R}$, $|f(t,x)| \le D$, $K_m \le K(t,x) \le K_M$, and $\alpha$, $\beta$, $D$, $K_m$ and $K_M$ are constant positive scalars. Then, $x(t)$ and $\dot{x}(t)$ will converge to zero in finite time if $\alpha > 1/K_m(D + (\beta^2/2))$.

*Theorem 3.1:* Under Assumption 2.3, using (3) for (2), $\inf_{y(t)\in\Omega(t)} \|x_i(t) - y(t)\| \to 0$ and $\inf_{y(t)\in\Upsilon(t)} \|v_i(t) - y(t)\| \to 0$ in finite time if $\alpha > \eta_b + (\beta^2/2)$, $k_3 > \eta_a$ and there exist $k_1 > 0$, $k_2 > 0$ and a symmetric positive-definite matrix $P$ such that (4) is satisfied, where $\epsilon = \eta_b$. In particular, $\|x_i(t) - x_{di}(t)\| \to 0$ and $\|v_i(t) - \dot{x}_{di}(t)\| \to 0$, $i = 1, \cdots, n$, in finite time.

*Proof:* Note from Lemma 3.1 that there exists a $T_1 > 0$ such that $\hat{x}_i(t) = x_{di}(t)$, $i = 1, \cdots, n$, for all $t \ge T_1$. We next show that $x_i(t)$, $v_i(t)$, $\hat{x}_i(t)$, $z_{0i}(t)$ and $z_{1i}(t)$, $i = 1, \cdots, n$, will not diverge to infinity for all $t \in [0, T_1]$. Because from (3a) $\|u_i(t)\|_\infty \le \alpha$, it is easy to see that $x_i(t)$ and $v_i(t)$ are bounded for all $t \in [0, T_1]$. Because from (3d) $\|\dot{\hat{x}}_i(t)\|_\infty \le k_3$, it follows that $\hat{x}_i(t)$ is bounded for all $t \in [0, T_1]$, which implies that $x_i(t) - \hat{x}_i(t)$ is bounded for all $t \in [0, T_1]$. Because from (3c) $\|\dot{z}_{1i}(t)\|_\infty \le k_2 + \alpha$, it follows that $z_{1i}(t)$ is bounded for all $t \in [0, T_1]$. From (3b) we have that $\dot{\bar{z}}_{0i}(t) = z_{1i}(t) - [v_i(t) - \dot{\hat{x}}_i(t)] - k_1 \mathrm{sig}[\bar{z}_{0i}(t)]^{1/2}$. Because $z_{1i}(t)$, $v_i(t)$ and $\dot{\hat{x}}_i(t)$ are bounded, we assume that $\|z_{1i}(t) - [v_i(t) - \dot{\hat{x}}_i(t)]\|_\infty < \gamma$ for all $t \in [0, T_1]$. Suppose that $|\bar{z}_{0i(l)}(t_1)| > (\gamma^2/k_1^2)$ at a certain $t_1 \in [0, T_1]$. If $\bar{z}_{0i(l)}(t_1) > (\gamma^2/k_1^2)$, then it follows that:

$$\begin{aligned}\dot{\bar{z}}_{0i(l)}(t_1) = z_{1i(l)}(t_1) - \left[v_{i(l)}(t_1) - \dot{\hat{x}}_{i(l)}(t_1)\right] \\ -k_1 \left|\bar{z}_{0i(l)}(t_1)\right|^{\frac{1}{2}} < \gamma - k_1 \left|\bar{z}_{0i(l)}(t_1)\right|^{\frac{1}{2}} < 0.\end{aligned}$$

.

If $\bar{z}_{0i(l)}(t_1) < -(\gamma^2/k_1^2)$, then it follows that:

$$\begin{aligned}\dot{\bar{z}}_{0i(l)}(t_1) = z_{1i(l)}(t) - \left[v_{i(l)}(t_1) - \dot{\hat{x}}_{i(l)}(t_1)\right] \\ +k_1 \left|\bar{z}_{0i(l)}(t_1)\right|^{\frac{1}{2}} > -\gamma + k_1 \left|\bar{z}_{0i(l)}(t_1)\right|^{\frac{1}{2}} > 0.\end{aligned}$$

.

Therefore, because $\bar{z}_{0i(l)}(0)|$ is bounded, $\bar{z}_{0i(l)}(t)$ will not diverge to infinity for all $t \in [0, T_1]$, which implies that $z_{0i(l)}(t)$ will not diverge to infinity for all $t \in [0, T_1]$. Thus $x_{di}(t)$ can be used to replace $\hat{x}_i(t)$ for $t \ge T_1$.

For $t \ge T_1$, because $\hat{x}_i(t) \equiv x_{di}(t)$, it follows from (2) and (3) that:

$$\begin{aligned}\dot{\tilde{z}}_{0i}(t) &= \tilde{z}_{1i}(t) - k_1 \mathrm{sig}\left[\tilde{z}_{0i}(t)\right]^{\frac{1}{2}},\\ \dot{\tilde{z}}_{1i}(t) &= -k_2 \mathrm{sgn}\left[\tilde{z}_{0i}(t)\right] + \ddot{x}_{di}(t)\end{aligned}$$

where $\tilde{z}_{0i}(t) \triangleq z_{0i}(t) - \tilde{x}_i(t)$, $\tilde{z}_{1i}(t) \triangleq z_{1i}(t) - \dot{\tilde{x}}_i(t)$, $\tilde{x}_i(t) \triangleq x_i(t) - x_{di}(t)$. If there exists a symmetric positive definite matrix $P$ such that (4) is satisfied, where $\epsilon = \eta_b$, it follows from Lemma 3.2 that there exists $T_2 > T_1$ such that $\tilde{z}_{0i}(t) = 0$ and $\tilde{z}_{1i}(t) = 0$ for all $t \ge T_2$, which implies that $z_{0i}(t) = \tilde{x}_i(t)$ and $z_{1i}(t) = \dot{\tilde{x}}_i(t)$ for all $t \ge T_2$. It follows from a similar statement to the above that $x_i(t)$, $v_i(t)$, $z_{0i}(t)$, $z_{1i}(t)$ are all bounded for all $t \in [T_1, T_2]$. Thus $\dot{\tilde{x}}_i(t)$ can be used to replace $z_{1i}(t)$ for $t \ge T_2$.

For $t \ge T_2$, because $z_{1i}(t) \equiv \dot{\tilde{x}}_i(t)$, the closed-loop system of (2) using (3a) becomes $\ddot{\tilde{x}}_i(t) = -\alpha \mathrm{sgn}\{\dot{\tilde{x}}_i(t) + \beta \mathrm{sig}[\tilde{x}_i(t)]^{1/2}\} - \ddot{x}_{di}(t)$. Because $\alpha > \eta_b + (\beta^2/2)$, it follows from Lemma 3.3 that there exists $T_3 > T_2$ such that $\tilde{x}_i(t) = 0$ and $\dot{\tilde{x}}_i(t) = 0$ for all $t \ge T_3$, which implies that $\|x_i(t) - x_{di}(t)\|$ and $\|v_i(t) - \dot{x}_{di}(t)\|$ will converge to zero in finite time. It follows from Lemma 2.2 that $\inf_{y(t)\in\Omega(t)} \|x_i(t) - y(t)\| \to 0$ and $\inf_{y(t)\in\Upsilon(t)} \|v_i(t) - y(t)\| \to 0$ in finite time. ■

Next we show how to choose the gains $k_1$ and $k_2$ in (3) such that there exists a symmetric positive-definite matrix $P$ such that (4) is satisfied, where $\epsilon = \eta_b$.

*Lemma 3.4:* Given a constant $\varepsilon > 0$, there exists a symmetric-positive definite matrix $P$ such that (4) is satisfied if $k_2 > \varepsilon$ and $k_2 + 2 - \sqrt{k_2^2 - \varepsilon^2} < k_1 < k_2 + 2 + \sqrt{k_2^2 - \varepsilon^2}$.

*Proof:* Let $P = \begin{bmatrix} z & -2 \\ -2 & 1 \end{bmatrix}$. It is easy to see that $P$ is symmetric positive definite if and only if $z > 4$. For a given constant $\varepsilon > 0$, we have that

$$\begin{aligned}&A^T P + PA + \varepsilon^2 C^T C + PBB^T P\\ &\quad = \begin{bmatrix} -k_1 z + 4k_2 + 4 + \varepsilon^2 & k_1 - k_2 + \frac{1}{2}z - 2 \\ k_1 - k_2 + \frac{1}{2}z - 2 & -1 \end{bmatrix}.\end{aligned}$$

Suppose that $k_2 > \varepsilon$ and $k_2 + 2 - \sqrt{k_2^2 - \varepsilon^2} < k_1 < k_2 + 2 + \sqrt{k_2^2 - \varepsilon^2}$. It is easy to check that there exists $z > 4$ such that $-k_1 z + 4k_2 + 4 + \varepsilon^2 + (k_1 - k_2 + (1/2)z - 2)^2 < 0$, which implies that $A^T P + PA + \varepsilon^2 C^T C + PBB^T P < 0$. ■

### *B. Containment Control Using Relative Position Measurements*

In this section, we assume that the relative position measurements of the vehicles are available but the velocity and acceleration measurements are not available. We propose the following algorithm:

$$\begin{aligned}u_i(t) =& - D_i(t)\mathrm{sgn}\left\{\sum_{j=1}^{n+s} a_{ij}\left[x_i(t) - x_j(t)\right]\right\}\\ &- k_1\left\{\sum_{j=1}^{n+s} a_{ij}\left[x_i(t) - x_j(t)\right]\right\} - k_2 \hat{v}_i(t) \qquad (5a)\\ \dot{\hat{v}}_i(t) =& - D_i(t)\mathrm{sgn}\left\{\sum_{j=1}^{n+s} a_{ij}\left[x_i(t) - x_j(t)\right]\right\}\\ &- k_1\left\{\sum_{j=1}^{n+s} a_{ij}\left[x_i(t) - x_j(t)\right]\right\} - k_2 \hat{v}_i(t),\\ i =& 1, \cdots, n \qquad (5b)\end{aligned}$$

where $\hat{v}_i(0) = 0$ for $i = 1, \cdots, n$, $D_i(t) \triangleq \mathrm{diag}[d_{i1}(t), \cdots, d_{im}(t)]$ with

$$\begin{aligned}d_{il}(t) \triangleq& \left|\sum_{j=1}^{n+s} a_{ij}\left[x_{i(l)}(t) - x_{j(l)}(t)\right]\right|\\ &+ \int_0^t \left|\sum_{j=1}^{n+s} a_{ij}\left[x_{i(l)}(\tau) - x_{j(l)}(\tau)\right]\right| d\tau \qquad (6)\end{aligned}$$

for $l = 1, \cdots, m$, and $k_1$ and $k_2$ are constant positive scalars.

*Remark 3.2:* Because the velocities of the followers are not available, we use $\hat{v}_i(t)$ to estimate $v_i(t)$ for $i = 1, \cdots, n$. Since only relative position measurements are available, it is difficult to estimate $v_i(t)$ accurately. Therefore, we let $\dot{\hat{v}}_i(t) \equiv \dot{v}_i(t)$ for $i = 1, \cdots, n$, so that $v_i(t) - \hat{v}_i(t) \equiv v_i(0) - \hat{v}_i(0)$ for $i = 1, \cdots, n$. In the following analysis we will show that this is enough to guarantee that the followers move into the convex hull formed by the leaders.

Define $\psi_i(t) \triangleq \sum_{j=1}^{n+s} a_{ij}[x_i(t) - x_j(t)]$, $\phi_i(t) \triangleq \sum_{j=n+1}^{n+s} a_{ij}[k_2 v_j(t) + f_j(t)]$ and $\bar{v}_i(t) \triangleq \hat{v}_i(t) - v_i(t)$, $i = 1, \cdots, n$. Also define $\Psi(t) \triangleq [\psi_1^T(t), \cdots, \psi_n^T(t)]^T$,

$\Phi(t) \triangleq [\phi_1^T(t), \cdots, \phi_n^T(t)]^T$ and $\bar{v}(t) \triangleq [\bar{v}_1^T(t), \cdots, \bar{v}_n^T(t)]^T$. Because $\|v_i(t)\|_\infty$, $\|f_i(t)\|_\infty$ and $\|\dot{f}_i(t)\|_\infty$, $i = n+1, \cdots, n+s$, are all bounded, it is easy to see that $\Phi(t)$ and $\dot{\Phi}(t)$ are also bounded.

*Lemma 3.5:* Under Assumption 2.3, consider the function

$$V_1(t) = V_2 + \int_0^t \left[\Psi(\tau) + \dot{\Psi}(\tau)\right]^T \times \left\{k^* \operatorname{sgn}\left[\Psi(\tau)\right] + \left(L_1^{-1} \otimes I_m\right)\Phi(\tau) + k_2 \bar{v}(0)\right\} d\tau$$

where $k^*$ is a constant positive scalar and $V_2 \triangleq k^* \Psi^T(0)\operatorname{sgn}[\Psi(0)] + \Psi^T(0)(L_1^{-1} \otimes I_m)\Phi(0) + k_2 \Psi^T(0)\bar{v}(0)$. $V_1(t) \geq 0$ if $k^*$ satisfies

$$k^* > \max\left\{\left\|\left(L_1^{-1} \otimes I_m\right)\left[\Phi(t) - \dot{\Phi}(t)\right]\right\|_\infty, \left\|\left(L_1^{-1} \otimes I_m\right)\Phi(t)\right\|_\infty\right\} + k_2 \|\bar{v}(0)\|_\infty. \quad (7)$$

*Proof:* See the Appendix. ■

*Theorem 3.2:* Under Assumption 2.3, using (5) for (2), $\inf_{y(t)\in\Omega(t)} \|x_i(t) - y(t)\| \to 0$ and $\inf_{y(t)\in\Upsilon(t)} \|v_i(t) - y(t)\| \to 0$ as $t \to \infty$ if $k_1 > 0$ and $k_2 > 1$. In particular, $\|x_i(t) - x_{di}(t)\| \to 0$ and $\|v_i(t) - \dot{x}_{di}(t)\| \to 0$, $i = 1, \cdots, n$, as $t \to \infty$.

*Proof:* From (2) and (5) we know that $\dot{\bar{v}}_i(t) = \dot{\hat{v}}_i(t) - \dot{v}_i(t) \equiv 0$, $i = 1, \cdots, n$. It follows that $\bar{v}_i(t) \equiv \bar{v}_i(0)$, $i = 1, \cdots, n$. Equation (5a) can be rewritten as $u_i(t) = -D_i(t)\operatorname{sgn}[\psi_i(t)] - k_1\psi_i(t) - k_2\hat{v}_i(t)$, $i = 1, \cdots, n$. It follows that:

$$\begin{aligned}
\ddot{\psi}_i(t) &= \sum_{j=1}^{n+s} a_{ij}\left[\ddot{x}_i(t) - \ddot{x}_j(t)\right] \\
&= -\sum_{j=1}^{n+s} a_{ij}\left\{D_i(t)\operatorname{sgn}\left[\psi_i(t)\right] + k_1\psi_i(t) + k_2\hat{v}_i(t)\right\} \\
&\quad + \sum_{j=1}^{n} a_{ij}\left\{D_j(t)\operatorname{sgn}\left[\psi_j(t)\right] + k_1\psi_j(t) + k_2\hat{v}_j(t)\right\} \\
&\quad - \sum_{j=n+1}^{n+s} a_{ij} f_j(t) \\
&= -\sum_{j=1}^{n+s} a_{ij}\left\{D_i(t)\operatorname{sgn}\left[\psi_i(t)\right] + k_1\psi_i(t)\right\} \\
&\quad - k_2 \sum_{j=1}^{n+s} a_{ij}\left[v_i(t) + \bar{v}_i(0)\right] + k_2 \sum_{j=1}^{n} a_{ij}\bar{v}_j(0) \\
&\quad + \sum_{j=1}^{n} a_{ij}\left\{D_j(t)\operatorname{sgn}\left[\psi_j(t)\right] + k_1\psi_j(t)\right\} \\
&\quad + k_2 \sum_{j=1}^{n+s} a_{ij} v_j(t) - \phi_i(t). \qquad (8)
\end{aligned}$$

Note that (8) can be rewritten in a vector form as

$$\ddot{\Psi}(t) = -(L_1 \otimes I_m)D(t)\operatorname{sgn}\left[\Psi(t)\right] - k_1(L_1 \otimes I_m)\Psi(t) - k_2\dot{\Psi}(t) - k_2(L_1 \otimes I_m)\bar{v}(0) - \Phi(t) \quad (9)$$

where $D(t)$ is a block diagonal matrix of all $D_i(t)$, $i = 1, \cdots, n$.

Consider the following Lyapunov function candidate:

$$\begin{aligned}
V(t) = &\frac{1}{2}\left[\Psi(t) + \dot{\Psi}(t)\right]^T \left(L_1^{-1} \otimes I_m\right)\left[\Psi(t) + \dot{\Psi}(t)\right] + V_1(t) \\
&+ \frac{1}{2}\Psi^T(t)\left[k_1 I_{nm} + (k_2 - 1)\left(L_1^{-1} \otimes I_m\right)\right]\Psi(t) \\
&+ \frac{1}{2}\left[D(t)\mathbf{1}_{nm} - k^*\mathbf{1}_{nm}\right]^T \left[D(t)\mathbf{1}_{nm} - k^*\mathbf{1}_{nm}\right]
\end{aligned}$$

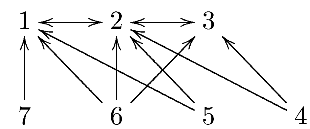



Fig. 1. Network topology associated with vehicles 1 to 7. Here $i$ denotes vehicle $i$, $i = 1, \cdots, 7$.

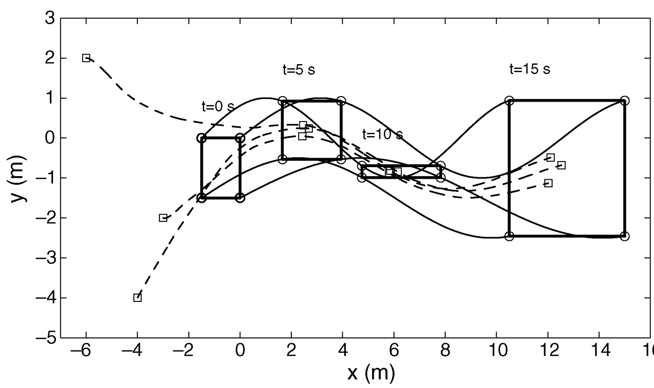



Fig. 2. Trajectories of vehicles 1 to 7 using (3). The circles denote the leaders while the squares denote the followers.

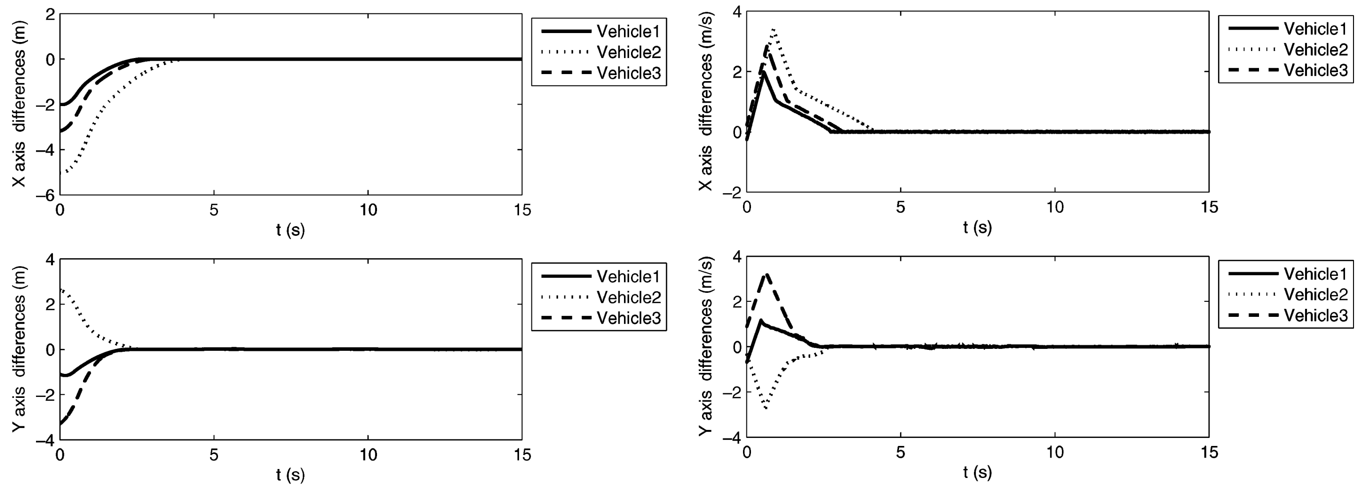
where $k^*$ is a constant satisfying (7). Under Assumption 2.3, it follows from Lemma 2.1 that $L_1$ is symmetric positive definite, which means that $L_1^{-1}$ is also symmetric positive definite. Because $k^*$ satisfies (7), it follows from Lemma 3.5 that $V_1(t) \geq 0$. Because $k_1 > 0$ and $k_2 > 1$, we have that $k_1 I_{nm} + (k_2 - 1)(L_1^{-1} \otimes I_m)$ is symmetric positive definite. Therefore, $V(t)$ is symmetric positive definite with respect to $\Psi(t)$, $\dot{\Psi}(t)$ and $D(t)\mathbf{1}_{nm} - k^*\mathbf{1}_{nm}$.

For $i = 1, \cdots, n$, $l = 1, \cdots, m$, from (6) we have that

$$\begin{aligned}
\dot{d}_{il}(t) = &\left\{\sum_{j=1}^{n+s} a_{ij}\left[v_{i(l)}(t) - v_{j(l)}(t)\right]\right\} \\
&\times \operatorname{sgn}\left\{\sum_{j=1}^{n+s} a_{ij}\left[x_{i(l)}(t) - x_{j(l)}(t)\right]\right\} \\
&+ \left\{\sum_{j=1}^{n+s} a_{ij}\left[x_{i(l)}(t) - x_{j(l)}(t)\right]\right\} \\
&\times \operatorname{sgn}\left\{\sum_{j=1}^{n+s} a_{ij}\left[x_{i(l)}(t) - x_{j(l)}(t)\right]\right\}.
\end{aligned}$$

It follows that:

$$\begin{aligned}
&\left[D(t)\mathbf{1}_{nm} - k^*\mathbf{1}_{nm}\right]^T \dot{D}(t)\mathbf{1}_{nm} \\
&= \sum_{i=1}^{n}\sum_{l=1}^{m}\left[d_{il}(t) - k^*\right]\dot{d}_{il}(t) \\
&= \sum_{i=1}^{n}\sum_{l=1}^{m}\left[d_{il}(t) - k^*\right] \\
&\quad \times \left\{\sum_{j=1}^{n+s} a_{ij}\left[x_{i(l)}(t) - x_{j(l)}(t)\right] + \sum_{j=1}^{n+s} a_{ij}\left[v_{i(l)}(t) - v_{j(l)}(t)\right]\right\} \\
&\quad \times \operatorname{sgn}\left\{\sum_{j=1}^{n+s} a_{ij}\left[x_{i(l)}(t) - x_{j(l)}(t)\right]\right\} \\
&= \left[\Psi(t) + \dot{\Psi}(t)\right]^T \left[D(t) - k^* I_{nm}\right]\operatorname{sgn}\left[\Psi(t)\right].
\end{aligned}$$




(a) (b)

Fig. 3. Trajectories of $x_i(t) - x_{di}(t)$ and $v_i(t) - \dot{x}_{di}(t)$ using (3), $i = 1, 2, 3$. (a) $x_i(t) - x_{di}(t)$; $v_i(t) - \dot{x}_{di}(t)$.

Taking the derivative of $V(t)$, we have that

$$\begin{aligned}\dot{V}(t) =& \left[\Psi(t) + \dot{\Psi}(t)\right]^T \left(L_1^{-1} \otimes I_m\right)\\ &\times \Big\{ -(L_1 \otimes I_m)D(t)\mathrm{sgn}\left[\Psi(t)\right] - k_1(L_1 \otimes I_m)\Psi(t)\\ &\quad -(k_2 - 1)\dot{\Psi}(t) - k_2(L_1 \otimes I_m)\bar{v}(0) - \Phi(t)\Big\}\\ &+ \left[\Psi(t) + \dot{\Psi}(t)\right]^T\\ &\times \left\{k^* \mathrm{sgn}\left[\Psi(t)\right] + \left(L_1^{-1} \otimes I_m\right)\Phi(t) + k_2\bar{v}(0)\right\}\\ &+ \Psi^T(t)\left[k_1 I_{nm} + (k_2 - 1)\left(L_1^{-1} \otimes I_m\right)\right]\dot{\Psi}(t)\\ &+ \left[\Psi(t) + \dot{\Psi}(t)\right]^T \left[D(t) - k^* I_{nm}\right]\mathrm{sgn}\left[\Psi(t)\right]\\ =& -k_1\Psi(t)^T\Psi(t) - (k_2 - 1)\dot{\Psi}(t)^T\\ &\times \left(L_1^{-1} \otimes I_m\right)\dot{\Psi}(t). \end{aligned} \tag{10}$$

Because $k_1 > 0$ and $k_2 > 1$, we have that $\dot{V}(t)$ is negative semidefinite. It follows that $V(t)$ is bounded, which implies that $\Psi(t)$, $\dot{\Psi}(t)$ and $D(t)$ are all bounded. Because $\bar{v}(0)$ and $\Phi(t)$ are also bounded, it follows from (9) that $\ddot{\Psi}(t)$ is bounded. From (10) we have that

$$\ddot{V}(t) = -2k_1\Psi(t)^T\dot{\Psi}(t) - 2(k_2 - 1)\dot{\Psi}(t)^T\left(L_1^{-1} \otimes I_m\right)\ddot{\Psi}(t).$$

Therefore, $\ddot{V}(t)$ is bounded. By Barbalatt's Lemma we have that $\dot{V}(t) \to 0$ as $t \to \infty$, which implies that $\Psi(t) \to 0$ and $\dot{\Psi}(t) \to 0$ as $t \to \infty$. Let $x_F(t) \triangleq [x_1^T(t), \cdots, x_n^T(t)]^T$, and $v_F(t) \triangleq [v_1^T(t), \cdots, v_n^T(t)]^T$, we have that $(L_1 \otimes I_m)x_F(t) + (L_2 \otimes I_m)x_L(t) \to 0$ and $(L_1 \otimes I_m)v_F(t) + (L_2 \otimes I_m)v_L(t) \to 0$ as $t \to \infty$. It follows that $\|x_F(t) - x_d(t)\| \to 0$ and $\|v_F(t) - \dot{x}_d(t)\| \to 0$ as $t \to \infty$. It follows from Lemma 2.2 that $\inf_{y(t)\in\Omega(t)}\|x_i(t) - y(t)\| \to 0$ and $\inf_{y(t)\in\Upsilon(t)}\|v_i(t) - y(t)\| \to 0$ as $t \to \infty$.

## IV. Numerical Simulations

This section gives simulation results to illustrate the theoretical results in Section III. Consider a group of three followers and four leaders in the 2-D space. We assume that $x_4(t) = [t, \sin(t/2)]^T$, $x_5(t) = [0.8t - 1.5, \sin(t/2)]^T$, $x_6(t) = [0.8t - 1.5, \sin(t/3) - 1.5]^T$ and $x_7(t) = [t, \sin(t/3) - 1.5]^T$. The network topology associated with the seven agents is shown by Fig. 1. We let $a_{ij} = 1$ if $(j, i) \in \mathcal{E}$ and $a_{ij} = 0$ otherwise. The initial positions and velocities of the followers are chosen as $x_1(0) = [-3, -2]^T$, $v_1(0) = [0.6, -0.3]^T$, $x_2(0) = [-6, 2]^T$, $v_2(0) = [0.8, 0.1]^T$, $x_3(0) = [-4, -4]^T$ and $v_3(0) = [1.1, 0.1]^T$.

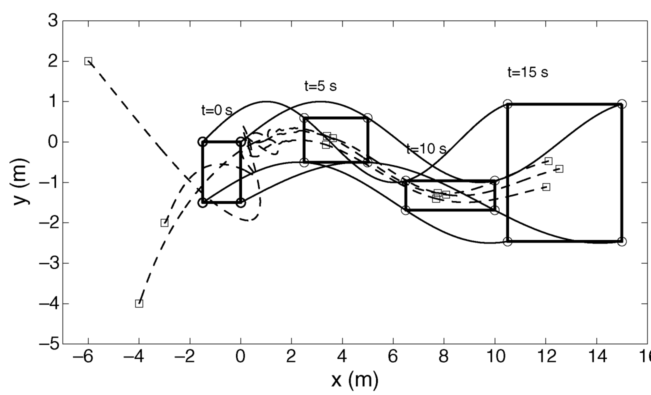
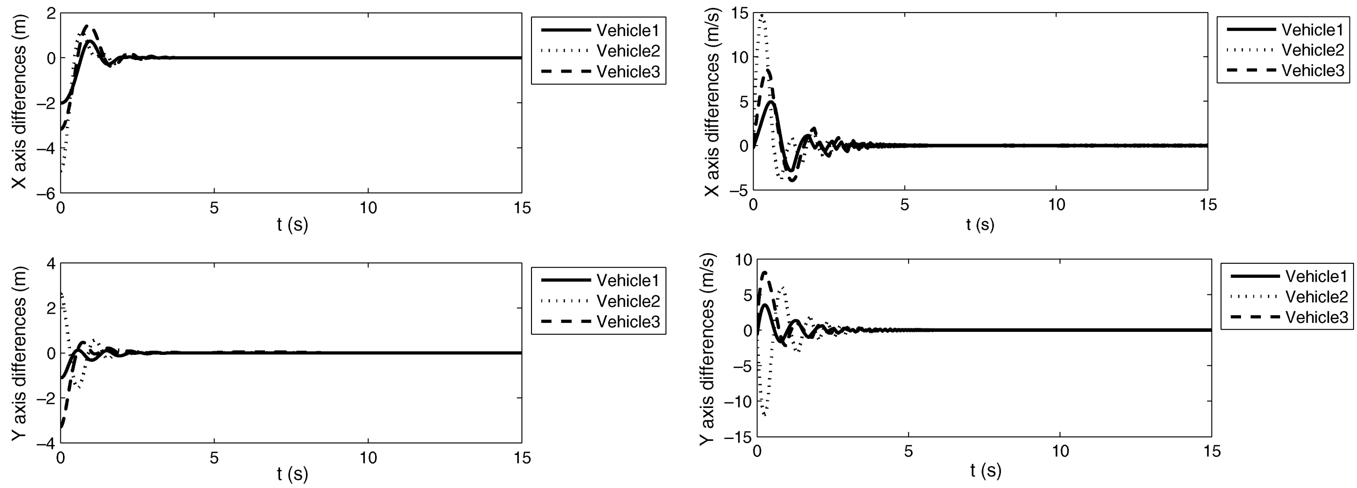



Fig. 4. Trajectories of vehicles 1 to 7 using (5). The circles denote the leaders while the squares denote the followers.

For the algorithm (3), we choose $\alpha = 4, \beta = 1, k_1 = 4, k_2 = 2$ and $k_3 = 3$. It can be seen that the conditions in Theorem 3.1 are satisfied. Fig. 2 shows the trajectories of vehicles 1 to 7 using (3). It can be seen that vehicles 1 to 3 move into the convex hull spanned by vehicles 4 to 7. Fig. 3 shows the difference between $x_i(t)$ and $x_{di}(t)$ and the difference between $v_i(t)$ and $\dot{x}_{di}(t)$, $i = 1, 2, 3$. For the algorithm (5), we choose $k_1 = 5$ and $k_2 = 3$. Fig. 4 shows the trajectories of vehicles 1 to 7 using (5). It can be seen that vehicles 1 to 3 move into the convex hull spanned by vehicles 4 to 7. Fig. 5 shows the differences between $x_i(t)$ and $x_{di}(t)$ and the difference between $v_i(t)$ and $\dot{x}_{di}(t)$, $i = 1, 2, 3$.

## V. Conclusion

In this note, the containment control problem has been investigated for multiple autonomous vehicles with double-integrator dynamics in the presence of multiple dynamic leaders. Two distributed containment control algorithms have been derived under different constraints. Different from the related results in the literature, the proposed algorithms use only the position measurements of the leaders and the followers. Therefore, they can be realized more easily.




(a) (b)

Fig. 5. Trajectories of $x_i(t) - x_{di}(t)$ and $v_i(t) - \dot{x}_{di}(t)$ using (5), $i = 1, 2, 3$.

## APPENDIX

*Proof of Lemma 3.5:* Because $k^* > \|(L_1^{-1} \otimes I_m)\Phi(t)\|_\infty + k_2\|\bar{v}(0)\|_\infty$, we have that

$$\begin{aligned}
&\int_0^t \dot{\Psi}^T(\tau)\left\{k^*\mathrm{sgn}\left[\Psi(\tau)\right] + \left(L_1^{-1} \otimes I_m\right)\Phi(\tau) + k_2\bar{v}(0)\right\}d\tau \\
&= \int_0^t \left\{\left(L_1^{-1} \otimes I_m\right)\Phi(\tau) + k_2\bar{v}(0)\right\}d\Psi(\tau) \\
&\quad + \int_0^t k^*\dot{\Psi}^T(\tau)\mathrm{sgn}\left[\Psi(\tau)\right]d\tau \\
&= \Psi^T(\tau)\left\{k^*\mathrm{sgn}\left[\Psi(\tau)\right] + \left(L_1^{-1} \otimes I_m\right)\Phi(\tau) + k_2\bar{v}(0)\right\}|_0^t \\
&\quad - \int_0^t \Psi(\tau)d\left\{\left(L_1^{-1} \otimes I_m\right)\Phi(\tau) + k_2\bar{v}(0)\right\} \\
&= \Psi^T(t)\left\{k^*\mathrm{sgn}\left[\Psi(t)\right] + \left(L_1^{-1} \otimes I_m\right)\Phi(t) + k_2\bar{v}(0)\right\} \\
&\quad - V_2 - \int_0^t \Psi^T(\tau)(L_1^{-1} \otimes I_m)\dot{\Phi}(\tau)d\tau \\
&\geq -V_2 - \int_0^t \Psi^T(\tau)\left(L_1^{-1} \otimes I_m\right)\dot{\Phi}(\tau)d\tau. \qquad (11)
\end{aligned}$$

Because $k^* > \|(L_1^{-1} \otimes I_m)[\Phi(t) - \dot{\Phi}(t)]\|_\infty + k_2\|\bar{v}(0)\|_\infty$, it then follows that:

$$\begin{aligned}
V_1(t) \geq V_2 &+ \int_0^t \Psi^T(\tau)\left\{k^*\mathrm{sgn}\left[\Psi(\tau)\right] + \left(L_1^{-1} \otimes I_m\right)\Phi(\tau)\right. \\
&\qquad\qquad \left. + k_2\bar{v}(0)\right\}d\tau \\
&- V_2 - \int_0^t \Psi^T(\tau)\left(L_1^{-1} \otimes I_m\right)\dot{\Phi}(\tau)d\tau \\
&= \int_0^t \Psi^T(\tau)\Big\{k^*\mathrm{sgn}\left[\Psi(\tau)\right] + \left(L_1^{-1} \otimes I_m\right)\Big[\Phi(\tau) - \dot{\Phi}(\tau)\Big] \\
&\qquad\qquad + k_2\bar{v}(0)\Big\}d\tau \geq 0. \qquad (12)
\end{aligned}$$

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# Extended $\mathbf{H_2}$ Controller Synthesis for Continuous Descriptor Systems

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***Abstract*—This technical note presents a complete solution to the nonstandard $\mathbf{H_2}$ output feedback control problem for continuous descriptor systems where unstable and nonproper weighting functions are used. In such a problem, the desired controller has to satisfy two conditions simultaneously: (i) the closed-loop is admissible and has a minimum $\mathbf{H_2}$ norm, (ii) only the internal stability of a part of the closed-loop is sought. The condition of the existence of such a controller is deduced. An explicit characterization of the optimal solution is also formulated, based on two generalized algebraic Riccati equations (GAREs) and two generalized Sylvester equations. A numerical example is included to illustrate the validity of the proposed results.**

***Index Terms*—Comprehensive admissibility, descriptor systems, $\mathbf{H_2}$ norm, unstable and nonproper weights.**

## I. Introduction

Descriptor (Singular, Implicit) Systems Have Been Attracting The Attention Of Many Researchers Over Recent Decades Due To Their Capacity To Preserve The Structure Of Physical Systems And To Describe Static Constraints And Impulsive Behaviors. A Number Of Control Issues Have Been Successfully Extended To Descriptor Systems And The Related Results Have Been Reported, For Instance In [1]–[3] And The References Therein.

The standard $H_2$ output feedback control problem for descriptor systems was investigated in [4], and the optimal controller was characterized based on two GAREs. Later, the authors proposed an explicit formulation of all optimal controllers for the full information and the state-feedback cases in [5]. They showed that, in contrast with the state-space case, the usual gain matrix defined as an affine function of the GARE solution can be non-optimal. In both papers, sufficient conditions about the solvability of GAREs were given. However, to the best of the authors' knowledge, solutions to the nonstandard $H_2$ output feedback control problem for descriptor systems, where unstable and nonproper weights are considered in the overall feedback model, have not yet been studied in the literature. In fact, the $H_2$ control problem requires the definition of a standard model, which is necessarily based on the physical model of the system, the models of disturbances and reference signals together with the control objectives. In this context (as for many control problems), it is often desirable to take unstable, even nonproper, weighting filters to meet the design specifications [6], [7]. These choices generally result in a nonstandard design problem for plants having unstabilizable (undetectable) finite dynamics, or even uncontrollable (unobservable) impulsive elements due to the weights involved. These undesirable elements can of course be treated, for example, by slight perturbation to render the problem standard [8]. This approach is, however, vulnerable to the troubles related to lightly-damped poles and may lead to higher order and non strictly proper controllers. Moreover, the methodology of filter absorption [7], [9] and the theory of quasi-stabilizing solutions of Riccati equations [10], [11] have also been proposed for solving these nonstandard problems. In addition, the authors have equally treated this problem for descriptor systems with the presence of unstable weights via state feedback in [12]. Moreover, it is worth noting that the well-known regulation problems, see [13]–[15] and the references therein, can also be handled by the use of unstable weighting filters.

The main contribution of this technical note is an investigation of the "extended" $H_2$ output feedback control problem for continuous descriptor systems. Systems and their weights are all described within the descriptor framework. Hence, it is possible to take into account not only unstable weights, but nonproper weights as well. This case results in nonstandard $H_2$ control problems for which the standard solution procedures fail. In the current technical note, the existence of a solution to this extended problem (the "extended" term indicates here that the desirable controller can and must stabilize a part of the generalized closed-loop) is characterized in terms of two GAREs together with two generalized Sylvester equations.

This technical note is organized as follows. Section II recalls some basic notations of descriptor systems and formulates the extended $H_2$ control problem. Then, based on two generalized Sylvester equations, quasi-admissible solutions to the GAREs are deduced in Section III. Section IV characterizes explicitly the optimal $H_2$ output feedback controllers. Finally, a numerical example is given in Section V to illustrate the proposed results.

*Notation:* The superscripts "$\top$" and "$*$" represent the transpose and complex conjugate transpose, respectively. The notations $\mathcal{F}_l(\cdot,\cdot)$ and $\otimes$ stand for the lower linear fractional transformation and Kronecker product, respectively. $RH_\infty$ denotes the set of all proper rational stable transfer matrices. Moreover, the column vector $Col(P)$ denotes an ordered stack of the columns of the matrix $P$ from left to right starting with the first column.

## II. Problem Formulation

### A. Preliminaries

Consider the following continuous descriptor system:

$$\begin{cases} E\dot{x}(t) = Ax(t) + Bu(t), \\ y(t) = Cx(t) + Du(t) \end{cases} \tag{1}$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^p$ and $u \in \mathbb{R}^m$ are the descriptor variable, measurement and control input vector, respectively. The matrix $E \in \mathbb{R}^{n\times n}$ may be singular, i.e. $rank(E) = r \leq n$.

The descriptor system (1) is said to be regular if $det(sE - A)$ is not identically null. If the descriptor system is regular, then it has a unique solution for any initial condition and any continuous input function [16], [17]. It is said to be impulse-free if $deg(det(sE - A)) = rank(E)$. It is said to be stable if all the roots of $det(sE - A) = 0$ have negative real parts. If the descriptor system is regular, impulse-free and stable, then it is admissible. In addition, the descriptor system (1)

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