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Consensus of linear multi-agent systems with reduced-order observer-based protocols

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ABSTRACT

This paper considers the consensus problems for both continuous- and discrete-time linear multi-agent systems with directed communication topologies. Distributed reduced-order observer-based consensus protocols are proposed, based on the relative outputs of neighboring agents. A multi-step algorithm is presented to construct a reduced-order protocol, under which a continuous-time multi-agent system whose communication topology contains a directed spanning tree can reach consensus. This algorithm is further modified to achieve consensus with a prescribed convergence rate. These two algorithms have a favorable decoupling property. In light of the modified algebraic Riccati equation, an algorithm is then given to construct a reduced-order protocol for the discrete-time case.

1. Introduction

Cooperative control of a group of agents has received compelling attention from various scientific communities. A group of autonomous agents can coordinate with each other via communication or sensing networks to perform certain challenging tasks, which cannot be well accomplished by a single agent. Its potential applications include spacecraft formation flying, sensor networks, and cooperative surveillance [1,2]. In the area of cooperative control of multi-agent systems, consensus is an important and fundamental problem, which is closely related to formation control [3] and flocking problems [4,5]. The main idea of consensus is to develop distributed control policies that enable a group of agents to reach an agreement on certain quantities of interest.

Consensus problems have been extensively studied by numerous researchers from various perspectives. A theoretical explanation is provided in [6] for the alignment behavior observed in [7] by using graph theory. In [8], a general framework of the consensus problem for networks of dynamic agents with fixed or switching topologies is addressed. The conditions given by [6,8] are further relaxed in [9]. The controlled agreement problem for multi-agent networks is considered from a graph-theoretic perspective in [10]. Tracking control for multi-agent consensus with an active leader is considered in [11] by using a neighbor-based state-estimation rule. A distributed algorithm is proposed in [12] to achieve consensus in finite time. The distributed H_∞ control and consensus problems are investigated in [13,14] for networks of agents subject to external disturbances. The consensus problem of networks of double- and high-order integrators is studied in [15–18]. Sampled-data control protocols are proposed in [19,20] to achieve consensus for fixed and switching agent networks. One limitation in the aforementioned works is that the agent dynamics are assumed to be first-, second-, or high-order integrators, which might be restrictive in many cases.

This paper extends to consider the distributed consensus problems for multi-agent systems with continuous- and discrete-time general linear dynamics and directed communication topologies by expanding on our preliminary work [21]. Distributed reduced-order observer-based dynamic consensus protocols, relying on the relative outputs of neighboring agents, are proposed for both the continuous- and discrete-time cases. The dynamic protocols here can be regarded as extensions of the traditional reduced-order observer-based controller for a single system to those for multi-agent systems. It is shown that the separation principle of traditional observer-based controllers still holds in the multi-agent setting. Previous works related to this paper include [22–26]. In contrast to the static consensus protocol based on the relative states in [22], the protocols in the current paper rely on the relative outputs. In contrast to the dynamic protocols in [23–26], whose dimensions are equal to or even higher than that of a single agent, the

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protocols in the current paper are reduced-order and hence have lower dimensions. In particular, the full-order observer-based protocol in [25] possesses a certain degree of redundancy, which stems from the fact that while the observer constructs an estimate of the entire state, part of the state information is already reflected in the system outputs. The reduced-order protocol proposed here eliminates this redundancy and thereby can considerably reduce the dimension of the protocol especially for the case where the agents are MIMO systems.

For the continuous-time case, a multi-step algorithm is presented to construct a reduced-order observer-based consensus protocol for a multi-agent system whose communication topology contains a directed spanning tree. It is shown that a sufficient condition for the existence of such a protocol is that each agent is stabilizable and detectable. Another algorithm is further proposed to construct a protocol, under which the agents can reach consensus with a prescribed convergence rate. These two algorithms have a favorable decoupling feature. Specifically, the first three steps in these algorithms deal with only the agent dynamics, while the last step tackles the communication topology. The case with discrete-time agent dynamics is also considered, where the row-stochastic matrix, rather than the Laplacian matrix as in the continuous-time case, is utilized to characterize the communication topology. In light of the modified algebraic Riccati equation, an algorithm is given to construct a reduced-order protocol to solve the consensus problem for a discrete-time multi-agent system whose communication topology contains a directed spanning tree. It is observed that the nonzero eigenvalue with the smallest real part of the Laplacian matrix plays a key role in the continuous-time case, while the non-one eigenvalue of the stochastic matrix with the largest magnitude is critical in the discrete-time case.

The rest of this paper is organized as follows. Some basic notation and useful results of the graph theory are reviewed in Section 2. The consensus problems of continuous- and discrete-time multi-agent systems are investigated in, respectively, Sections 3 and 4. Section 5 concludes the paper.

2. Concepts and notation

Let $\mathbf{R}^{n\times n}$ and $\mathbf{C}^{n\times n}$ be the set of $n\times n$ real matrices and complex matrices, respectively. The superscript T means transpose for real matrices and H means conjugate transpose for complex matrices. I_N represents the identity matrix of dimension N. Matrices, if not explicitly stated, are assumed to have compatible dimensions. Denote by $\mathbf{1}$ the column vector with all entries equal to one. $\mathrm{Re}(\zeta)$ denotes the real part of $\zeta\in\mathbf{C}$. $A\otimes B$ denotes the Kronecker product of matrices A and B. The matrix inequality $A>(\geq)B$ means that A and B are square Hermitian matrices and that $A-B$ is positive (semi-)definite. A matrix is Hurwitz (in the continuous-time sense) if all of its eigenvalues have negative real parts, while it is Schur stable (in the discrete-time sense) if all of its eigenvalues have magnitude less than 1.

A directed graph $\mathcal{G}$ is a pair $(\mathcal{V},\mathcal{E})$, where $\mathcal{V}$ is a nonempty finite set of nodes and $\mathcal{E}\subseteq\mathcal{V}\times\mathcal{V}$ is a set of edges, in which an edge is represented by an ordered pair of distinct nodes. For an edge (i,j), node i is called the parent node, node j the child node, and i is a neighbor of j. A graph with the property that $(i,j)\in\mathcal{E}$ implies $(j,i)\in\mathcal{E}$ is said to be undirected. A path on $\mathcal{G}$ from node i_1 to node i_l is a sequence of ordered edges of the form (i_k,i_{k+1}), $k=1,\ldots,l-1$. A directed graph has or contains a directed spanning tree if there exists a node called the root, which has no parent node, such that there exists a directed path from this node to every other node in the graph.

Suppose that there are m nodes in a graph. The adjacency matrix $\mathcal{A}=(a_{ij})\in\mathbf{R}^{m\times m}$ is defined by $a_{ii}=0$, $a_{ij}=1$ if $(j,i)\in\mathcal{E}$ and $a_{ij}=0$ otherwise. The Laplacian matrix $\mathcal{L}\in\mathbf{R}^{m\times m}$ is defined as $\mathcal{L}_{ii}=\sum_{j\neq i}a_{ij}$, $\mathcal{L}_{ij}=-a_{ij}$ for $i\neq j$. Let $\mathcal{D}\in\mathbf{R}^{m\times m}$ be a row-stochastic matrix with the additional assumption that $d_{ii}>0$, $d_{ij}>0$ if $(j,i)\in\mathcal{E}$ and $d_{ij}=0$ otherwise.

Lemma 2.1 ([8,9,27]). *Zero is an eigenvalue of $\mathcal{L}$ with $\mathbf{1}$ and a nonnegative vector $r^T\in\mathbf{R}^{1\times N}$, respectively, as the corresponding right and left eigenvectors, and all nonzero eigenvalues have positive real parts. Furthermore, zero is a simple eigenvalue of $\mathcal{L}$ if and only if the graph $\mathcal{G}$ has a directed spanning tree.*

Lemma 2.2 ([9]). *One is an eigenvalue of $\mathcal{D}$ with $\mathbf{1}$ and a nonnegative vector $\hat{r}^T\in\mathbf{R}^{1\times N}$, respectively, as the corresponding right and left eigenvectors, and all other eigenvalues of $\mathcal{D}$ are in the open unit disk. Furthermore, one is a simple eigenvalue of $\mathcal{D}$ if and only if $\mathcal{G}$ contains a directed spanning tree.*

3. Continuous-time multi-agent systems

Consider a group of N identical agents with general continuous-time linear dynamics. The dynamics of the i-th agent are described by

$$\begin{aligned}\dot{x}_i&=Ax_i+Bu_i,\\ y_i&=Cx_i,\quad i=1,\ldots,N,\end{aligned}\tag{1}$$

where $x_i\in\mathbf{R}^n$ is the state, $u_i\in\mathbf{R}^p$ the control input, and $y_i\in\mathbf{R}^q$ the measured output. A, B, C, are constant matrices with compatible dimensions, where C is assumed to have full row rank.

It is assumed that each agent has access to the relative output measurements with respect to its neighbors. Differing from the dynamic protocols in [23–26], whose dimensions are equal to or even higher than that of a single agent, we introduce here a reduced-order observer-based consensus protocol as

$$\begin{aligned}\dot{v}_i&=Fv_i+Gy_i+TBu_i,\\ u_i&=cKQ_1\sum_{j=1}^{N}a_{ij}(y_i-y_j)\\ &\quad+cKQ_2\sum_{j=1}^{N}a_{ij}(v_i-v_j),\quad i=1,\cdots,N,\end{aligned}\tag{2}$$

where $v_i\in\mathbf{R}^{n-q}$ is the protocol state, $c>0$ is the coupling strength, a_{ij} is the (i,j)-th entry of the adjacency matrix $\mathcal{A}$ of a directed graph $\mathcal{G}$, $F\in\mathbf{R}^{(n-q)\times(n-q)}$ is Hurwitz and has no eigenvalues in common with those of A, $G\in\mathbf{R}^{(n-q)\times q}$, $T\in\mathbf{R}^{(n-q)\times n}$ is the unique solution to the following Sylvester equation:

$$TA-FT=GC,\tag{3}$$

which further satisfies that $\begin{bmatrix}C\\T\end{bmatrix}$ is nonsingular, $Q_1\in\mathbf{R}^{n\times q}$ and $Q_2\in\mathbf{R}^{n\times(n-q)}$ are given by $[Q_1\ Q_2]=\begin{bmatrix}C\\T\end{bmatrix}^{-1}$, and $K\in\mathbf{R}^{p\times n}$ is the feedback gain matrix to be designed. Note that protocol (2) is distributed, since it is based only on the relative information of neighboring agents.

Let $z_i=[x_i^T,v_i^T]^T$ and $z=[z_1^T,\ldots,z_N^T]^T$. Then, the closed-loop network dynamics resulting from (1) and (2) can be written as

$$\dot{z}=(I_N\otimes\mathcal{M}+c\mathcal{L}\otimes\mathcal{R})z,\tag{4}$$

where $\mathcal{L}\in\mathbf{R}^{N\times N}$ is the Laplacian matrix of $\mathcal{G}$, and

$$\mathcal{M}=\begin{bmatrix}A&0\\GC&F\end{bmatrix},\qquad\mathcal{R}=\begin{bmatrix}BKQ_1C&BKQ_2\\TBKQ_1C&TBKQ_2\end{bmatrix}.$$

We say that the protocol (2) solves the consensus problem for (1), if the states of (4) satisfy $\lim_{t\to\infty}\|x_i(t)-x_j(t)\|=0,\forall\,i,j=1,\ldots,N$.

Next, an algorithm is presented to select the control parameters in (2).

Algorithm 3.1. Given that (A, B, C) is stabilizable and detectable, the protocol (2) can be constructed as follows:

(1) Choose a Hurwitz matrix F having no eigenvalues in common with those of A. Select G such that (F, G) is stabilizable.
(2) Solve (3) to get a solution T, which satisfies that $\begin{bmatrix} C \\ T \end{bmatrix}$ is nonsingular. Then, compute matrices Q_1 and Q_2 by $\begin{bmatrix} Q_1 & Q_2 \end{bmatrix} = \begin{bmatrix} C \\ T \end{bmatrix}^{-1}$.
(3) Solve the following linear matrix inequality (LMI):

$$AP + PA^T - 2BB^T < 0, \tag{5}$$

to get one solution $P > 0$. Then, choose the matrix $K = -B^T P^{-1}$.
(4) Select the coupling strength $c \geq \frac{1}{\min_{\lambda_i \neq 0} \{\mathrm{Re}(\lambda_i)\}}$, where λ_i is the i-th eigenvalue of $\mathcal{L}$.

Remark 3.2. By Theorem 8.M6 in [28], a necessary condition for the matrix T to the unique solution to (3) and further to satisfy that $\begin{bmatrix} C \\ T \end{bmatrix}$ is nonsingular is that (F, G) is stabilizable, (A, C) is detectable, and F and A have no common eigenvalues. In the case where the agent in (1) is single-input single-output (SISO), this condition is also sufficient. Under such a condition, it is shown for the general multi-input multi-output (MIMO) case that the probability for $\begin{bmatrix} C \\ T \end{bmatrix}$ to be nonsingular is 1 [28]. If $\begin{bmatrix} C \\ T \end{bmatrix}$ is singular in step (2), we need to go back to step (1) and repeat the process. As shown in [25], a necessary and sufficient condition for the existence of a positive-definite solution to the LMI (5) is that (A, B) is stabilizable. Therefore, a sufficient condition for Algorithm 3.1 to successfully construct a protocol (2) is that (A, B, C) is stabilizable and detectable.

Theorem 3.3. *For the multi-agent network (4) whose communication topology $\mathcal{G}$ contains a directed spanning tree, the dynamic protocol (2) constructed by Algorithm 3.1 solves the consensus problem. Specifically,*

$$x_i(t) \to \varpi(t) \triangleq (r^T \otimes e^{At}) \begin{bmatrix} x_1(0) \\ \vdots \\ x_N(0) \end{bmatrix}, \tag{6}$$

$$v_i(t) \to GC\varpi(t), \; i = 1, \ldots, N, \quad \text{as } t \to \infty,$$

where $r \in \mathbf{R}^N$ is a nonnegative vector such that $r^T \mathcal{L} = 0$ and $r^T \mathbf{1} = 1$.

Proof. Let $\xi = ((I_N - \mathbf{1}r^T) \otimes I_{2n-q})z$. Then, it follows from (4) that ξ satisfies the following dynamics:

$$\dot{\xi} = (I_N \otimes \mathcal{M} + c\mathcal{L} \otimes \mathcal{R})\xi. \tag{7}$$

Clearly, 0 is a simple eigenvalue of $I_N - \mathbf{1}r^T$ with $\mathbf{1}$ as the right eigenvector, and 1 is the other eigenvalue with multiplicity $N-1$. Thus, by the definition of ξ, $\xi = 0$ if and only if $z_1 = \cdots = z_N$, i.e., the consensus problem is solved if system (7) is asymptotically stable.

Because $\mathcal{G}$ contains a directed spanning tree, it follows from Lemma 2.1 that zero is a simple eigenvalue of $\mathcal{L}$ and all other eigenvalues have positive real parts. Let $U \in \mathbf{R}^{N\times N}$ be such a unitary matrix that $U^T \mathcal{L} U = \Lambda = \begin{bmatrix} 0 & 0 \\ 0 & \Delta \end{bmatrix}$, where the diagonal entries of Δ are the nonzero eigenvalues of $\mathcal{L}$. Since the right and left eigenvectors corresponding to the zero eigenvalue of $\mathcal{L}$ are, respectively, $\mathbf{1}$ and r^T, we can choose $U = \begin{bmatrix} \frac{\mathbf{1}}{\sqrt{N}} & Y_1 \end{bmatrix}$, $U^T = \begin{bmatrix} r^T \\ Y_2 \end{bmatrix}$, with $Y_1 \in \mathbf{R}^{N\times(N-1)}$, $Y_2 \in \mathbf{R}^{(N-1)\times N}$. Let $\zeta \triangleq [\zeta_1^T, \ldots, \zeta_N^T]^T = (U^T \otimes I_{2n-q})\xi$. Then, (7) can be rewritten as

$$\dot{\zeta} = (I_N \otimes \mathcal{M} + c\Lambda \otimes \mathcal{R})\zeta. \tag{8}$$

By the definition of ξ, it is easy to see that $\zeta_1 = (r^T \otimes I_{2n-q})\xi = 0$. Note that the state matrix of (8) is block uppertriangular. Hence, ζ_i, $i = 2, \ldots, N$, converge asymptotically to zero, if and only if the $N-1$ subsystems

$$\dot{\zeta}_i = (\mathcal{M} + c\lambda_i \mathcal{R})\zeta_i, \quad i = 2, \ldots, N, \tag{9}$$

are asymptotically stable. Multiplying the left and right sides of the matrix $\mathcal{M} + c\lambda_i\mathcal{R}$ by $Q = \begin{bmatrix} I & 0 \\ -T & I \end{bmatrix}$ and Q^{-1}, respectively, and in virtue of (3), we get

$$Q(\mathcal{M} + c\lambda_i \mathcal{R})Q^{-1} = \begin{bmatrix} A + c\lambda_i BK & c\lambda_i BKQ_2 \\ 0 & F \end{bmatrix}. \tag{10}$$

By steps (3) and (4) in Algorithm 3.1, we can obtain that there exists a $P > 0$ satisfying

$$\begin{aligned}(A + c\lambda_i BK)P + P(A + c\lambda_i BK)^H \\ &= AP + PA^T - 2c\mathrm{Re}(\lambda_i)BB^T \\ &\leq AP + PA^T - 2BB^T < 0, \quad i = 2, \ldots, N.\end{aligned}$$

That is, $A + c\lambda_i BK$, $i = 2, \ldots, N$, are Hurwitz. Therefore, the $N-1$ systems in (9) are asymptotically stable, implying that system (7) is asymptotically stable, i.e., the consensus problem is solved.

Next, the solution of (4) can be obtained as

$$\begin{aligned} z(t) &= e^{(I_N \otimes \mathcal{M} + c\mathcal{L} \otimes \mathcal{R})t} z(0) \\ &= (U \otimes I) e^{(I_N \otimes \mathcal{M} + c\Lambda \otimes \mathcal{R})t} (U^T \otimes I) z(0) \\ &= (U \otimes I) \begin{bmatrix} e^{\mathcal{M}t} & 0 \\ 0 & e^{(I_{N-1} \otimes \mathcal{M} + c\Delta \otimes \mathcal{R})t} \end{bmatrix} (U^T \otimes I) z(0). \end{aligned} \tag{11}$$

It has been shown above that $I_{N-1} \otimes \mathcal{M} + c\Delta \otimes \mathcal{R}$ is Hurwitz. Thus,

$$\begin{aligned} z(t) &\to (\mathbf{1} \otimes I) e^{\mathcal{M}t} (r^T \otimes I) z(0) \\ &= (\mathbf{1}r^T) \otimes e^{\mathcal{M}t} z(0), \quad \text{as } t \to \infty, \end{aligned}$$

implying that

$$z_i(t) \to r^T \otimes e^{\mathcal{M}t} z(0), \quad \text{as } t \to \infty. \tag{12}$$

Since F is Hurwitz, (12) directly leads to (6). □

Remark 3.4. The consensus protocol (2) can be regarded as an extension of the traditional reduced-order observer-based controller for a single system to the one for multi-agent systems. The separation principle of the traditional observer-based controllers still holds in the multi-agent setting, as shown in (10). Some observations on the final consensus value in (6) can be concluded as follows: If A in (1) has eigenvalues located in the open right-half plane, then the consensus value $\varpi(t)$ reached by the agents will tend to infinity exponentially. If A is Hurwitz, then $\varpi(t) \to 0$, as $t \to \infty$. On the other hand, if A has eigenvalues in the closed left-half plane, then the agents in (1) may reach consensus nontrivially. That is, some states of each agent might approach a common nonzero value. Typical examples belonging to the last case include the commonly-studied first-, second-, and high-order integrators.

Remark 3.5. Algorithm 3.1 has a favorable decoupling feature. Specifically, the first three steps deal with only the agent dynamics and the feedback gain matrices of (2), while the last step tackles the communication topology. Therefore, the consensus protocol (2) constructed via Algorithm 3.1 for a given communication graph can be directly used for any other communication graph containing a directed spanning tree, with the only additional task of appropriately adjusting the coupling strength c.

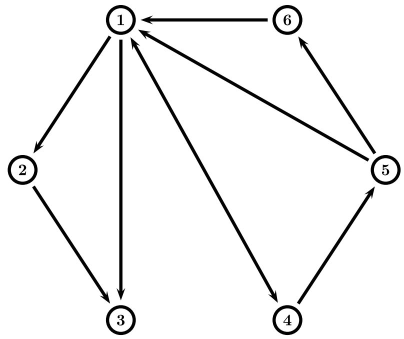

Fig. 1. The communication topology.

Algorithm 3.1 constructs a protocol to achieve consensus. In the following, the protocol (2) will be redesigned to achieve consensus with a given convergence rate. From the proof of Theorem 3.3, it is easy to see that the convergence rate of the N agents in (1) reaching consensus under the protocol (2) is equal to the minimal decay rate of the $N-1$ systems in (9). The decay rate of the system $\dot{x} = Ax$ is defined as the maximum of negative real parts of the eigenvalues of A [29]. Thus, by noticing (10), the convergence rate of agents (1) reaching consensus can be manipulated by properly assigning the eigenvalues of $A + c\lambda_i BK$, $i = 2, \dots, N$, and F.

Algorithm 3.6. Given that (A, B, C) is stabilizable and detectable, the protocol (2) can be constructed as follows:

(1) Choose the matrix F whose eigenvalues lie in the left-half plane of $x = -\alpha$. Select G such that (F, G) is stablizable.
(2) Step 2 in Algorithm 3.1.
(3) Solve the following LMI:

$$AQ + QA^T - 2BB^T + 2\alpha Q < 0, \tag{13}$$

to get one solution $Q > 0$. Then, choose the matrix $K = -B^T Q^{-1}$.
(4) Step 4 in Algorithm 3.1.

Theorem 3.7. *For the multi-agent network (4) with $\mathcal{G}$ containing a directed spanning tree, the protocol (2) constructed by Algorithm 3.6 solves the consensus problem with a convergence rate larger than α. The final consensus values are the same as in (6).*

Proof. It can be shown by following similar steps to those in Theorem 3.3, and by further noting the fact: The decay rate of the system $\dot{x} = Ax$ is larger than $\alpha > 0$, if and only if there exists a matrix $Q > 0$ such that $AQ + QA^T + 2\alpha Q < 0$ [29]. □

Example 3.8. Consider a network of second-order integrators, i.e., the agent dynamics in (1) are given by

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

A first-order dynamic protocol based only on the relative positions is in the form of (2).

Take $F = -2$ and $G = -1$. Using the function `lyap` in Matlab to solve the Sylvester Eq. (3) gives $T = \begin{bmatrix} -0.5 & 0.25 \end{bmatrix}$, which obviously satisfies that $\begin{bmatrix} C \\ T \end{bmatrix}$ is nonsingular. Then, the matrices Q_1 and Q_2 can be obtained as $Q_1 = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$ and $Q_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Solving the LMI (5) by using the Sedumi toolbox [30], we have $K = \begin{bmatrix} -0.8543 & -2.5628 \end{bmatrix}$. Assume that the communication graph is given by Fig. 1. The corresponding Laplacian matrix is

$$\mathcal{L} = \begin{bmatrix} 3 & 0 & 0 & -1 & -1 & -1 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 2 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix},$$

whose nonzero eigenvalues are 1, $1.3376 \pm 0.5623j$, 2, 3.3247. By Algorithm 3.1 and Theorem 3.3, the protocol (2) with feedback gain matrices given as above solves the consensus problem for the communication graph in Fig. 1, if the coupling strength $c \geq 1$.

Algorithm 3.6 can be utilized to construct a protocol achieving consensus with a prescribed convergence rate, e.g., larger than 1. The matrices in (2) except K remain the same. Solving the LMI (13) with $\alpha = 1$ gives $K = \begin{bmatrix} -5.0141 & -3.7372 \end{bmatrix}$. For the communication graph in Fig. 1, select $c = 1$ for simplicity. The states of the network (4) with the protocol (2) given by Algorithm 3.6 as above are depicted in Fig. 2. The convergence rate of the agents reaching consensus can be obtained as 1.5301.

4. Discrete-time multi-agent systems

This section focuses on the discrete-time counterpart of the last section. Consider a network of N identical discrete-time linear agents, with the dynamics of the i-th agent described by

$$\begin{aligned} x_i^+ &= Ax_i + Bu_i, \\ y_i &= Cx_i, \quad i = 1, \dots, N, \end{aligned} \tag{14}$$

where $x_i = x_i(k) \in \mathbf{R}^{n\times n}$ is the state, $x_i^+ = x_i(k+1)$ is the state at the next time instant, $u_i \in \mathbf{R}^p$ is the control input, and $y_i \in \mathbf{R}^q$ is the measured output. It is assumed that C is of full row rank.

Similar to the continuous-time case, the following reduced-order observer-based consensus protocol is proposed

$$\begin{aligned} \hat{v}_i^+ &= F\hat{v}_i + Gy_i + TBu_i, \\ u_i &= KQ_1 \sum_{j=1}^{N} d_{ij}(y_i - y_j) + KQ_2 \sum_{j=1}^{N} d_{ij}(\hat{v}_i - \hat{v}_j), \\ &\quad i = 1, \dots, N, \end{aligned} \tag{15}$$

where $\hat{v}_i \in \mathbf{R}^{n-q}$ is the protocol state, $F \in \mathbf{R}^{(n-q)\times(n-q)}$ is Schur stable and has no eigenvalues in common with those of A, $G \in \mathbf{R}^{(n-q)\times q}$, $T \in \mathbf{R}^{(n-q)\times n}$ is the unique solution to (3), satisfying that $\begin{bmatrix} C \\ T \end{bmatrix}$ is nonsingular, $\begin{bmatrix} Q_1 & Q_2 \end{bmatrix} = \begin{bmatrix} C \\ T \end{bmatrix}^{-1}$, $K \in \mathbf{R}^{p\times n}$ is the feedback gain matrix to be designed, and d_{ij} is the (i, j)-th entry of the row-stochastic matrix $\mathcal{D}$ associated with the graph $\mathcal{G}$.

Let $\hat{z}_i = [x_i^T, \hat{v}_i^T]^T$ and $\hat{z} = [\hat{z}_1^T, \dots, \hat{z}_N^T]^T$. Then, the collective network dynamics can be written as

$$\hat{z}^+ = (I_N \otimes \mathcal{M} + (I_N - \mathcal{D}) \otimes \mathcal{R})\hat{z}, \tag{16}$$

where matrices $\mathcal{M}$ and $\mathcal{R}$ are defined in (4).

We say that the protocol (15) solves the consensus problem for (1) if the states of (16) satisfy $\lim_{k\to\infty} \|x_i(k) - x_j(k)\| = 0$, $\forall\, i, j = 1, \dots, N$.

Before moving forward, we introduce the following modified algebraic Riccati equation (MARE) [31,32]:

$$P = A^T PA - \delta A^T PB(B^T PB + I)^{-1} B^T PA + Q. \tag{17}$$

For $\delta = 1$, the MARE (17) is reduced to the commonly-used discrete-time Riccati equation discussed in, e.g., [33].

The following lemma shows the existence of solutions for the MARE.

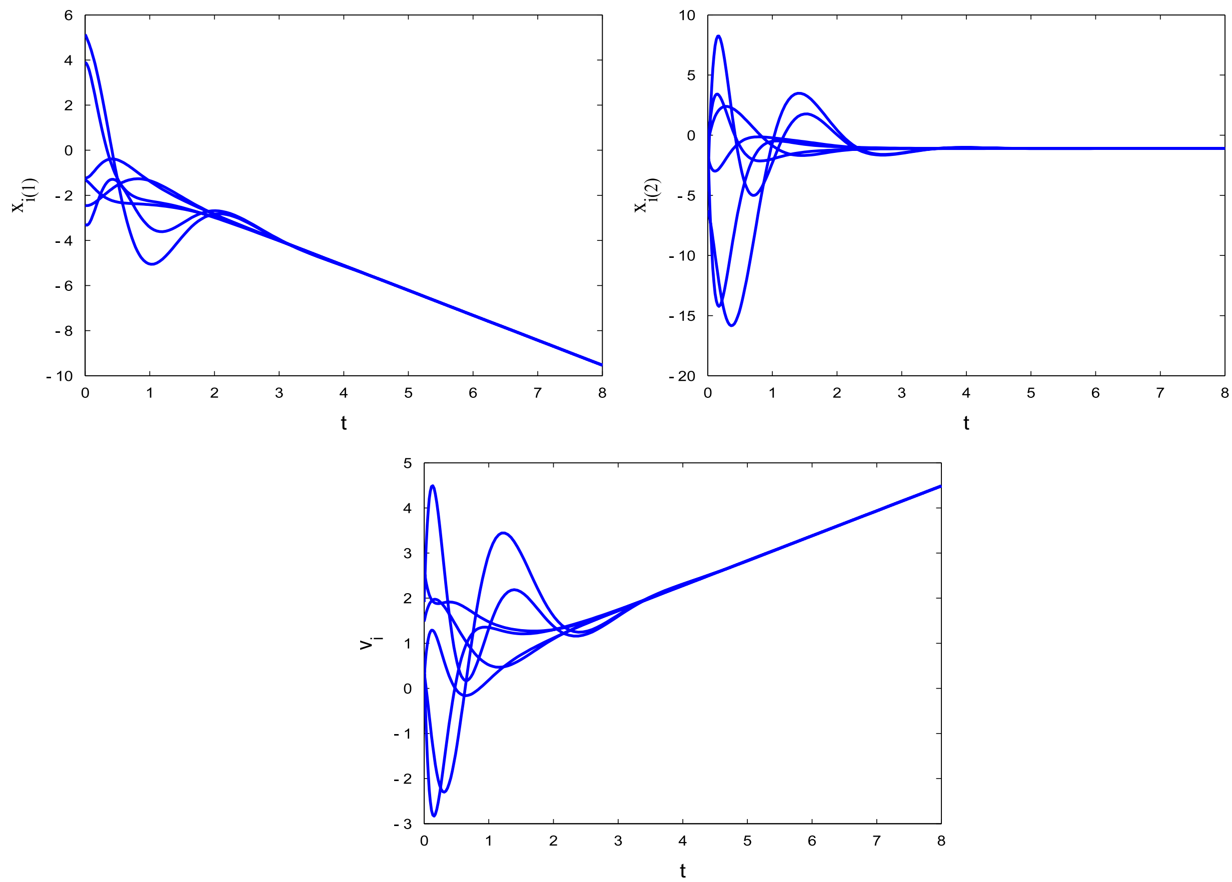

Fig. 2. The states of the network (4) under the protocol (2) constructed via Algorithm 3.6. Here $x_{i(k)}$ denotes the k-th component of x_i.

Lemma 4.1 (*[31,32]*)**.** *For $0 < \delta < 1$, the MARE (17) has a unique positive-definite solution P, if the matrix A has no eigenvalues with magnitude larger than 1, $(A, Q^{1/2})$ is stablizable, and (A, C) is detectable. Furthermore, $P = \lim_{k\to\infty} P_k$ for any initial condition $P_0 \geq 0$, where P_k satisfies*

$$P(k+1) = A^T P(k) A - \delta A^T P(k) B (B^T P(k) B + I)^{-1} B^T P(k) A + Q.$$

Next, an algorithm for the protocol (15) is presented, which will be used later.

Algorithm 4.1. Given that (A, B, C) is stabilizable and detectable, the protocol (15) can be constructed as follows:

(1) Select a Schur stable matrix F having no eigenvalues in common with those of A, and G such that (F, G) is stablizable.

(2) Solve (3) to get a solution T, satisfying that $\begin{bmatrix} C \\ T \end{bmatrix}$ is nonsingular. Then, compute the matrices Q_1 and Q_2 by $\begin{bmatrix} Q_1 & Q_2 \end{bmatrix} = \begin{bmatrix} C \\ T \end{bmatrix}^{-1}$.

(3) Choose $K = -(B^T P B + I)^{-1} B^T P A$, where $P > 0$ is the unique solution of the following MARE:

$$P = A^T P A - (1 - \max_{|\hat{\lambda}_i|<1} |\hat{\lambda}_i|^2) A^T P B (B^T P B + I)^{-1} B^T P A + Q, \tag{18}$$

with $Q > 0$ and $\hat{\lambda}_i$ being the i-th eigenvalue of $\mathcal{D}$.

Remark 4.2. A sufficient condition for the existence of the consensus protocol by using Algorithm 4.1 is that (A, B, C) is stabilizable and detectable, and A has no eigenvalues with magnitude larger than 1 which is required here to ensure the solvability of the MARE (18).

Theorem 4.3. *Assume that A has no eigenvalues with magnitude larger than 1. For the multi-agent network (16) with $\mathcal{G}$ containing a directed spanning tree, the protocol (15) constructed by Algorithm 4.1 solves the consensus problem. Specifically,*

$$x_i(k+1) \to \psi(k+1) \triangleq (\hat{r}^T \otimes A^k) \begin{bmatrix} x_1(0) \\ \vdots \\ x_N(0) \end{bmatrix}, \tag{19}$$

$$\hat{v}_i(k+1) \to GC\psi(k+1), \quad i = 1, \ldots, N, \quad \text{as } k \to \infty,$$

where $\hat{r} \in \mathbf{R}^N$ is nonnegative such that $\hat{r}^T (I_N - \mathcal{D}) = 0$ and $\hat{r}^T \mathbf{1} = 1$.

Proof. Let $\hat{\xi} = ((I_N - \mathbf{1}\hat{r}^T) \otimes I_{2n-q})\hat{z}$. As demonstrated in the proof of Theorem 3.3, the consensus problem can be reduced to the asymptotical stability of $\hat{\xi}$, which evolves according to the following dynamics:

$$\hat{\xi}^+ = (I_N \otimes \mathcal{M} + (I_N - \mathcal{D}) \otimes \mathcal{R})\hat{\xi}. \tag{20}$$

For any graph containing a directed spanning tree, it follows from Lemma 2.2 that 0 is a simple eigenvalue of $I_N - \mathcal{D}$ and all other eigenvalues lie within a disk of radius 1 centered at the point $1 + 0j$ in the complex plane. Let $\hat{U} = \begin{bmatrix} \frac{\mathbf{1}}{\sqrt{N}} & \hat{Y}_1 \end{bmatrix}$, $\hat{U}^T = \begin{bmatrix} \hat{r}^T \\ \hat{Y}_2 \end{bmatrix}$, with $\hat{Y}_1 \in \mathbf{R}^{N\times(N-1)}$, $\hat{Y}_2 \in \mathbf{R}^{(N-1)\times N}$, be such unitary matrices that $\hat{U}^T (I_N - \mathcal{D}) \hat{U} = \hat{\Lambda} = \begin{bmatrix} 0 & 0 \\ 0 & \hat{\Delta} \end{bmatrix}$, where the diagonal entries of $\hat{\Delta}$ are the nonzero eigenvalues of $I_N - \mathcal{D}$. Let $\hat{\zeta} \triangleq [\hat{\zeta}_1^T, \ldots, \hat{\zeta}_N^T]^T = (\hat{U}^T \otimes I_{2n-q})\hat{\xi}$. Then, (20) can be rewritten as

$$\hat{\zeta}^+ = (I_N \otimes \mathcal{M} + (I_N - \hat{\Lambda}) \otimes \mathcal{R})\hat{\zeta}. \tag{21}$$

Clearly, $\hat{\zeta}_1 = (\hat{r}^T \otimes I_{2n-q})\,\hat{\xi} = 0$. By noting that the state matrix of (21) is block uppertriangular, $\hat{\zeta}_i$, $i = 2, \ldots, N$, converge to zero asymptotically, if and only if the $N-1$ subsystems

$$\hat{\zeta}_i^+ = (\mathcal{M} + (1 - \hat{\lambda}_i)\mathcal{R})\hat{\zeta}_i, \quad i = 2, \ldots, N, \tag{22}$$

are asymptotically stable. It is known that $\mathcal{M} + (1-\hat{\lambda}_i)\mathcal{R}$ is similar to $\begin{bmatrix} A + (1-\hat{\lambda}_i)BK & (1-\hat{\lambda}_i)BKQ_2 \\ 0 & F \end{bmatrix}$. In light of step (3) in Algorithm 4.1, we can obtain

$$\begin{aligned}
&(A + (1 - \hat{\lambda}_i)BK)^H P(A + (1 - \hat{\lambda}_i)BK) - P \\
&\quad = A^T PA - 2\mathbf{Re}(1 - \hat{\lambda}_i)A^T PB(B^T PB + I)^{-1}B^T PA - P \\
&\qquad + |1 - \hat{\lambda}_i|^2 A^T PB(B^T PB + I)^{-1}B^T PB(B^T PB + I)^{-1}B^T PA \\
&\quad = A^T PA + (-2\mathbf{Re}(1 - \hat{\lambda}_i) + |1 - \hat{\lambda}_i|^2)A^T \\
&\qquad \times PB(B^T PB + I)^{-1}B^T PA - P + |1 - \hat{\lambda}_i|^2 A^T PB(B^T PB + I)^{-1} \\
&\qquad \times (-I + B^T PB(B^T PB + I)^{-1})B^T PA \\
&\quad = A^T PA + (|\hat{\lambda}_i|^2 - 1)A^T PB(B^T PB + I)^{-1}B^T PA - P \\
&\qquad - |1 - \hat{\lambda}_i|^2 A^T PB(B^T PB + I)^{-2}B^T PA \\
&\quad \le A^T PA - (1 - \max_{|\hat{\lambda}_i|<1} |\hat{\lambda}_i|^2)A^T PB(B^T PB + I)^{-1}B^T PA - P \\
&\quad = -Q < 0,
\end{aligned} \tag{23}$$

where the identity $-I + B^T PB(B^T PB + I)^{-1} = -(B^T PB + I)^{-1}$ has been applied. Then, (23) implies that $A + (1 - \hat{\lambda}_i)BK$, $i = 2, \ldots, N$, are Schur stable. Therefore, the $N-1$ systems in (22) are asymptotically stable, implying that the consensus problem is solved.

By noting that $I_{N-1} \otimes \mathcal{M} + \hat{\Delta} \otimes \mathcal{R}$ is Schur stable, the solution of (16) can be obtained as

$$\begin{aligned}
\hat{z}(k+1) &= (I_N \otimes \mathcal{M} + (I_N - \mathcal{D}) \otimes \mathcal{R})^k \hat{z}(0) \\
&= (\hat{U} \otimes I)(I_N \otimes \mathcal{M} + \hat{\Lambda} \otimes \mathcal{R})^k (\hat{U}^T \otimes I)\hat{z}(0) \\
&= (\hat{U} \otimes I)\begin{bmatrix} \mathcal{M}^k & 0 \\ 0 & (I_{N-1} \otimes \mathcal{M} + \hat{\Delta} \otimes \mathcal{R})^k \end{bmatrix} (\hat{U}^T \otimes I)\hat{z}(0) \\
&\to (\mathbf{1}\hat{r}^T) \otimes \mathcal{M}^k \hat{z}(0), \quad \text{as } k \to \infty.
\end{aligned} \tag{24}$$

Therefore, we have

$$\hat{z}_i(k+1) \to \hat{r}^T \otimes \mathcal{M}^k \hat{z}(0), \quad \text{as } k \to \infty,$$

which directly leads to (19). □

Remark 4.4. Theorem 4.3 gives the discrete-time counterpart of the results in Theorem 3.3. The Laplacian matrix $\mathcal{L}$ is used in the last section to represent the communication graph for continuous-time multi-agent systems. In contrast, the row-stochastic matrix $\mathcal{D}$ is utilized here for the discrete-time case. By observing Algorithms 3.1, 3.6 and 4.1, it can be concluded that the nonzero eigenvalue with the smallest real part of the Laplacian matrix plays a key role in continuous-time multi-agent systems, while the non-one eigenvalue of the stochastic matrix with the largest magnitude is critical for the discrete-time case. It can be observed from (19) that the consensus value $\psi(k+1)$ reached by the agents will tend to infinity exponentially, if A in (14) has an eigenvalue with magnitude larger than 1. Therefore, the assumption on A in Theorem 4.3 does not involve much conservatism. Similar to Theorem 3.3, A in (14) with eigenvalues with a unit magnitude is critical for the agents to reach consensus nontrivially.

5. Conclusion

In this paper, the consensus problems for multi-agent systems with continuous- and discrete-time linear dynamics and directed communication topologies have been considered. Distributed reduced-order consensus protocols, based on the information of relative outputs of neighboring agents, have been proposed. Several multi-step algorithms have been presented to construct the consensus protocols, which solve the consensus problem for both the continuous- and discrete-time cases. In this paper, we did not consider the issues of time delays, switching topologies, or random graphs. However, these issues are interesting topics that deserve further investigation in future work.

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