# High-Order and Model Reference Consensus Algorithms in Cooperative Control of Multi-Vehicle Systems

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## Abstract

In this paper we study $\ell^{\text{th}}$-order ($\ell >= 3$) consensus algorithms, which generalize the existing first-order and second-order consensus algorithms in the literature. We show necessary and sufficient conditions under which each information variable and their higher-order derivatives converge to common values. We also present the idea of higher-order consensus with a leader and introduce the concept of an $\ell^{\text{th}}$-order model-reference consensus problem, where each information variable and their high-order derivatives not only reach consensus but also converge to the solution of a prescribed dynamic model. The effectiveness of these algorithms is demonstrated through simulations and a multi-vehicle cooperative control application which mimics a flocking behavior in birds.

# I. INTRODUCTION

Cooperative control for multi-vehicle systems has been applied both to formation control problems, with applications to mobile robots, unmanned air vehicles (UAVs), autonomous underwater vehicles (AUVs), satellites, aircraft, spacecraft, and automated highway systems[1–6,20], and to non-formation cooperative control problems such as task assignment, payload transport, role assignment, air traffic control, timing, search, and adaptive scheduling[7–10]. For cooperative control strategies to be successful, numerous issues must be addressed, including the definition and management of shared information among a group of vehicles. The cooperation of a multi-vehicle team is often facilitated if the team members can form a consistent view of the shared information.

In plain language, when several entities or vehicles agree on a common value of a variable of interest they are said to have come to *consensus*. For a group of networked mobile vehicles with a common mission or task, information consensus can play a pivotal role, particularly when the communication capability for each vehicle is limited and/or purposely constrained. For example, when the dynamic environment changes, the vehicles in a team must be in agreement as to what changes have taken place, even when every vehicle cannot talk directly to every other vehicle. To achieve consensus, there must be a shared variable of interest as well as appropriate algorithmic methods for negotiating to reach consensus on the value of that variable (called a consensus algorithm or protocol). Consensus algorithms have a historical perspective in Refs. 11–14, to name a few. Recently, consensus algorithms have been studied extensively in the field of cooperative control using algebraic graph theory[15–19] and nonlinear mathematical tools[21–23]. Some results in consensus algorithms can be understood in the context of connective stability[24]. Optimality issues in consensus algorithms are also considered in the literature[25]. In addition, information consensus is studied in the context of random networks[26], flocking[27,28], and asynchronous communication[29].

Notice that in the literature, most consensus algorithms focus on the case where the communicating/cooperating vehicles come to consensus about the value of the consensus variable (e.g., Refs. 16,17,19,21). Although the consensus variable may be a vector, such algorithms are effectively first-order, as the typical consensus algorithm adapts the first derivative of the consensus variable on each node based on the value of the consensus variable of its neighbors. The idea of a second-order consensus algorithm under directed information exchange

has been suggested in Ref. 30, where each vehicle adapts the second-order derivative of its local consensus variable based on both zero-order and first-order derivatives of the consensus variables of their nearest neighbors. In the current paper we generalize the first-order and second-order consensus algorithms in the literature. Expanding on our earlier work reported in Ref. 31, we show *necessary* and *sufficient* conditions under which the consensus information variable and its higher-order derivatives converge to common values. We also extend these ideas to include setpoint tracking in higher-order derivatives of the consensus variable (higher-order consensus with a leader) and we consider an $\ell^{\text{th}}$-order model-reference consensus problem, where each information variable and their high-order derivatives not only reach consensus but also converge to the solution of a prescribed dynamic model.

When considering the problem of consensus among a group of cooperating entities, a natural question is: "*consensus on what?*." A similar question "*formation to what form or shape?*" has been asked in a general way[32] within the context of mobile actuator and sensor networks[33]. However, in most work on information consensus the answer is application-dependent. Likewise, in our case, with the question "*consensus on what*" in mind, our motivation for studying higher-order consensus comes from observing the behavior of flocks of birds. It is often noted that such flocks fly somewhat in formation, maintaining a nominal separation from each other, but each traveling with the same velocity vector. In Ref. 34 it is shown how second-order consensus algorithms can produce the behavior of a separation and common velocity under directed information exchange. However, sometimes a bird flock abruptly changes direction, perhaps when one of them suddenly perceives a source of danger or food. Clearly the birds in this setting need to build consensus on not only their relative position and their velocity, but also on acceleration. This motivates the idea of higher-order consensus. Higher-order consensus also makes sense in swarm-on-swarm scenarios, where a cooperative team of "friendly" UAVs confront another team of "hostile" UAVs, necessitating abrupt collective motions of the team. Finally, again driven by the question "*consensus on what*" we are also motivated to consider what we call the $\ell^{\text{th}}$-order model-reference consensus problem, where each information variable and their high-order derivatives not only reach consensus but also converge to the solution of a prescribed dynamic model. Convergence not to something arbitrary as defined by initial conditions, but to the solution of a prescribed problem, as defined by the reference model, is important in applications where the team of vehicles has a global task to achieve but does not have centralized communication.

We introduce this model-reference consensus problem and establish sufficient conditions for consensus convergence for the case where all or some of the team members have complete knowledge of the reference model.

The remainder of the paper is organized as follows. Section II presents some background materials and some mathematical preliminaries for our later development. Higher-order consensus algorithms are given in Section III, including the standard, unforced case and the case of setpoint tracking, which leads to the idea of higher-order consensus with a leader, and model-reference consensus. The effectiveness of the proposed algorithms is illustrated throughout the paper by simulations, including an example of flocking behavior in Section IV. Section V concludes the paper.

## II.   BACKGROUND AND PRELIMINARIES

It is natural to model information exchange among vehicles by directed/undirected graphs. A digraph (directed graph) consists of a pair $(\mathcal{N}, \mathcal{E})$, where $\mathcal{N}$ is a finite nonempty set of nodes and $\mathcal{E} \in \mathcal{N}^2$ is a set of ordered pairs of nodes, called edges. As a comparison, the pairs of nodes in an undirected graph are unordered. If there is a directed edge from node $v_i$ to node $v_j$, then $v_i$ is defined as the parent node and $v_j$ is defined as the child node. A directed path is a sequence of ordered edges of the form $(v_{i_1}, v_{i_2}), (v_{i_2}, v_{i_3}), \cdots$, where $v_{i_j} \in \mathcal{N}$, in a digraph. An undirected path in an undirected graph is defined accordingly. In a digraph, a cycle is a path that starts and ends at the same node. A digraph is called strongly connected if there is a directed path from every node to every other node. An undirected graph is called connected if there is a path between any distinct pair of nodes. A directed tree is a digraph, where every node has exactly one parent except for one node, called root, which has no parent, and the root has a directed path to every other node. Note that in a directed tree, each edge has a natural orientation away from the root, and no cycle exists. In the case of undirected graphs, a tree is a graph in which every pair of nodes is connected by exactly one path. A directed spanning tree of a digraph is a directed tree formed by graph edges that connect all the nodes of the graph. A graph has (or contains) a directed spanning tree if there exists a directed spanning tree being a subset of the graph. Note that the condition that a digraph has a directed spanning tree is equivalent to the case that there exists at least one node having a directed path to all the other nodes. In the case

of undirected graphs, having an undirected spanning tree is equivalent to being connected. However, in the case of digraphs, having a directed spanning tree is a weaker condition than being strongly connected.

The adjacency matrix $A = [a_{ij}] \in I\!\!R^{n \times n}$ of a weighted digraph is defined as $a_{ii} = 0$ and $a_{ij} > 0$ if $(j, i) \in \mathcal{E}$ where $i \neq j$. The adjacency matrix of a weighted undirected graph is defined accordingly except that $a_{ij} = a_{ji}$, $\forall i \neq j$, since $(j, i) \in \mathcal{E}$ implies $(i, j) \in \mathcal{E}$. Let matrix $L = [\ell_{ij}]$ be defined as $\ell_{ii} = \sum_{j \neq i} a_{ij}$ and $\ell_{ij} = -a_{ij}$, where $i \neq j$. The matrix $L$ satisfies the following conditions:

$$
\begin{aligned}
\ell_{ij} \leq 0, \quad & i \neq j \\
\sum_{j=1}^{n} \ell_{ij} = 0, \quad & i = 1, \ldots, n.
\end{aligned}
\tag{1}
$$

For an undirected graph, $L$ is called the Laplacian matrix[35], which has the property that it is symmetric positive semi-definite. However, $L$ for a digraph does not have this property. In both cases of undirected graphs and digraphs, 0 is an eigenvalue of $L$ with an associated eigenvector $\mathbf{1}$, where $\mathbf{1} \triangleq [1, \ldots, 1]^T$ is an $n \times 1$ column vector of all ones, since all of the row sums of $L$ are 0. In the case of undirected graphs, all of the nonzero eigenvalues of $L$ are positive. In the case of digraphs, all of the nonzero eigenvalues of $L$ have positive real parts following Gershgorin's disc theorem[37]. In the case of undirected graphs, 0 is a simple eigenvalue of $L$ if and only if the undirected graph is connected[36]. In the case of digraphs, 0 is a simple eigenvalue of $L$ if and only if the digraph has a directed spanning tree[38].

Let $\mathbf{0}$ denote the $n \times 1$ column vector of all zeros. Let $I_n$ denote the $n \times n$ identity matrix and $0_n$ denote the $n \times n$ zero matrix. Let $M_n(I\!\!R)$ represent the set of all $n \times n$ real matrices. Given a matrix $A = [a_{ij}] \in M_n(I\!\!R)$, the digraph of $A$, denoted by $\Gamma(A)$, is the digraph on $n$ nodes $v_i$, $i \in \mathcal{I}$, such that there is a directed edge in $\Gamma(A)$ from $v_j$ to $v_i$ if and only if $a_{ij} \neq 0$[37].

## III.  HIGHER-ORDER CONSENSUS ALGORITHMS

We begin by presenting the general $\ell^{\text{th}}$-order extension to the standard consensus algorithm, followed by two extensions: (1) setpoint tracking and higher-order consensus with a leader, and (2) model-reference consensus.

## A.  $\ell^{\text{th}}$-order consensus

Consider information variables with $\ell^{\text{th}}$-order dynamics given by

$$\dot{\xi}_i^{(0)} = \xi_i^{(1)}$$

$$\vdots$$

$$\dot{\xi}_i^{(\ell-2)} = \xi_i^{(\ell-1)} \qquad (2)$$

$$\dot{\xi}_i^{(\ell-1)} = u_i \quad i \in \{1, \ldots, n\},$$

where $\xi_i^{(k)} \in I\!\!R^m$, $k = 0, 1, \ldots, \ell - 1$, are the states, $u_i \in I\!\!R^m$ is the control input, and $\xi_i^{(k)}$ denotes the $k^{\text{th}}$ derivative of $\xi_i$ with $\xi_i^{(0)} = \xi_i$, $i = 1, \ldots, n$. Consensus is said to be reached among the $n$ vehicles if $\xi_i^{(k)} \to \xi_j^{(k)}$, $k = 0, 1, \ldots, \ell - 1$, $\forall i \neq j$. The goal of a consensus algorithm is to derive a control law $u_i$ such that consensus is reached among the vehicles.

In the case of $\ell = 1$, a consensus algorithm is proposed in Refs. 15–18,38 as

$$u_i = -\sum_{j=1}^{n} g_{ij} k_{ij} (\xi_i - \xi_j), \quad i \in \{1, \ldots, n\}, \qquad (3)$$

where $k_{ij} > 0$, $g_{ii} \overset{\triangle}{=} 0$, and $g_{ij}$ is 1 if information flows from vehicle $j$ to vehicle $i$ and 0 otherwise. The motivation behind (3) is to drive the information variable of each vehicle toward the information variables of its neighbors.

Motivated by (3), we propose the following higher-order consensus algorithm:

$$u_i = -\sum_{j=1}^{n} g_{ij} k_{ij} \left[ \sum_{k=0}^{\ell-1} \gamma_k (\xi_i^{(k)} - \xi_j^{(k)}) \right], \quad i \in \{1, \ldots, n\}, \qquad (4)$$

where $k_{ij} > 0$, $\gamma_k > 0$, and $g_{ij}$ is defined in Eq. (3). The motivation behind (4) is to drive each vehicle's information variable and its high-order derivatives toward the states of its neighbors. Note that the linear consensus strategies reported in the literature can be considered special cases of (4) when $l = 1$ or $l = 2$.

Let $\xi^{(k)}$ be an $mn \times 1$ column vector with components $\xi_i^{(k)}$, $i = 1, \ldots, n$, where $\xi_i^{(k)}$ is defined in Eq. (2). By applying (4), Eq. (2) can be written in matrix form as

$$\begin{bmatrix} \dot{\xi}^{(0)} \\ \dot{\xi}^{(1)} \\ \vdots \\ \dot{\xi}^{(\ell-1)} \end{bmatrix} = (\Gamma \otimes I_m) \begin{bmatrix} \xi^{(0)} \\ \xi^{(1)} \\ \vdots \\ \xi^{(\ell-1)} \end{bmatrix}, \qquad (5)$$

where $\otimes$ denotes the Kronecker product, and

$$
\Gamma = \begin{bmatrix}
0_n & I_n & 0_n & \cdots & 0_n \\
0_n & 0_n & I_n & \cdots & 0_n \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
0_n & 0_n & 0_n & \cdots & I_n \\
-\gamma_0 L & -\gamma_1 L & -\gamma_2 L & \cdots & -\gamma_{\ell-1} L
\end{bmatrix},
$$

where $L = [\ell_{ij}] \in M_n(\mathbb{R})$ with $\ell_{ii} = \sum_{j \neq i} g_{ij} k_{ij}$ and $\ell_{ij} = -g_{ij} k_{ij}$, $\forall i \neq j$. Note that $L$ satisfies the property (1).

In the following, we assume $m = 1$ for simplicity. However, all the results hereafter remain valid for $m > 1$. In addition, we only consider the case when $\ell = 3$. Similar analyses are applicable to the case when $\ell > 3$.

Before stating our main results, we need the following lemma.

**Lemma III.1** *In the case of $\ell = 3$, $\Gamma$ has at least three zero eigenvalues. It has exactly three zero eigenvalues if and only if $-L$ has a simple zero eigenvalue. Moreover, if $-L$ has a simple zero eigenvalue, the zero eigenvalue of $\Gamma$ has geometrical multiplicity equal to one.*

*Proof:* Let $\lambda$ be an eigenvalue of $\Gamma$ and $q = [p^T, r^T, s^T]^T$ be its associated eigenvector, where $p$, $r$, and $s$ are $n \times 1$ column vectors. Note that

$$
\Gamma q = \begin{bmatrix}
0_n & I_n & 0_n \\
0_n & 0_n & I_n \\
-\gamma_0 L & -\gamma_1 L & -\gamma_2 L
\end{bmatrix}
\begin{bmatrix} p \\ r \\ s \end{bmatrix}
= \lambda \begin{bmatrix} p \\ r \\ s \end{bmatrix},
$$

which implies that

$$
r = \lambda p
$$

$$
s = \lambda r
$$

$$
-\gamma_0 L p - \gamma_1 L r - \gamma_2 L s = \lambda s.
$$

Thus it follows that $q = [p^T, \lambda p^T, \lambda^2 p^T]^T$. It also follows that $-\gamma_0 L p - \gamma_1 L \lambda p - \gamma_2 L \lambda^2 p = \lambda^3 p$, which can be written as

$$
-L p = \frac{\lambda^3}{\gamma_0 + \gamma_1 \lambda + \gamma_2 \lambda^2} p.
$$

Thus it follows that $\frac{\lambda^3}{\gamma_0+\gamma_1\lambda+\gamma_2\lambda^2}$ is an eigenvalue of $-L$ with an associated eigenvector $p$. Let $\mu = \frac{\lambda^3}{\gamma_0+\gamma_1\lambda+\gamma_2\lambda^2}$, where $\mu$ is an eigenvalue of $-L$ with $p$ being its associated eigenvector. Therefore, it follows that

$$\lambda^3 - \gamma_2\mu\lambda^2 - \gamma_1\mu\lambda - \gamma_0\mu = 0, \tag{6}$$

which implies that there are three roots for $\lambda$ corresponding to each $\mu$. That is, each eigenvalue of $-L$ corresponds to three eigenvalues of $\Gamma$.

Let $\mu_i$, $i = 1, \ldots, n$, be the $i^{\text{th}}$ eigenvalue of $-L$. Also let $\lambda_{3i-2}$, $\lambda_{3i-1}$, and $\lambda_{3i}$, $i = 1, \ldots, n$, be the eigenvalues of $\Gamma$ corresponding to $\mu_i$. From Eq. (6), we can see that $\mu_j = 0$ implies that $\lambda_{3j-2} = \lambda_{3j-1} = \lambda_{3j} = 0$. It is straightforward to see that $-L$ has at least one zero eigenvalue with an associated eigenvector $\mathbf{1}$ since all of its row sums are equal to zero. Therefore, we know that $\Gamma$ has at least three zero eigenvalues.

From Eq. (6) we can also see that $-L$ has a simple zero eigenvalue if and only if $\Gamma$ has exactly three zero eigenvalues. In addition, if $-L$ has a simple zero eigenvalue, denoted as $\mu_1 = 0$ without loss of generality, then there is only one linearly independent eigenvector $p$ for $-L$ associated with the eigenvalue zero. Note that $\mu_1 = 0$ implies that $\lambda_1 = \lambda_2 = \lambda_3 = 0$, which in turn implies that $q = [p^T, \mathbf{0}^T, \mathbf{0}^T]^T$. Therefore, there is only one linearly independent eigenvector $q$ for $\Gamma$ associated with eigenvalue zero. That is, the zero eigenvalue of $\Gamma$ has geometric multiplicity equal to one if $-L$ has a simple zero eigenvalue. ∎

Using this lemma we can prove the following result.

**Theorem III.1** *In the case of $\ell = 3$, the algorithm (4) achieves consensus exponentially if and only if $\Gamma$ has exactly three zero eigenvalues and all of the other eigenvalues have negative real parts.*

*Proof:* (Sufficiency.) If $\Gamma$ has exactly three zero eigenvalues, we know that the eigenvalue zero has geometric multiplicity equal to one from Lemma III.1. As a result, it follows that $\Gamma$ can be written in Jordan canonical form as

$$\Gamma = \underbrace{[w_1, \ldots, w_{3n}]}_{P} \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0_{1\times(3n-3)} \\ 0 & 0 & 1 & 0_{1\times(3n-3)} \\ 0 & 0 & 0 & 0_{1\times(3n-3)} \\ 0_{(3n-3)\times1} & 0_{(3n-3)\times1} & 0_{(3n-3)\times1} & J' \end{bmatrix}}_{J} \underbrace{\begin{bmatrix} \nu_1^T \\ \vdots \\ \nu_{3n}^T \end{bmatrix}}_{P^{-1}}, \tag{7}$$

8

where $w_j \in \mathbb{R}^{3n}$, $j = 1, \ldots, 3n$, can be chosen to be the right eigenvectors or generalized eigenvectors of $\Gamma$, $\nu_j \in \mathbb{R}^{3n}$, $j = 1, \ldots, 3n$, can be chosen to be the left eigenvectors or generalized eigenvectors of $\Gamma$, and $J'$ is the Jordan upper diagonal block matrix corresponding to $3n - 3$ non-zero eigenvalues of $\Gamma$.

Without loss of generality, we choose $w_1 = [\mathbf{1}^T, \mathbf{0}^T, \mathbf{0}^T]^T$, $w_2 = [\mathbf{0}^T, \mathbf{1}^T, \mathbf{0}^T]^T$, and $w_3 = [\mathbf{0}^T, \mathbf{0}^T, \mathbf{1}^T]^T$, where it can be verified that $w_1$, $w_2$, and $w_3$ are an eigenvector and two generalized eigenvectors of $\Gamma$ associated with the eigenvalue 0 respectively. Noting that $\Gamma$ has exactly three zero eigenvalues, denoted as $\lambda_1 = \lambda_2 = \lambda_3 = 0$ without loss of generality, we know that $-L$ has a simple zero eigenvalue, which in turn implies that there exists a nonnegative $n \times 1$ vector $p$ such that $p^T L = 0$ and $p^T \mathbf{1} = 1$ as shown in Ref. 38. It can be verified that $\nu_1 = [p^T, \mathbf{0}^T, \mathbf{0}^T]^T$, $\nu_2 = [\mathbf{0}^T, p^T, \mathbf{0}^T]^T$, and $\nu_3 = [\mathbf{0}^T, \mathbf{0}^T, p^T]^T$ are two generalized left eigenvectors and a left eigenvector of $\Gamma$ associated with eigenvalue 0 respectively, where $\nu_j^T w_j = 1$, $j = 1, 2, 3$. Note that

$$
e^{\Gamma t} = \underbrace{[w_1, \ldots, w_{3n}]}_{P} \underbrace{\begin{bmatrix} 1 & t & \frac{1}{2}t^2 & 0_{1 \times (3n-3)} \\ 0 & 1 & t & 0_{1 \times (3n-3)} \\ 0 & 0 & 1 & 0_{1 \times (3n-3)} \\ 0_{(3n-3) \times 1} & 0_{(3n-3) \times 1} & 0_{(3n-3) \times 1} & e^{J't} \end{bmatrix}}_{e^{Jt}} \underbrace{\begin{bmatrix} \nu_1^T \\ \vdots \\ \nu_{3n}^T \end{bmatrix}}_{P^{-1}}.
$$

Also note that $\lim_{t \to \infty} e^{J't} \to 0_{3n-3}$ exponentially since the eigenvalues $\lambda_{3i-2}$, $\lambda_{3i-1}$, and $\lambda_{3i}$, $i = 2, \ldots, n$, have negative real parts. Therefore, it follows by computation that for large $t$, the dominant terms in $e^{\Gamma t}$ are

$$
\begin{bmatrix} \mathbf{1}p^T & t\mathbf{1}p^T & \frac{1}{2}t^2\mathbf{1}p^T \\ 0_n & \mathbf{1}p^T & t\mathbf{1}p^T \\ 0_n & 0_n & \mathbf{1}p^T \end{bmatrix},
$$

where the rows $(kn + 1)$ to $(k + 1)n$, $k = 0, 1, 2$, are identical. Noting that

$$
\begin{bmatrix} \xi^{(0)}(t) \\ \xi^{(1)}(t) \\ \xi^{(2)}(t) \end{bmatrix} = e^{\Gamma t} \begin{bmatrix} \xi^{(0)}(0) \\ \xi^{(1)}(0) \\ \xi^{(2)}(0) \end{bmatrix},
$$

where $\xi^{(k)} = [\xi_1^{(k)}, \ldots, \xi_n^{(k)}]^T$, we know that $\xi_i^{(k)}(t) \to \xi_j^{(k)}(t)$ exponentially, $\forall i \neq j$, $k = 0, 1, 2$, as $t \to \infty$.

(Necessity.) Suppose that the sufficient condition that $\Gamma$ has exactly three zero eigenvalues and all the other eigenvalues have negative real parts does not hold. Noting that $\Gamma$ has at least three zero eigenvalues, the fact that the sufficient condition does not hold implies that $\Gamma$ has either more than three zero eigenvalues or it has three zero eigenvalues but has at least another eigenvalue having positive real part. In either case, we know that $\lim_{t\to\infty} e^{Jt}$ has a rank larger than three, which implies that $\lim_{t\to\infty} e^{\Gamma t}$ has a rank larger than three.

Note that consensus is reached asymptotically if and only if $\lim_{t\to\infty} e^{\Gamma t} \to \begin{bmatrix} \mathbf{1}q^T \\ \mathbf{1}r^T \\ \mathbf{1}s^T \end{bmatrix}$, where $q$, $r$, and $s$ are $3n \times 1$ vectors. As a result, the rank of $\lim_{t\to\infty} e^{\Gamma t}$ cannot exceed three. This results in a contradiction. $\blacksquare$

In the case of $\ell = 3$, let $\lambda_k$, $k = 1, \ldots, 3n$, be the eigenvalues of $\Gamma$. Note that consensus is reached exponentially if and only if $\Gamma$ has exactly three zero eigenvalues and all of the other eigenvalues have negative real parts. The convergence speed of the algorithm (4) is related to the nonzero eigenvalues of $\Gamma$. Let $\lambda_j$ be the nonzero eigenvalue such that $|\text{Re}(\lambda_j)| = \min |\text{Re}(\lambda_k)|$, $\forall \lambda_k \neq 0$, where $\text{Re}(\cdot)$ represents the real part of a number. In particular, for $\ell = 3$, $e^{\lambda_j t}$ is the dominant term when $e^{\Gamma t}$ converges to a matrix of the form $\begin{bmatrix} \mathbf{1}q^T \\ \mathbf{1}r^T \\ \mathbf{1}s^T \end{bmatrix}$, where $q$, $r$, and $s$ are $3n \times 1$ vectors.

Note that $L$ has a simple zero eigenvalue and all of the other eigenvalues have positive real parts if and only if the information-exchange topology has a directed spanning tree[38]. In the case of $\ell = 3$, if (4) achieves consensus exponentially, we know that $\Gamma$ has exactly three zero eigenvalues following Theorem III.1. Thus we see that $-L$ has a simple zero eigenvalue, which in turn implies that the information-exchange topology has a directed spanning tree. Therefore, in the case of $\ell = 3$, having a directed spanning tree is a necessary condition for consensus seeking. However, similar to the case of $l = 2^{30}$, having a directed spanning tree is not a sufficient condition for consensus seeking. Both the information-exchange topology and values of $\gamma_k$, $k = 0, 1, 2$, will affect the convergence of the $\ell^{\text{th}}$-order consensus algorithm with $l \geq 3$. In contrast, in the case of $\ell = 1$, having a directed spanning tree is a necessary and sufficient condition for consensus seeking[38].

From Eq. (6) we can see that $\gamma_k$, $k = 0, 1, 2$, plays an important role in the eigenvalues

of $\Gamma$. In the case of $\ell = 2$, we know that if $-L$ has a simple zero eigenvalue and all of the other eigenvalues are real and therefore negative (e.g., an undirected connected information-exchange topology or a leader-follower information-exchange topology), then (4) achieves consensus for arbitrary $\gamma_k > 0$, $k = 0, 1$[30]. In contrast, this argument is no longer valid for the case of $\ell = 3$. In the case that the digraph of $-L$ is itself a directed spanning tree (i.e., a leader-follower information-exchange topology), $-L$ has a simple zero eigenvalue and all of the other eigenvalues are real and therefore negative[38]. As a result, all of the coefficients of the polynomial (6) are positive real numbers when $\mu \neq 0$, where $\mu$ is the eigenvalue of $-L$. From Routh criterion, there always exist $\gamma_k$, $k = 0, 1, 2$, such that all of the roots of (6) have negative real parts when $\mu \neq 0$. It is straightforward to see that when a graph has a directed spanning tree, the graph can be constructed by adding information-exchange links to a graph that is itself a directed spanning tree. Noting that $\mu$ is continuously dependent on the entries of $-L$ (i.e., the digraph of $-L$) and the roots of (6) are continuously dependent on its coefficients, we know that for each $-L$ whose graph has a directed spanning tree, there always exist $\gamma_k$, $k = 0, 1, 2$, such that all of the roots of (6) have negative real parts when $\mu \neq 0$. The parameters $\gamma_k$, $k = 0, 1, 2$, can be chosen according to Routh-Hurwitz theorem. As a result, the conditions of Theorem III.1 are satisfied.

To illustrate these points, consider the following simulation example. Let $L$ be given by

$$L_0 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}. \tag{8}$$

Note that the digraph of $L$ has a directed spanning tree. In fact, the graph of $L$ is itself a directed spanning tree in this case. Also note that $\mu_j = -1$, where $\mu_j$ is the nonzero eigenvalue of $-L$. In Case 1, we choose $\gamma_0 = 2$, $\gamma_1 = 1$, and $\gamma_2 = 2$. Then from Eq. (6) we see that $\lambda_{3j-2} = -2$, $\lambda_{3j-1} = i$, and $\lambda_{3j} = -i$, where $\lambda_*$ is the eigenvalue of $\Gamma$ corresponds to $\mu_j$. As a result, consensus cannot be achieved. However, if we choose $\gamma_0 = 1$, $\gamma_1 = 2$, and $\gamma_2 = 3$ in Case 2, then $\lambda_{3j-2} = -2.3247$, $\lambda_{3j-1} = -0.3376 + 0.5623i$, and $\lambda_{3j} = -0.3376 - 0.5623i$. As a result, consensus can be reached. Fig. 1 shows the plots of $\xi_i^{(2)}$, $i = 1, \ldots, 4$, for Cases 1 and 2 with different $\gamma_j$, $j = 0, 1, 2$, values. Note that although the digraph of $L$ has a directed spanning tree in both cases, the consensus system is not stable in Case 1, whereas it is stable in Case 2. Thus, the gains $\gamma_k$ must be chosen properly to ensure that consensus
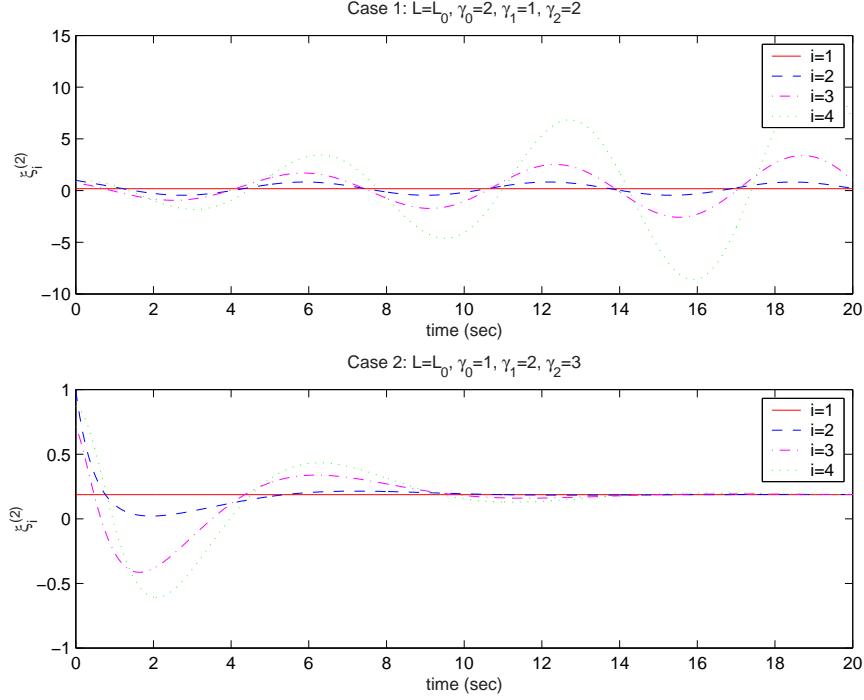
11

is achieved.



FIG. 1: Plots of $\xi_i^{(k)}$, $k = 2$, for Cases 1 and 2 with different $\gamma_j$, $j = 0, 1, 2$, values.

## B. Setpoint tracking and higher-order consensus with a leader

In Ref. 39 the idea of a leader-node is introduced, whereby a single node is chosen that ignores all the other nodes, but continues to broadcast, and the controllability properties of the resulting graph are explored. In Ref. 40 it was shown how to modify the first-order consensus algorithm to introduce setpoint tracking, but with a less stringent requirement than full state controllability (and not requiring that the leader ignore all the other nodes, though this is effectively what happens). The algorithm in Ref. 40 caused all the nodes to converge to the leader's setpoint. This is called *consensus with a leader*.

Though Ref. 40 considered only first-order consensus, in the same way for higher-order consensus we can modify Eq. (4) as follows:

$$u_i = -\sum_{j=1}^{n} g_{ij} k_{ij} \Big[\sum_{k=0}^{\ell-1} \gamma_k (\xi_i^{(k)} - \xi_j^{(k)})\Big] - \alpha_i (\xi_i^{(\ell-1)} - \xi_i^{(\ell-1)*}), \quad i \in \{1, \ldots, n\}, \qquad (9)$$

where $\alpha > 0$, and $\xi_i^{(\ell-1)*}$ is the local setpoint on node $i$. Using Eq. (9) we claim that if

$\alpha_i = 0$ for all but node $k$, with $\alpha_k = 1$, (i.e., higher-order consensus with a leader), and node $k$ has a directed path to all the other nodes, then $\xi_i^{(l-1)} \to \xi_k^{(\ell-1)*}$, $\forall i \in \{1, \ldots, n\}$ by following a similar argument in Ref. 40. Note that this assertion requires $\xi_k^{(\ell-1)*}$ to be piecewise constant. However, in the next subsection we generalize these ideas further by extending them to include setpoints for all the derivatives, where the setpoints come from a reference model.

To illustrate, we choose $\xi_1^{(2)*} = 0.5$, $\alpha_1 = 1$, and $\alpha_i = 0$, $\forall i \neq 1$. Fig. 2 shows the plots of $\xi_i^{(2)}$, where $L = L_0$ is given by Eq. (8). Note that each $\xi_i^{(2)}$ converges to the setpoint.
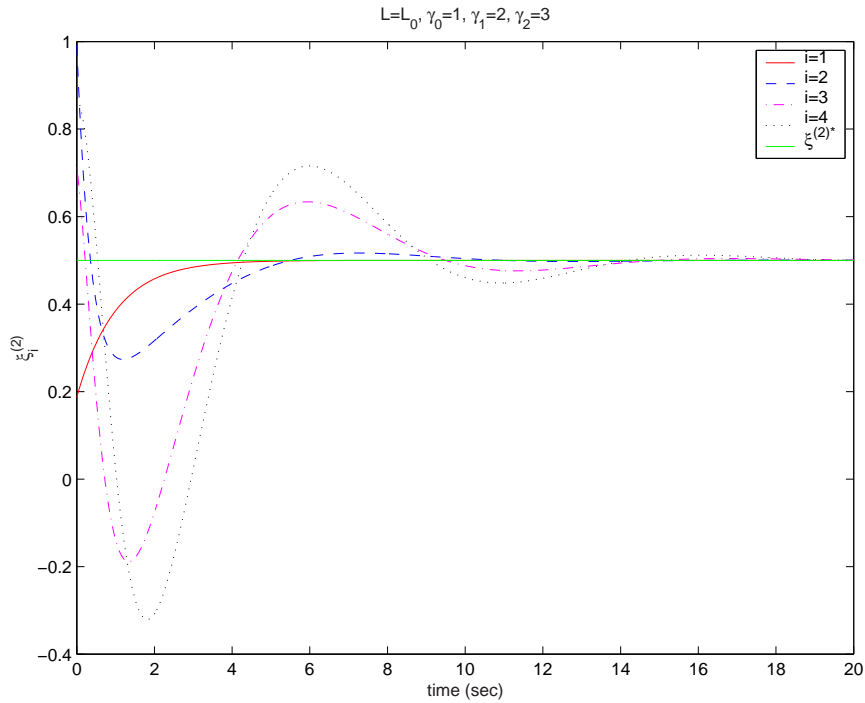


FIG. 2: Plots of $\xi_i^{(k)}$, $k = 2$, with $\xi_1^{(2)*} = 0.5$.

### C. Model-reference consensus

Consider a prescribed reference dynamic model given by

$$\dot{\xi}_r^{(0)} = \xi_r^{(1)}$$

$$\vdots$$

$$\dot{\xi}_r^{(\ell-2)} = \xi_r^{(\ell-1)}$$

$$\dot{\xi}_r^{(\ell-1)} = u_r \tag{10}$$

13

where $\xi_r^{(k)} \in I\!R^m$, $k = 0, 1, \ldots, \ell - 1$, are the reference states, and $u_r \in I\!R^m$ is the reference control input.

A model reference consensus problem is said to be solved if $\xi_i^{(k)} \to \xi_r^{(k)}$, $k = 0, \ldots, \ell - 1$, asymptotically and $\xi_i^{(k)} \to \xi_j^{(k)}$, $\forall i \neq j$, during the transition.

### 1. Full Access to the Reference Model

In the case that the reference model is available to each vehicle in the team, we propose the following model-reference consensus algorithm:

$$
u_i = -\sum_{j=1}^{n} g_{ij} k_{ij} \left[ \sum_{k=0}^{\ell-1} \gamma_k (\xi_i^{(k)} - \xi_j^{(k)}) \right]
$$

$$
- \eta \sum_{k=0}^{\ell-1} \gamma_k (\xi_i^{(k)} - \xi_r^{(k)}) + u_r \quad i \in \{1, \ldots, n\} \tag{11}
$$

where $\eta > 0$.

Let $\tilde{\xi}_i^{(k)} = \xi_i^{(k)} - \xi_r^{(k)}$, $k = 0, \ldots, \ell$, and $\tilde{\xi}^{(k)}$ be an $mn \times 1$ column vector with components $\tilde{\xi}_i^{(k)}$, $i = 1, \ldots, n$. By applying (11), Eq. (2) can be written in matrix form as

$$
\begin{bmatrix} \dot{\tilde{\xi}}^{(0)} \\ \dot{\tilde{\xi}}^{(1)} \\ \vdots \\ \dot{\tilde{\xi}}^{(\ell-1)} \end{bmatrix} = (\Sigma \otimes I_m) \begin{bmatrix} \tilde{\xi}^{(0)} \\ \tilde{\xi}^{(1)} \\ \vdots \\ \tilde{\xi}^{(\ell-1)} \end{bmatrix}, \tag{12}
$$

where

$$
\Sigma = \begin{bmatrix} 0_n & I_n & 0_n & \cdots & 0_n \\ 0_n & 0_n & I_n & \cdots & 0_n \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0_n & 0_n & 0_n & \cdots & I_n \\ -\gamma_0 M & -\gamma_1 M & -\gamma_2 M & \cdots & -\gamma_{\ell-1} M \end{bmatrix},
$$

with $M = L + \eta I_n$.

Letting $\mu_i$ and $\rho_i$ be the $i^{\text{th}}$ eigenvalue of $-L$ and $-M$ respectively, then we see that $\rho_i = \mu_i - \eta$. Following the argument of Theorem III.1, we know that each eigenvalue of $-M$ corresponds to three eigenvalues of $\Sigma$. Letting $\varsigma_{3i-j}$, $j = 1, 2, 3$, be the eigenvalue of $\Sigma$ corresponding to $\rho_i$, then they are related by the following equation:

$$
\varsigma^3 - \gamma_2 \rho \varsigma^2 - \gamma_1 \rho \varsigma - \gamma_0 \rho = 0. \tag{13}
$$

Note that if $\mathrm{Re}(\varsigma_i) < 0$, $i = 1, \ldots, 3n$, that is, $\Sigma$ is a stable matrix, then $\tilde{\xi}_i^{(k)} \to 0$ asymptotically, $k = 0, \ldots, \ell - 1$, which in turn implies that $\xi_i^{(k)} \to \xi_r^{(k)}$ asymptotically.

It is straightforward to see that $\mathrm{Re}(\rho_i) < 0$, $i = 1, \ldots, 3n$, due to the fact that $\mathrm{Re}(\mu_i) \leq 0$ as shown in Section II and $\eta > 0$. Similar to Eq. (6), $\gamma_k$, $k = 0, 1, 2$, plays an important role in the eigenvalues of $\Sigma$ in Eq. (13). Note that even if the information-exchange topology does not have a directed spanning tree (e.g., the worse case of no information exchange among vehicles, that is, $L = 0_n$), it is still possible to choose $\gamma_k$, $k = 0, 1, 2$, according to Routh-Hurwitz theorem such that all of the eigenvalues of $\Sigma$ have negative real parts. However, having a directed spanning tree guarantees that $\xi_i^{(k)} \to \xi_j^{(k)}$, $k = 0, \ldots, \ell - 1$, $\forall i \neq j$ during the transition when $\xi_i^{(k)} \to \xi_r^{(k)}$.

To give a simulation example of model-reference consensus, let $u_r = \sin(t)$ and $\xi_r^{(k)}(0) = 0$, $k = 0, 1, 2$. Also let $\eta = 0.3$, $\gamma_0 = 1$, $\gamma_1 = 2$, and $\gamma_2 = 3$. Fig. 3 shows the plots of $\xi_i^{(2)}$, $i = 1, \ldots, 4$, for Case 1, where $L = L_0$ is given by Eq. (8), and Case 2, where $L = 0_n$. Note that although $\xi_i^{(2)}$ approaches $\xi_r^{(2)}$ asymptotically in both Cases 1 and 2, we can see that $\max_{i \neq j} |\xi_i^{(2)} - \xi_j^{(2)}|$ during the transition in Case 1 is smaller than that in Case 2.
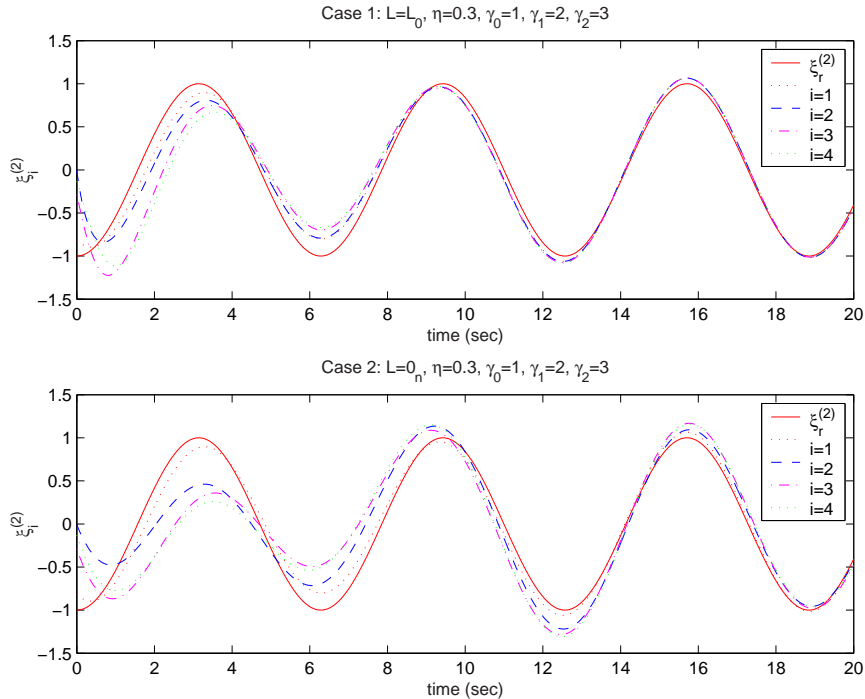


FIG. 3: Plots of $\xi_i^{(k)}$, $k = 2$, for Cases 1 and 2 with different information exchange topologies.

Also note that the value of $\eta$ also has an effect on convergence. Let $u_r$ and $\xi_r^{(k)}(0) = 0$ be

defined the same as above. Also let $L = L_0$ be given by Eq. (8), $\gamma_0 = 2$, $\gamma_1 = 1$, and $\gamma_2 = 3$. Fig. 4 shows the plots of $\xi_i^{(2)}$, $i = 1, \ldots, 4$, for Case 1, where $\eta = 0.3$, and Case 2, where $\eta = 2$. Note that $\xi_i^{(2)}$ does not approach $\xi_r^{(2)}$ in Case 1 due to small $\eta$. However, when we increase $\eta$ to 2, convergence to $\xi_r^{(r)}$ is guaranteed in Case 2.
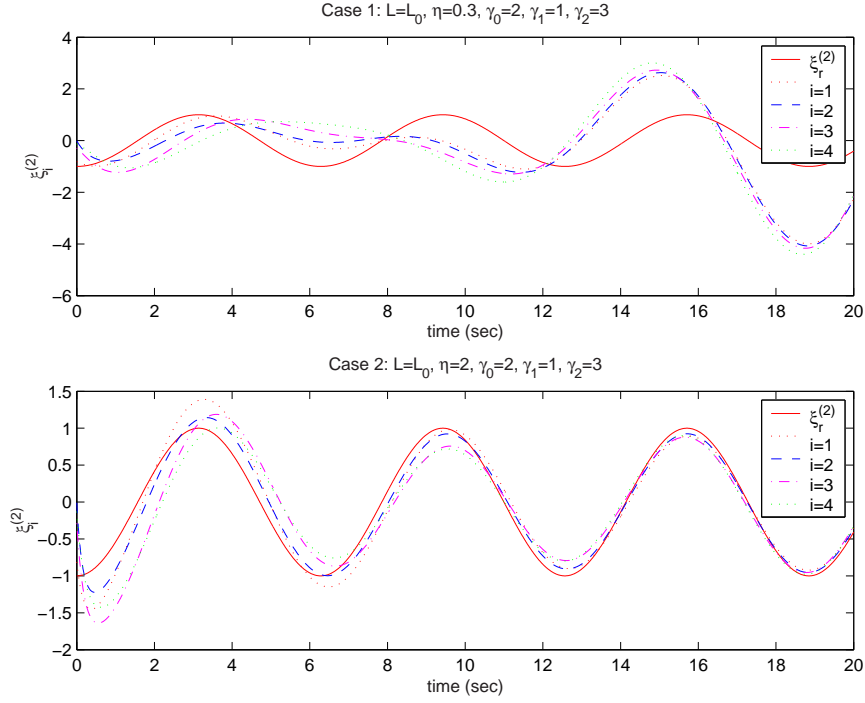


FIG. 4: Plots of $\xi_i^{(k)}$, $k = 2$, for Cases 1 and 2 with different $\eta$ values.

In the case of $\ell = 2$, if $L = 0_n$, then $\mathrm{Re}(\varsigma_i) < 0$ for any $\gamma_k > 0$, $k = 0, 1$, due to the fact that $\rho_i = -\eta$ is real. That is, the model reference consensus problem is solved for arbitrary $\gamma_k > 0$, $k = 0, 1$. However, in the case of $\ell = 3$, this argument is no longer valid and the gains $\gamma_k$ must be chosen properly to ensure that consensus is achieved.

16

*2. Partial Access to the Reference Model*

In the general case that the reference model is only available to a portion of the vehicles in the team, we propose the following model-reference consensus algorithm:

$$u_i = \frac{1}{\kappa_i}\{\sum_{j=1}^{n} g_{ij}k_{ij}u_j - \sum_{j=1}^{n} g_{ij}k_{ij}[\sum_{k=0}^{\ell-1}\gamma_k(\xi_i^{(k)} - \xi_j^{(k)})]$$

$$+ g_{ir}k_{ir}[u_r - \sum_{k=0}^{\ell-1}\gamma_k(\xi_i^{(k)} - \xi_r^{(k)})]\} \quad i \in \{1,\ldots,n\}, \tag{14}$$

where $g_{ir} = 1$ if the reference model is available to vehicle $i$ and 0 otherwise, and $\kappa_i = g_{ir}k_{ir} + \sum_{j=1}^{n} g_{ij}k_{ij}$.

Let node $n+1$ be a virtual vehicle with states $\xi_r^{(k)}$. Also let $\xi_{n+1} \equiv \xi_r$, $g_{i(n+1)} \equiv g_{ir}$, and $k_{i(n+1)} \equiv k_{ir}$. With the algorithm (14) and the reference model (10), Eq. (2) can be rewritten as

$$\sigma_i^{(3)} + \gamma_2\sigma_i^{(2)} + \gamma_1\sigma_i^{(1)} + \gamma_0\sigma_i^{(0)} = 0, \tag{15}$$

where $\sigma_i = \sum_{j=1}^{n+1} g_{ij}k_{ij}(\xi_i - \xi_j)$.

Note that the characteristic equation of the system (15) is given by $s^3 + \gamma_2 s^2 + \gamma_1 s + \gamma_0 = 0$. It is straightforward to choose appropriate $\gamma_k > 0$, $k = 0, 1, 2$, according to Routh criterion such that the system (15) is stable, which in turn implies that $\sigma_i^{(k)} \to 0$, $\forall i$, $k = 0, 1, 2$, as $t \to \infty$. If there exists a directed path from virtual vehicle $n+1$ to all of the vehicles in the team, then $\sigma_i^{(k)} \to 0$, $\forall i$, $k = 0, 1, 2$, implies that $\xi_i^{(k)} \to \xi_j^{(k)}$, $\forall i, j \in \{1,\ldots,n+1\}$, $k = 0, 1, 2$[38]. That is, $\xi_i^{(k)} \to \xi_r^{(k)}$, $\forall i$, $k = 0, 1, 2$, since $\xi_{n+1} \equiv \xi_r$.

## IV.  MULTI-VEHICLE COORDINATION EXAMPLE

In this section we illustrate via simulation how the higher-order consensus ideas presented above can be used in a formation control scenario for multiple vehicles, whereby a desired separation is maintained among vehicles, each vehicle travels at a common velocity, and a leader vehicle has access to a reference model. We assume that there are five vehicles,

exchanging information through a topology defined by the matrix

$$-L = \begin{bmatrix} -2 & 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 1 & 0 & -2 & 0 & 1 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & -1 \end{bmatrix}.$$

The equation running on each vehicle $i$ is given by:

$$\dot{q}_i = v_i$$

$$\dot{v}_i = a_i$$

$$\dot{a}_i = \frac{1}{\kappa_i}(\sum_{j=1}^{n} g_{ij}\dot{a}_j - \sum_{j=1}^{n} g_{ij}\{\gamma_0[(q_i - q_j) - (\delta_i - \delta_j)] + \gamma_1(v_i - v_j) + \gamma_2(a_i - a_j)\}$$

$$+ g_{ir}[\dot{a}_r - \gamma_0(q_i - \delta_i - q_r) - \gamma_1(v_i - v_r) - \gamma_2(a_i - a_r)]),$$

where $q_i = [q_{ix}, q_{iy}]^T$, $v_i = [v_{ix}, v_{iy}]^T$, and $a_i = [a_{ix}, a_{iy}]^T$ denote the two dimensional position, velocity, and acceleration of vehicle $i$ respectively, $q_r = [q_{rx}, q_{ry}]^T$, $v_r = [v_{rx}, v_{ry}]^T$, and $a_r = [a_{rx}, a_{ry}]^T$ denote the two dimensional reference position, velocity, and acceleration respectively, and the terms $\delta_i = [\delta_{ix}, \delta_{iy}]^T$ denote the desired formation separations (so that $(\delta_i - \delta_j)$ is the desired separation between vehicle $i$ and vehicle $j$[34]), again in two dimensions.

In the simulation, we choose $\delta_{ix} = 0$ and $\delta_{iy} = 100 * (i-1)$, $i = 1, \ldots, 5$. Thus the desired separation distances in x-axis and y-axis are 0 and 100 respectively. We also choose $\gamma_0 = 1$ and $\gamma_1 = \gamma_2 = 3$, which, together with the fact that $L$ defines an information-exchange topology that contains a directed spanning tree, results in a convergent consensus process. Further, we let vehicle 1 be the "leader" to which the reference model is available. That is, $g_{1r} = 1$ and $g_{ir} = 0$, $\forall i \neq 1$. Note that vehicle 1 has a directed path to all of the other vehicles. In the simulation initially all of the vehicles have different starting positions, velocities, and accelerations, in both x-axis and y-axis.

Figs. 5 and 6 show respectively the x-axis and y-axis reference position, velocity, and acceleration available to vehicle 1. We see that the system is presented with a sinusoidal-varying acceleration in the $x$-axis. Along the $y$-axis the system is "bumped" at 30 and 60 seconds.

The resulting positions, velocities, and accelerations of each vehicle are shown in Figs. 7, 8, and 9 respectively. Note that the positions of each vehicle converge to the reference value
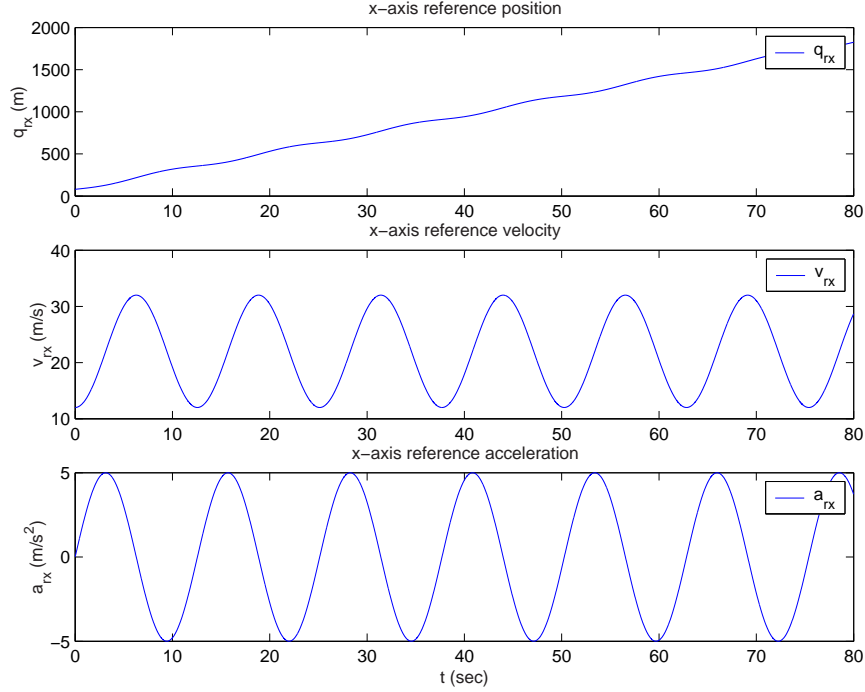
FIG. 5: The x-axis reference states available to vehicle 1.

with the desired separations given by $\delta_i$ as shown in Fig. 7. Also note that the velocities and accelerations of each vehicle converge to the reference velocity and acceleration as shown in Figs. 8 and 9 respectively.

Fig. 10 shows the resulting motion in the $x - y$ plane, where the squares denote the starting positions of each vehicle and plus signs denote the positions of each vehicle at $t = 20, 40, 60, 80$ seconds. It can be seen that the vehicles maintain the desired separation between each other. We see that the higher-order model reference consensus algorithm allows all the vehicles to respond to the reference states received by vehicle 1, while maintaining their formation. This behavior can be seen to be similar to that of a flock of birds moving as a group in formation, but periodically having large changes in direction.

## V. CONCLUSION

In this paper we have defined a class of $\ell^{\text{th}}$-order ($\ell >= 3$) consensus algorithms and have shown necessary and sufficient conditions under which each information variable and their higher-order derivatives converge to common values. We also introduced the idea of higher-order consensus with a leader and the concept of an $\ell^{\text{th}}$ order model-reference consensus
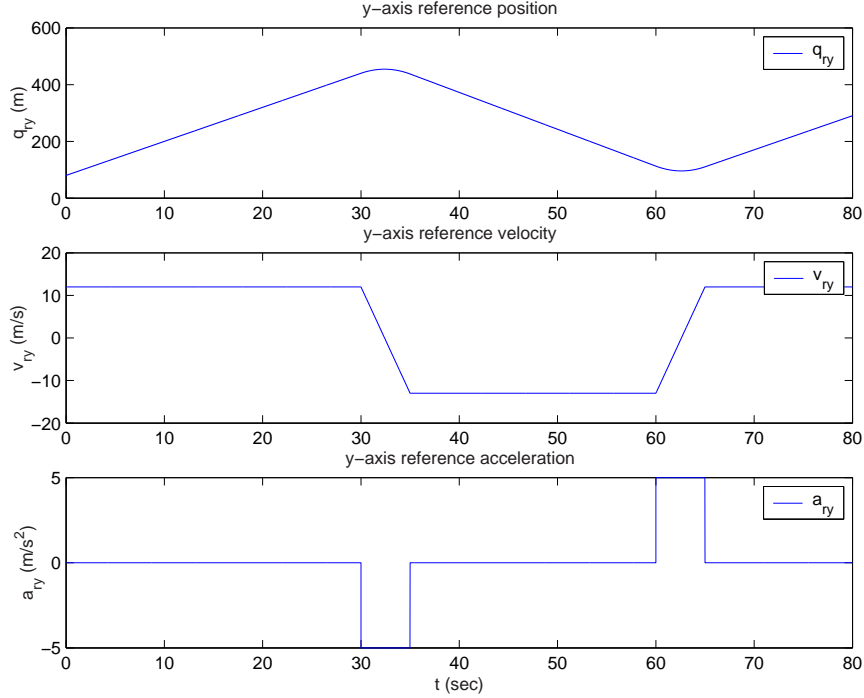
FIG. 6: The y-axis reference states available to vehicle 1.

problem, where each information variable and their high-order derivatives not only reach consensus but also converge to the solution of a prescribed dynamic model. Future research will focus on experimental application of these ideas, including demonstration of formation control via consensus on the Mobile Actuator and Sensor Networks (MAS-net) testbed[32]. It also remains to study the practical implementation of consensus algorithms. Though convergence is a function of the eigenvalues of $\Gamma \otimes I_m$ and will not be impacted as the number of vehicles increases, the effects of delays in the communication among vehicles as well as the effect of time-varying communication topologies must be addressed. Finally, we note that the model-reference consensus algorithm we introduced requires that some or all of the vehicles have *complete knowledge* of the reference model. We are also studying the problem where each vehicle has only *partial knowledge* of the reference model, but that collectively the sum of their knowledge is equivalent to complete knowledge of the reference model. We are interested in consensus algorithms whereby each node will converge to the solution of the reference model.
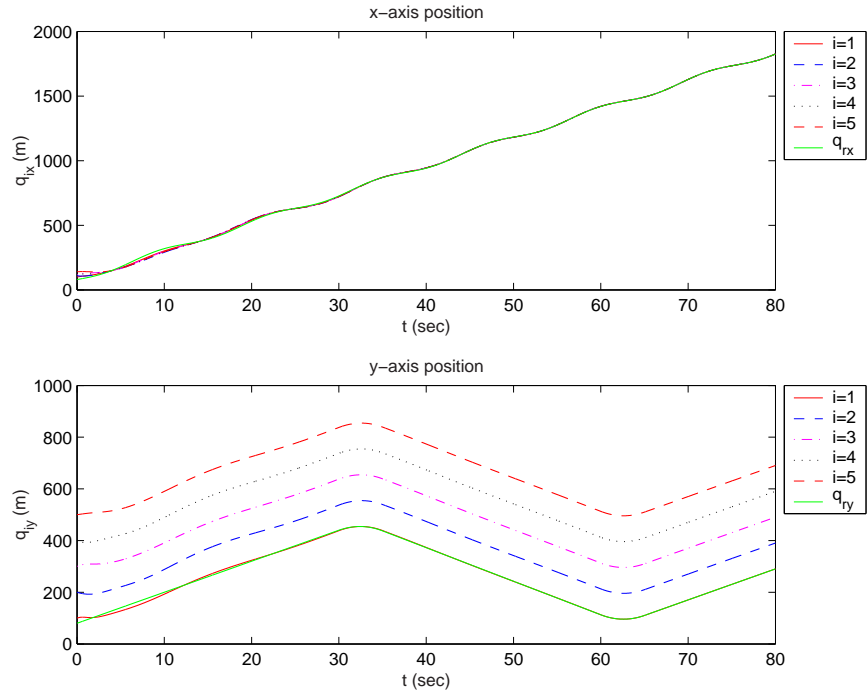
FIG. 7: The positions of each vehicle together with the reference position.
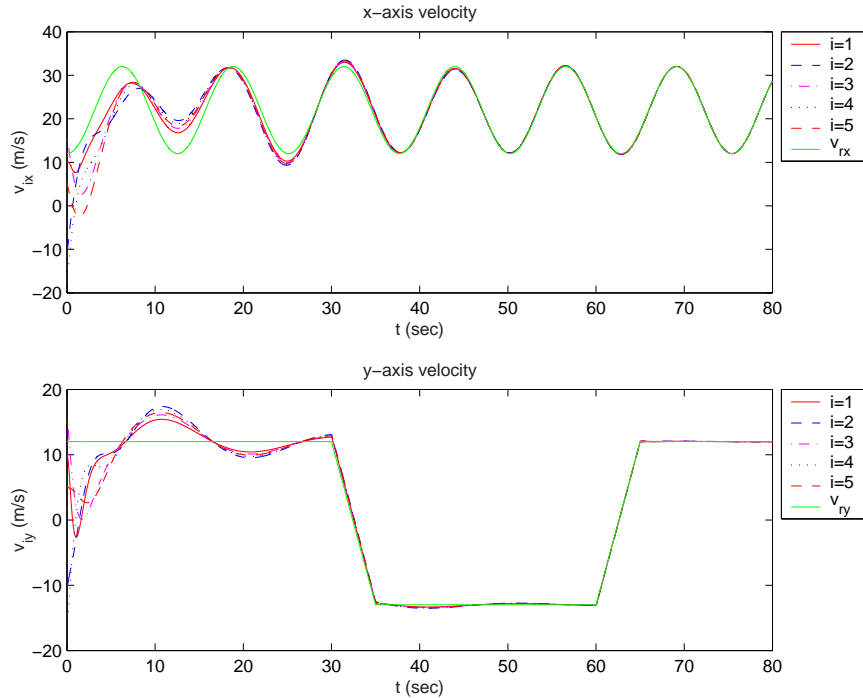
**Acknowledgments**

FIG. 8: The velocities of each vehicle together with the reference velocity.

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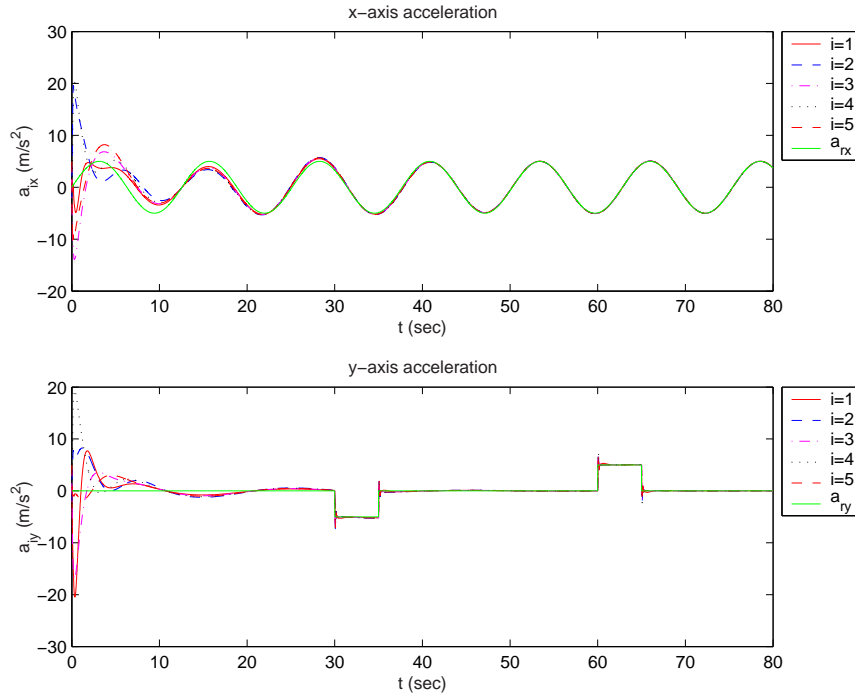
FIG. 9: The accelerations of each vehicle together with the reference acceleration.

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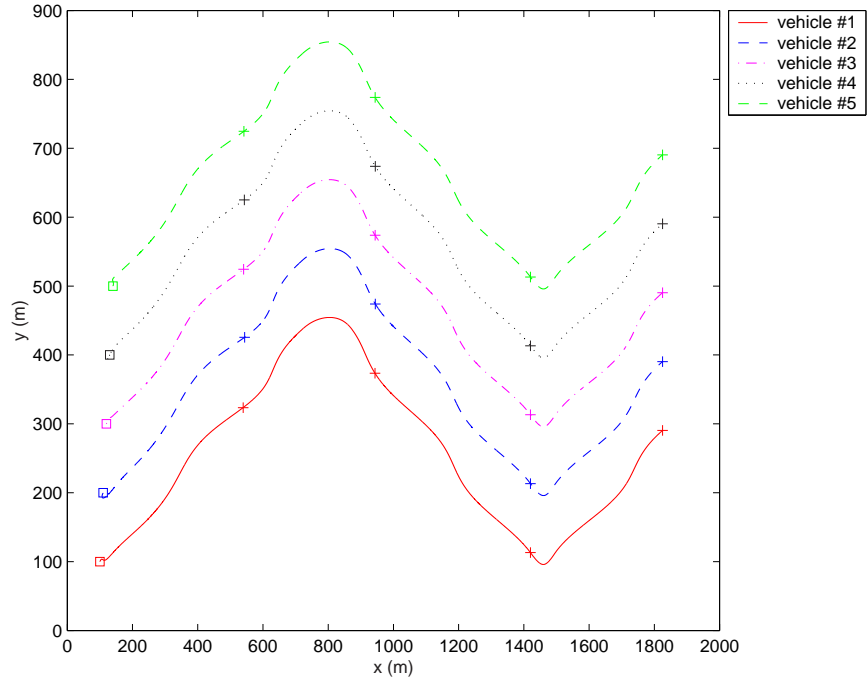
FIG. 10: Resulting $x - y$ motion.

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