



Decentralized finite-time sliding mode estimators and their applications in decentralized finite-time formation tracking

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ABSTRACT

In this paper, a simple but efficient framework is proposed to achieve finite-time decentralized formation tracking of multiple autonomous vehicles with the introduction of decentralized sliding mode estimators. First, we propose and study both first-order and second-order decentralized sliding mode estimators. In particular, we show that the proposed first-order decentralized sliding mode estimator can guarantee accurate position estimation in finite time and the proposed second-order decentralized sliding mode estimator can guarantee accurate position and velocity estimation in finite time. Then the decentralized sliding mode estimators are employed to achieve decentralized formation tracking of multiple autonomous vehicles. In particular, it is shown that formation tracking can be achieved for systems with both single-integrator kinematics and double-integrator dynamics in finite time. By using the decentralized sliding mode estimators, many formation tracking/flying scenarios can be easily decoupled into two subtasks, that is, decentralized sliding mode estimation and vehicle desired state tracking, without imposing a stringent condition on the information flow. Finally, several simulation results are presented as a proof of concept.

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1. Introduction

Recently, decentralized multi-vehicle cooperative control, including consensus [1–7], formation control [8–12], and flocking [13–16], has drawn significant research attention in the control society. Different from the traditional centralized control where a central station is employed to control all vehicles, no central station is required for decentralized cooperative control. Therefore, a number of benefits, for example, scalability, robustness, and easy maintenance, can be obtained by employing decentralized cooperative control.

A crucial task for decentralized cooperative control is to achieve a global group behavior through only local interaction. As a fundamental problem in decentralized cooperative control, consensus has been studied extensively. The main idea of consensus is for a group of vehicles to reach an agreement on some common features, e.g., positions, phases, and attitudes, by negotiating with their local (time-varying) neighbors. Consensus algorithms have been studied for systems with different dynamics under various scenarios. For systems with single-integrator kinematics, the authors in [1–4] studied synchronous consensus while the authors in [5–7] studied asynchronous consensus when the network topology is modeled in a deterministic setting. When

the network topology is modeled in a stochastic setting, the authors in [17–20] studied consensus in probability and derived corresponding convergence conditions. For systems with double-integrator dynamics, the authors in [21–24] studied consensus algorithms in both continuous-time and discrete-time settings. Consensus for systems with general linear dynamics was studied in [25,26]. In these references, consensus was studied in the absence of a virtual leader (or a group reference state).

Although consensus without a virtual leader is interesting, it is sometimes more meaningful and interesting to study consensus in the presence of a virtual leader when the virtual leader's state may represent the state of interest for the group. Here consensus in the presence of a virtual leader is also called *consensus tracking*. In [27,28], a proportional and derivative like consensus tracking algorithm was proposed and studied for single-integrator kinematics in the presence of a dynamic virtual leader in both continuous-time and discrete-time settings. In particular, [27] requires the availability of the velocity measurements of the virtual leader and [28] requires a small sampling period. In [29,30], consensus tracking algorithms were studied for double-integrator dynamics in the presence of a dynamic virtual leader. In particular, [29] requires the availability of the leader's acceleration to all followers and [30] requires the design of distributed observers. The authors in [31] studied a consensus tracking algorithm in the presence of time-varying delays. In particular, an estimator was designed to estimate the leader's velocity. Although these references only studied consensus tracking scenarios, the

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proposed approaches can be extended to the *formation tracking* scenarios where a group of vehicles tracks the virtual leader while maintaining a certain desired geometric formation simultaneously.

In this paper, we propose a simple but efficient framework to solve the finite-time formation tracking problem with the aid of decentralized sliding mode estimators by expanding on our preliminary work reported in [32]. Compared with the consensus tracking algorithms in [27–31], the proposed algorithms demonstrate the following advantages: (1) simple, (2) finite-time convergence, (3) reduced information measurements, and (4) mild requirement on the information flow. We first propose and study both first-order and second-order decentralized sliding mode estimators under a directed switching network topology. In particular, we show that the first-order decentralized sliding mode estimator can guarantee accurate position estimation and the second-order decentralized sliding mode estimator can guarantee accurate position and velocity estimation in finite time. Then we propose finite-time formation tracking algorithms based on the decentralized sliding mode estimators. Note that although finite-time consensus was solved in [33–35], the algorithms proposed in [33–35] cannot be applied to the case when there exists a dynamic virtual leader in the absence of velocity or acceleration measurements. Note also that a Lyapunov-based approach was used in [36] to show finite-time consensus. However, [36] focused on undirected fixed or switching network topologies. Furthermore, because accurate estimation can be achieved in finite time when employing the decentralized sliding mode estimators, many formation tracking/flying problems can be easily decoupled into two subtasks, namely, decentralized sliding mode estimation and vehicle desired state tracking, without imposing a stringent condition on the information flow.

2. Graph theory notions

For a group of n vehicles, the interaction for these vehicles can be modeled by a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{W})$, where $\mathcal{V} = \{v_1, v_2, \dots, v_n\}$ and $\mathcal{W} \subseteq \mathcal{V}^2$ represent, respectively, the vehicle set and the edge set. Each edge denoted as (v_i, v_j) means that vehicle j can access the state information of vehicle i , but not necessarily vice versa. Accordingly, vehicle i is a neighbor of vehicle j . We use \mathcal{N}_j to denote the neighbor set of vehicle j . A directed path is a sequence of edges in a directed graph of the form $(v_1, v_2), (v_2, v_3), \dots$, where $v_i \in \mathcal{V}$. A directed graph has a directed spanning tree if there exists at least one vehicle that has directed paths to all other vehicles.

Two matrices are frequently used to represent the interaction graph: the adjacency matrix $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{n \times n}$ with $a_{ij} > 0$ if $(v_j, v_i) \in \mathcal{W}$ and $a_{ij} = 0$ otherwise, and the (nonsymmetric) Laplacian matrix $\mathcal{L} = [\ell_{ij}] \in \mathbb{R}^{n \times n}$ with $\ell_{ii} = \sum_{j=1, j \neq i}^n a_{ij}$ and $\ell_{ij} = -a_{ij}$, $i \neq j$.

3. First-order decentralized sliding mode estimator

In this section, we propose and study a first-order decentralized sliding mode estimator. Here, we assume that all vehicles are in a one-dimensional space for the simplicity of presentation. However, all results are still valid for the high-dimensional space by the introduction of the Kronecker product.

Assume that there exist n estimators each of which is embedded in an individual vehicle. Assume also that there exists a *virtual leader* with a (time-varying) position given by $r_0(t) \in \mathbb{R}$ available to only a subset of the n vehicles. The objective of this section is to construct a first-order decentralized estimator such that an accurate estimate of the virtual leader's position can be achieved in finite time. It is assumed that $r_0(t)$ satisfies the following two conditions:

1. $r_0(t)$ is differentiable.
2. $\sup_t |\dot{r}_0(t)| \leq \gamma$, i.e., the velocity of the virtual leader is bounded.

Inspired by [37], we propose the following first-order decentralized sliding mode estimator as

$$\begin{aligned} \dot{\hat{r}}_i(t) = & -\alpha \sum_{j=0}^n a_{ij}(t) [\hat{r}_i(t) - \hat{r}_j(t)] \\ & - \beta \operatorname{sgn} \left\{ \sum_{j=0}^n a_{ij}(t) [\hat{r}_i(t) - \hat{r}_j(t)] \right\}, \quad i = 1, \dots, n, \end{aligned} \quad (1)$$

where $\hat{r}_i(t) \in \mathbb{R}$ is the i th vehicle's estimate of $r_0(t)$, α is a nonnegative constant, β is a positive constant, $a_{ij}(t)$ is the (i, j) th entry of the adjacency matrix \mathcal{A} at time t , $a_{i0}(t) > 0$ if $r_0(t)$ is available to the i th vehicle at time t and $a_{i0}(t) = 0$ otherwise, $\operatorname{sgn}(\cdot)$ is the signum function, and $\hat{r}_0(t) \triangleq r_0(t)$. The objective of (1) is to guarantee that $\hat{r}_i(t) \rightarrow r_0(t)$ in finite time.

Theorem 3.1. Assume that the virtual leader has directed paths to every other vehicle at each time instant. When $\beta > \gamma$, $\hat{r}_i(t) = r_0(t)$ for every $t \geq \bar{T}_f$, where $\bar{T}_f = \frac{\max_i \{|\hat{r}_i(0) - r_0(0)|\}}{\beta - \gamma}$.

Proof. Define $\tilde{r}_i(t) \triangleq \hat{r}_i(t) - r_0(t)$. It follows that (1) can be written as

$$\begin{aligned} \dot{\tilde{r}}_i(t) = & -\alpha \sum_{j=0}^n a_{ij}(t) [\tilde{r}_i(t) - \tilde{r}_j(t)] \\ & - \beta \operatorname{sgn} \left\{ \sum_{j=0}^n a_{ij}(t) [\tilde{r}_i(t) - \tilde{r}_j(t)] \right\} - \dot{r}_0(t), \\ & i = 1, \dots, n. \end{aligned} \quad (2)$$

Define $\tilde{r}^+(t) \triangleq \max_i \tilde{r}_i(t)$ and $\tilde{r}^-(t) \triangleq \min_i \tilde{r}_i(t)$. Then the convergence time can be obtained by studying the following three cases:

Case (i) $\tilde{r}^+(0) \leq 0$ and $\tilde{r}^-(0) \leq 0$. When $\tilde{r}^+(0) = \tilde{r}^-(0) = 0$. It is trivial to show that $\tilde{r}^+(t) = \tilde{r}^-(t) = 0$ for any $t > 0$. We next study the case when $\tilde{r}^+(0) \leq 0$ and $\tilde{r}^-(0) < 0$. It follows from (2) that $\tilde{r}^+(t) \leq 0$ for any $t > 0$ because $\dot{\tilde{r}}^+(t) \leq -(\beta - \dot{r}_0) \leq -(\beta - \gamma) < 0$ if $\tilde{r}^+(t) > 0$. By following a similar analysis, we can also get that $\tilde{r}^-(t)$ is nondecreasing. Note that when $\tilde{r}^-(t) < 0$, either $\dot{\tilde{r}}^-(t) = -\dot{r}_0(t)$ or $\dot{\tilde{r}}^-(t) > \beta - \gamma > 0$. We next show that when $\tilde{r}^-(T) < 0$ for some $T > 0$, $\dot{\tilde{r}}^-(t) = -\dot{r}_0(t)$ only happens at isolated time instants when $t \leq T$. We prove this by contradiction.

Assume that $\dot{\tilde{r}}^-(t) = -\dot{r}_0(t)$ for $t \in [t_1, t_2]$ where $t_1 < t_2 \leq T$. Then there exists some vehicle, labeled as j , with the state $\tilde{r}^-(t)$ satisfying $\dot{\tilde{r}}_j(t) = -\dot{r}_0(t)$ for $t \in [t_1, t_3]$, where $t_1 < t_3 \leq t_2$. Note that $\dot{\tilde{r}}_j(t) = -\dot{r}_0(t)$ implies that $\sum_{i=0}^n a_{ij}[\tilde{r}_j(t) - \tilde{r}_i(t)] = 0$ for $t \in [t_1, t_3]$ from (2). Because $\hat{r}_j(t) = \tilde{r}^-(t)$, it follows that $\dot{\tilde{r}}_j(t) = -\dot{r}_0(t)$ for $t \in [t_1, t_3]$ implies that $\hat{r}_i(t) = \tilde{r}^-(t)$, $\forall i \in \mathcal{N}_j$, for $t \in [t_1, t_3]$. By following a similar analysis, it follows that $\hat{r}_i(t) = \tilde{r}^-(t)$, $\forall i = 0, \dots, n$, for $t \in [t_1, t_3]$ if the virtual leader has directed paths to every other vehicle at each time instant, which results in a contradiction because $\tilde{r}^-(t) < 0$ for $t \leq T$. Therefore, when $\tilde{r}^-(T) < 0$, $\tilde{r}^-(t)$ will keep increasing at a speed larger than $\beta - \gamma$ when $t \leq T$ except for some isolated time instants. It follows that the maximal convergence time is given by $\frac{|\tilde{r}^-(0)|}{\beta - \gamma}$. Note that this bound also applies to the case when $\tilde{r}^-(0) = 0$.

Case (ii) $\tilde{r}^+(0) \geq 0$ and $\tilde{r}^-(0) \geq 0$. By following a similar analysis to that of Case (i), it can be computed that the maximal convergence time is given by $\frac{|\tilde{r}^+(0)|}{\beta - \gamma}$.

Case (iii). $\tilde{r}^+(0) \geq 0$ and $\tilde{r}^-(0) \leq 0$. By combining Cases (i) and (ii), it can be computed that the maximal convergence time is given by $\frac{\max\{|\tilde{r}^+(0)|, |\tilde{r}^-(0)|\}}{\beta - \gamma}$.

Combining the previous three cases completes the proof. \square

Remark 3.2. From the proof of [Theorem 3.1](#), both α and β in (1) can be chosen differently for different vehicle i . Assume that the control gains for the i th estimator are α_i and β_i . When $\alpha_i \geq 0$ and $\beta_i > \gamma$, the convergence time becomes $\bar{T}_f = \frac{\max_i\{|\hat{r}_i(0) - r_0(0)|\}}{\min_i\{\beta_i - \gamma\}}$.

Remark 3.3. Note that $\hat{r}_i(t)$ given in (1) is discontinuous, which is different from $\dot{r}_0(t)$ by noting that $\dot{r}_0(t)$ might be continuous. However, $\int_{t_1}^{t_2} \hat{r}_i(t) dt = \int_{t_1}^{t_2} \dot{r}_0(t) dt$, $i = 1, \dots, n$ for any $t_2 > t_1 \geq \bar{T}_f$. Therefore, when $t \geq \bar{T}_f$, $\hat{r}_i(t)$ can be used to replace $\dot{r}_0(t)$ in integration related applications for $t \geq \bar{T}_f$. Here, the replacement of $\hat{r}_i(t)$ for $\dot{r}_0(t)$ can also be interpreted as follows. Note that $\hat{r}_i(t)$ can be written as

$$\dot{\hat{r}}_i(t) = \dot{r}_0(t) + f(t), \quad (3)$$

where $f(t)$ is a switching signal satisfying $\int_{t_1}^{t_2} f(t) dt = 0$ for any $t_2 > t_1 \geq \bar{T}_f$ and the frequency of $f(t)$ is infinitely large. Note also that an integration-based system can be considered an ideal low-pass filter. Therefore, $f(t)$ will be filtered out with the low-pass filter.

Remark 3.4. Inspired by the analysis in [Theorem 3.1](#), we next study a finite-time leaderless consensus algorithm for single-integrator kinematics. Consider a group of n agents with single-integrator kinematics given by

$$\dot{r}_i(t) = u_i(t), \quad i = 1, \dots, n, \quad (4)$$

where $r_i(t)$ is the position of i th vehicle and $u_i(t)$ is the corresponding control input at time t . Leaderless consensus is achieved in finite time if there exists a positive \bar{t} such that $r_i(t) = r_j(t)$ for $t > \bar{t}$. Define

$$g(t) = \begin{cases} 1, & \mathcal{G}(t) \text{ has a directed spanning tree,} \\ 0, & \text{otherwise.} \end{cases}$$

By using

$$u_i = -\alpha \sum_{j=1}^n a_{ij}(t)(r_i - r_j) - \beta \operatorname{sgn} \left[\sum_{j=1}^n a_{ij}(t)(r_i - r_j) \right], \quad (5)$$

where α is a nonnegative constant and β is a positive constant for (4), $r_i(t) = r_j(t)$, $i, j \in \{1, \dots, n\}$, holds for every $t > \sigma$ (i.e., leaderless consensus is achieved in finite time), if $\int_0^\sigma g(t) dt > \frac{\max_i r_i(0) - \min_i r_i(0)}{2\beta}$. The proof is similar to that of [Theorem 3.1](#) by showing that over a time interval $[t_1, t_2]$ the maximal position will decrease and the minimal position will increase with a minimum speed β before leaderless consensus is achieved except for some isolated time instants if $\mathcal{G}(t)$ has a directed spanning tree over $[t_1, t_2]$. In addition, if $\mathcal{G}(t)$ does not have a directed spanning tree over a time interval, the maximal position will not increase and the minimal position will not decrease over the time interval. Different from the traditional linear leaderless consensus algorithm $u_i = -\alpha \sum_{j=1}^n a_{ij}(t)[r_i(t) - r_j(t)]$ [i.e., $\beta = 0$ in (5)] under which leaderless consensus can be achieved if the network topology has a directed spanning tree jointly [3], using (5) for (4), leaderless consensus might not be achieved when the network topology has a directed spanning jointly because $\max_i r_i(t) - \min_i r_i(t)$ might not decrease over time in this case. For example, suppose that there exist three agents in a team. At each time instant, there is only one agent who is a neighbor of another agent. Initially, agent 1 is a neighbor of agent 2 for a period of time that guarantees that agent 2's state will become equal to that of agent 1 due to finite-time convergence. Then agent 3 is a neighbor of agent 1 for a period of time that guarantees that agent 1's state will become equal to

that of agent 3 due to finite-time convergence. Then agent 2 is a neighbor of agent 3 for a period of time that guarantees that agent 3's state will become equal to that of agent 2 due to finite-time convergence. Then the previous sequence repeats. Note that the directed graph has a directed spanning tree jointly. However, consensus cannot be achieved.

Remark 3.5. For (4), there exist two other leaderless finite-time consensus algorithms

$$u_i = - \sum_{j=1}^n a_{ij}(t) \operatorname{sgn}(r_i - r_j) |r_i - r_j|^\gamma, \quad \gamma \in (0, 1) \quad (6)$$

which is proposed in [35,33] and

$$u_i = -\operatorname{sgn} \left[\sum_{j=1}^n a_{ij}(r_i - r_j) \right] \left| \sum_{j=1}^n a_{ij}(r_i - r_j) \right|^\gamma, \quad \gamma \in (0, 1) \quad (7)$$

which is proposed in [34]. Using (6) for (4), finite-time leaderless consensus was shown to be achieved under a fixed/switching undirected network topology in [35,33]. Using (7) for (4), finite-time leaderless consensus was shown to be achieved under a fixed directed network topology in [34]. Both (6) and (7) can only solve the finite-time leaderless consensus problem rather than the finite-time consensus tracking problem with a dynamic leader in the absence of velocity measurements.¹ That is, if $\dot{r}_0(t)$ is not known, (6) and (7) cannot solve the finite-time consensus tracking problem. Note also that the author in [36] used a Lyapunov-based approach to show that leaderless consensus can be achieved in finite time by using (5) with $\alpha = 0$ under an undirected network topology. We have shown that (5) can guarantee finite-time leaderless consensus for directed switching network topologies by analyzing how the maximal and minimal positions evolve.

4. Second-order sliding mode estimator

In this section, we propose and study a second-order decentralized sliding mode estimator. We also assume that all vehicles are in a one-dimensional space for the simplicity of presentation. However, all results are still valid for high-dimensional space by the introduction of the Kronecker product.

Assume also that there exist n estimators each of which is embedded in an individual vehicle. Assume that there also exists a *virtual leader*, whose position is given by $r_0(t)$ and velocity is given by $v_0(t)$ available to only a subset of the n vehicles. The objective here is to construct a second-order decentralized estimator such that accurate estimation of the virtual leader's position and velocity can be achieved in finite time. It is assumed that $v_0(t)$ satisfies the following two conditions:

1. $v_0(t)$ is differentiable.
2. $\sup_t |\dot{v}_0(t)| \leq \xi$, i.e., the acceleration of the virtual leader is bounded.

We propose the following second-order sliding mode estimator as

$$\dot{\hat{r}}_i(t) = \hat{v}_i(t) - \alpha \operatorname{sgn} \left\{ \sum_{j=0}^n a_{ij}(t) [\hat{r}_i(t) - \hat{r}_j(t)] \right\} \quad (8a)$$

$$\dot{\hat{v}}_i(t) = -\beta \operatorname{sgn} \left\{ \sum_{j=0}^n a_{ij}(t) [\hat{v}_i(t) - \hat{v}_j(t)] \right\}, \quad i = 1, \dots, n, \quad (8b)$$

¹ The proposed algorithm (1) can be used to solve the finite-time consensus tracking problem with a dynamic leader in the absence of velocity measurements.

where $\hat{r}_i(t)$ and $\hat{v}_i(t)$ are the i th vehicle's estimates of, respectively, the position and velocity of the virtual leader, α and β are positive constants, $a_{ij}(t)$ is defined the same as in (1), and $\hat{v}_0(t) \triangleq v_0(t)$. The objective of (8) is to guarantee that $\hat{r}_i(t) \rightarrow r_0(t)$ and $\hat{v}_i(t) \rightarrow v_0(t)$ in finite time.

Theorem 4.1. Assume that the virtual leader has directed paths to every other vehicle at each time instant. When $\beta > \xi$, $\hat{r}_i(t) = r_0(t)$ and $\hat{v}_i(t) = v_0(t)$ for every

$$t \geq \bar{T}_s, \quad (9)$$

where $\bar{T}_s = 2t_1 + \frac{(\xi + \beta)t_1^2}{2\alpha} + \frac{(|\hat{r}_i(0) - r_0(0)| + |\hat{v}_i(0) - v_0(0)|)t_1}{\alpha}$ and

$$t_1 = \frac{\max_i \{|\hat{v}_i(0) - v_0(0)|\}}{\beta - \xi}. \quad (10)$$

Proof. It follows from (8b) and Theorem 3.1 that $\hat{v}_i(t) = v_0(t)$ when $t \geq t_1$, where t_1 is given by (10). Similarly, suppose that the initial time of (8a) is $t = t_1$. Note that $\hat{v}_i(t) = v_0(t)$ for $t \geq t_1$. By replacing $\hat{v}_i(t)$ with $v_0(t)$ for $t \geq t_1$ and defining $\tilde{r}_i(t) \triangleq \hat{r}_i(t) - r_0(t)$, it follows that (8a) can be written as $\dot{\tilde{r}}_i(t) = -\alpha \operatorname{sgn}\{\sum_{j=0}^n a_{ij}(t)[\tilde{r}_i(t) - \tilde{r}_j(t)]\}$ for $t \geq t_1$. Note also that $\tilde{r}_0(t) = 0$. It then follows from the proof of Theorem 3.1 that $\tilde{r}_i(t) = 0$, i.e., $\hat{r}_i(t) = r_0(t)$ for any $t > t_1 + \frac{\max_i \{|\hat{r}_i(t_1) - r_0(t_1)|\}}{\alpha}$.

We next study $\max_i \{|\hat{r}_i(t_1) - r_0(t_1)|\}$. Define $\tilde{v}_i(t) \triangleq \hat{v}_i(t) - v_0(t)$. It follows from (8) that

$$\dot{\tilde{r}}_i(t) = \tilde{v}_i(t) - \alpha \operatorname{sgn}\left\{\sum_{j=0}^n a_{ij}(t)[\tilde{r}_i(t) - \tilde{r}_j(t)]\right\} \quad (11)$$

$$\dot{\tilde{v}}_i(t) = -\dot{v}_0(t) - \beta \operatorname{sgn}\left\{\sum_{j=0}^n a_{ij}(t)[\tilde{v}_i(t) - \tilde{v}_j(t)]\right\}. \quad (12)$$

Note that $\sup_t \{|\dot{v}_0(t)|\} \leq \xi$. We have that $|\dot{\tilde{v}}_i(t)| \leq \xi + \beta$. It follows from (12) that

$$|\tilde{v}_i(t)| \leq |\tilde{v}_i(0)| + (\xi + \beta)t$$

for $0 < t < t_1$. It then follows from (11) that

$$|\tilde{r}_i(t_1)| \leq |\tilde{r}_i(0)| + (|\tilde{v}_i(0)| + \alpha)t_1 + \frac{(\xi + \beta)t_1^2}{2}.$$

Combining the previous arguments shows that $\hat{r}_i(t) = r_0(t)$ and $\hat{v}_i(t) = v_0(t)$ for any t satisfying (9). \square

Remark 4.2. In Theorem 4.1, the estimation of the position and velocity can be decoupled into two independent problems because the decentralized sliding estimator (8b) can guarantee accurate estimation of the virtual leader's velocity in finite time. It is worthwhile to mention that the adjacency matrix in (8a) could be different from that in (8b). Accurate estimation can still be achieved in finite time if the corresponding graphs associated with the adjacency matrices in (8a) and (8b) satisfy that the virtual leader has directed paths to every other vehicle at each time instant.

Remark 4.3. Similar to the analysis in Remark 3.3, although \hat{r}_i (respectively, \hat{v}_i) is not equal to \hat{v}_i (respectively, \hat{v}_0) at each time instant, the integration of \hat{r}_i (respectively, \hat{v}_i) over $t \in (t_1, t_2)$ is equal to the integration of \hat{v}_i over $t \in (t_1, t_2)$ (respectively, \hat{v}_0) if $t_2 > t_1 > \bar{T}_s$, where \bar{T}_s is defined in Theorem 4.1. Therefore, for (14), \hat{v}_i (respectively, \hat{v}_i) can be used to replace \hat{r}_i (respectively, \hat{v}_0), and vice versa.

Remark 4.4. In Theorems 3.1 and 4.1, it is assumed that either the velocity of the virtual leader (cf. Theorem 3.1) or the acceleration of the virtual leader (cf. Theorem 4.1) is bounded and the bound is known beforehand. When the bound of the velocity or the acceleration of the virtual leader is unknown beforehand, β might be chosen as a nondecreasing switching signal and β increases if the estimation errors do not decrease.

5. Finite-time decentralized formation tracking

In this section, we study a finite-time decentralized formation tracking problem for both single-integrator kinematics and double-integrator dynamics by using the decentralized sliding mode estimators. Assume that there exists a (time-varying) virtual leader whose position is given by $r_0(t)$ and velocity is given by $v_0(t)$ available to only a subset of a group of n vehicles. The objective is to design a decentralized control law such that all followers track the virtual leader while maintaining a certain desired geometric formation with only local interaction. For the simplicity of presentation, we also assume that all vehicles are in a one-dimensional space.

5.1. Single-integrator kinematics

In this case, we assume that $\sup_t |v_0(t)| < \gamma$. Consider a group of n vehicles with single-integrator kinematics given by (4). Inspired by the first-order sliding mode estimator in Section 3, we propose the following formation tracking algorithm as

$$u_i(t) = -\alpha \sum_{j=0}^n a_{ij}(t)[r_i(t) - \delta_i - r_j(t) + \delta_j] - \beta \operatorname{sgn}\left\{\sum_{j=0}^n a_{ij}(t)[r_i(t) - \delta_i - r_j(t) + \delta_j]\right\}, \quad (13)$$

$$i = 1, \dots, n,$$

where $a_{ij}(t)$ is the (i, j) th entry of the adjacency matrix, δ_i is constant at time t , α is a nonnegative constant, and β is a positive constant. Define $\Delta_{ij} \triangleq \delta_i - \delta_j$. The objective of (13) is to guarantee that $r_i(t) - r_j(t) \rightarrow \Delta_{ij}$ in finite time.

Theorem 5.1. Assume that the virtual leader has directed paths to all vehicles 1 to n at each time instant. When $\beta > \gamma$, $r_i(t) - r_j(t) \rightarrow \Delta_{ij}$ in finite time.

Proof. Define $\tilde{r}_i \triangleq r_i - \delta_i$. By following the proof of Theorem 3.1, it is straightforward to show that $\tilde{r}_i \rightarrow \tilde{r}_j$ in finite time, which implies that $r_i(t) - r_j(t) \rightarrow \Delta_{ij}$ in finite time. \square

Remark 5.2. Another simple formation tracking algorithm for (4) is given by

$$u_i(t) = \beta \operatorname{sgn}[\hat{r}_i(t) - r_i(t) + \delta_i],$$

where $\beta > \gamma$ and $\hat{r}_i(t)$ is the i th vehicle's estimate of the virtual leader's position when using the decentralized sliding mode estimator given by (1).

5.2. Double-integrator dynamics

In this case, we assume that $\sup_t |\dot{v}_0(t)| < \xi$. For a group of n vehicles with double-integrator dynamics given by

$$\begin{aligned} \dot{r}_i(t) &= v_i(t) \\ \dot{v}_i(t) &= u_i(t), \quad i = 1, \dots, n \end{aligned} \quad (14)$$

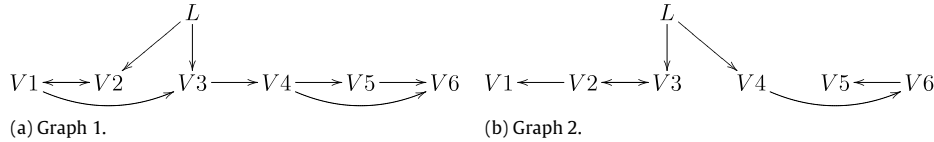


Fig. 1. Network topology for a group of six vehicles with a virtual leader. Here L denotes the virtual leader while $V_i, i = 1, \dots, 6$, denote the vehicles.

where $r_i(t)$ and $v_i(t)$ represent, respectively, the position and velocity of the i th vehicle, and $u_i(t)$ is the control input for the i th vehicle at time t . Inspired by [38], we propose the following formation tracking algorithm as

$$u_i(t) = \dot{\hat{v}}_i(t) - \text{sgn}[v_i(t) - \hat{v}_i(t)] |v_i(t) - \hat{v}_i(t)|^\gamma - \text{sgn}(\phi) |\phi|^{2-\gamma}, \quad (15)$$

where $\hat{r}_i(t)$ and $\hat{v}_i(t)$ are the i th vehicle's estimates of, respectively, the virtual leader's position and velocity when using the decentralized sliding mode estimator given by (8), $\gamma \in (0, 1)$ and $\phi = r_i(t) - \delta_i - \hat{r}_i(t) + \frac{1}{2-\gamma} \text{sgn}[v_i(t) - \hat{v}_i(t)] |v_i(t) - \hat{v}_i(t)|^{2-\gamma}$. The objective of (15) is to guarantee that $r_i(t) - r_j(t) \rightarrow \Delta_{ij}$ and $v_i(t) \rightarrow v_0(t)$ in finite time. Before moving on, we need the following lemma.

Lemma 5.1 ([38]). *For double-integrator dynamics*

$$\dot{x} = y, \quad \dot{y} = u,$$

the origin is a global finite-time-stable equilibrium under the feedback control law

$$u = -\text{sgn}(y) |y|^\gamma - \text{sgn}(\phi) |\phi|^{2-\gamma}$$

where $\gamma \in (0, 1)$ and $\phi = x + \frac{1}{2-\gamma} \text{sgn}(y) |y|^{2-\gamma}$.

Theorem 5.3. *Assume that the virtual leader has directed paths to all vehicles 1 to n at each time instant. Assume also that the conditions on the network topology and the control gains in Theorem 4.1 are satisfied. Using (15) for (14), $r_i(t) - r_j(t) \rightarrow \Delta_{ij}$ and $v_i(t) \rightarrow v_0(t)$ in finite time.*

Proof. From Theorem 4.1, the decentralized sliding mode estimator can guarantee that $\hat{r}_i(t) = r_0(t)$ and $\hat{v}_i(t) = v_0(t)$ for $t \geq \bar{T}_s$, where \bar{T}_s is defined in (9). Note that $\hat{v}_i(t)$ can be replaced with $\dot{v}_0(t)$ for $t \geq \bar{T}_s$ according to Remark 3.3. Define $\tilde{r}_i \triangleq r_i - \delta_i - \hat{r}_i$. For $t \geq \bar{T}_s$, (14) using (15) can be written as

$$\ddot{\tilde{r}}_i(t) = -\text{sgn}[\dot{\tilde{r}}_i(t)] |\dot{\tilde{r}}_i(t)|^\gamma - \text{sgn}(\phi) |\phi|^{2-\gamma}, \quad (16)$$

where $\phi = \tilde{r}_i(t) + \frac{1}{2-\gamma} \text{sgn}[\dot{\tilde{r}}_i(t)] |\dot{\tilde{r}}_i(t)|^{2-\gamma}$. It then follows from Lemma 5.1 that $\tilde{r}_i(t)$ will go to zero in finite time, which in turn implies that $r_i(t) - r_j(t) \rightarrow \Delta_{ij}$ and $v_i(t) \rightarrow v_0(t)$ in finite time. \square

Remark 5.4. Because the second-order decentralized sliding mode estimator can achieve accurate estimation in finite time, formation tracking for double-integrator dynamics can be decoupled into two subtasks:

- 1: A decentralized estimator to accurately estimate the virtual leader's information in finite time.
- 2: A local control to guarantee accurate tracking of the virtual leader in finite time.

Remark 5.5. The main difference between decentralized control and centralized control is whether some group information can be obtained by each group member instantaneously. If there exists a central station which can not only obtain the group information but also communicate with each vehicle in

the group, centralized control techniques can then be used to achieve various formation control scenarios. However, the requirement of the central station poses an obvious limitation and may not be feasible in real applications. Therefore, it is more practical to employ decentralized control techniques. By using the decentralized sliding mode estimators, accurate estimation of the group information can be achieved in finite time. Accordingly, centralized-control-like decentralized control techniques can be easily designed and implemented. For example, the decentralized virtual structure approach proposed in [39] can be revised by using the decentralized sliding mode approach and the formation control problem can be solved by employing a decentralized estimator and a local controller. From this perspective, this paper provides a new simple but efficient framework to solve the formation tracking/flying problem.

Remark 5.6. In [40], a proportional and derivative (PI) based decentralized estimation method was proposed for multi-agent coordination. However, the proposed decentralized estimator can guarantee only bounded tracking errors. Inspired by Theorem 3.1, each vehicle can establish $n - 1$ decentralized sliding mode estimators to estimate all other external time-varying states. According to Theorem 3.1, each time-varying external state can be estimated accurately by all vehicles in finite time. It then follows that all external states can be estimated accurately by all agents in finite time accordingly. Therefore, accurate estimation of the average of a group of time-varying references can be achieved in finite time accordingly. Furthermore, accurate estimation of an arbitrary function of a group of time-varying references can be achieved in finite time as well.

6. Simulation

In this section, we present several simulation results to validate the theoretical results in previous sections. We consider a group of six vehicles. The directed network topology switches from Fig. 1(a) to (b) every 1 s and then the process repeats. We choose $a_{ij} = 1$ if $(v_j, v_i) \in \mathcal{W}$ and $a_{ij} = 0$ otherwise.

For the first-order decentralized sliding mode estimator (1), the virtual leader's position is chosen as $r_0(t) = \sin(t) + \frac{t}{5}$ and the initial states of the estimators embedded in the six vehicles are given by $\hat{r}(0) = [-5, 2, -3, 5, 2, -4]^T$, where $\hat{r}(0)$ is a column stack vector of $\hat{r}_i(0), i = 1, \dots, 6$. We choose $\alpha = 2$ and $\beta = 4$. Fig. 2(a) and (b) show, respectively, the decentralized estimates of the virtual leader's position and the estimation errors using (1). It can be seen that each vehicle can obtain accurate estimation of the virtual leader's position in finite time under a directed switching network topology.

For the second-order decentralized sliding mode estimator (8), the virtual leader's position is chosen as $r_0(t) = \sin(t) + \frac{t}{2}$ and the initial states of the estimators are given by $\hat{r}(0) = [-5, 2, -3, 5, 2, -4]^T$ and $\hat{v}(0) = [-0.4, 1, -1.3, 0.2, 0.8, -1.4]^T$, where $\hat{v}(0)$ is a column stack vector of $\hat{v}_i(0), i = 1, \dots, 6$. We also choose $\alpha = 2$ and $\beta = 4$. Fig. 3(a) and (c) show the estimates of, respectively, the position and the velocity of the virtual leader using (8). Fig. 3(b) and (d) show, respectively, the position and the velocity estimation errors using (8). It can be seen that each vehicle can obtain accurate estimation of the virtual leader's position

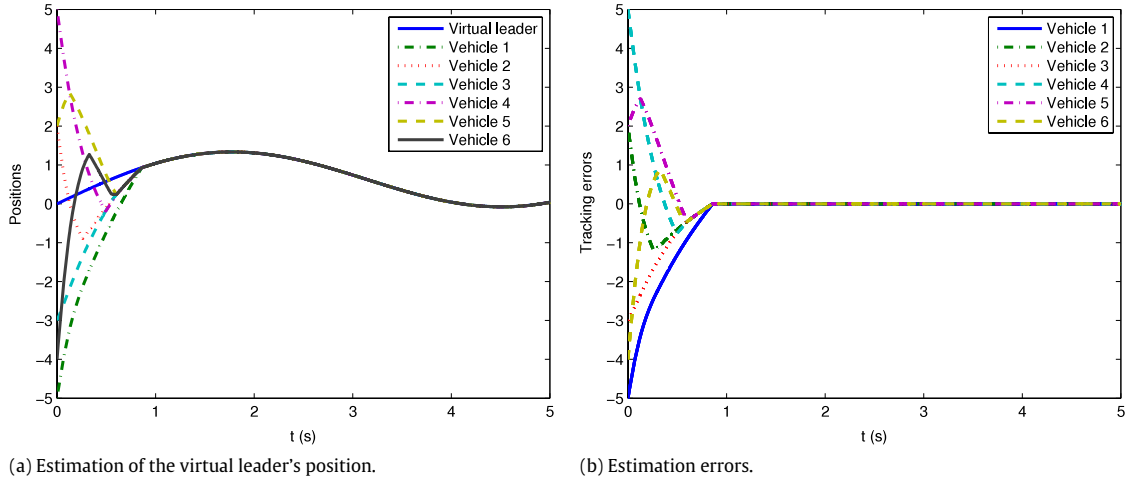


Fig. 2. Simulation results using (1).

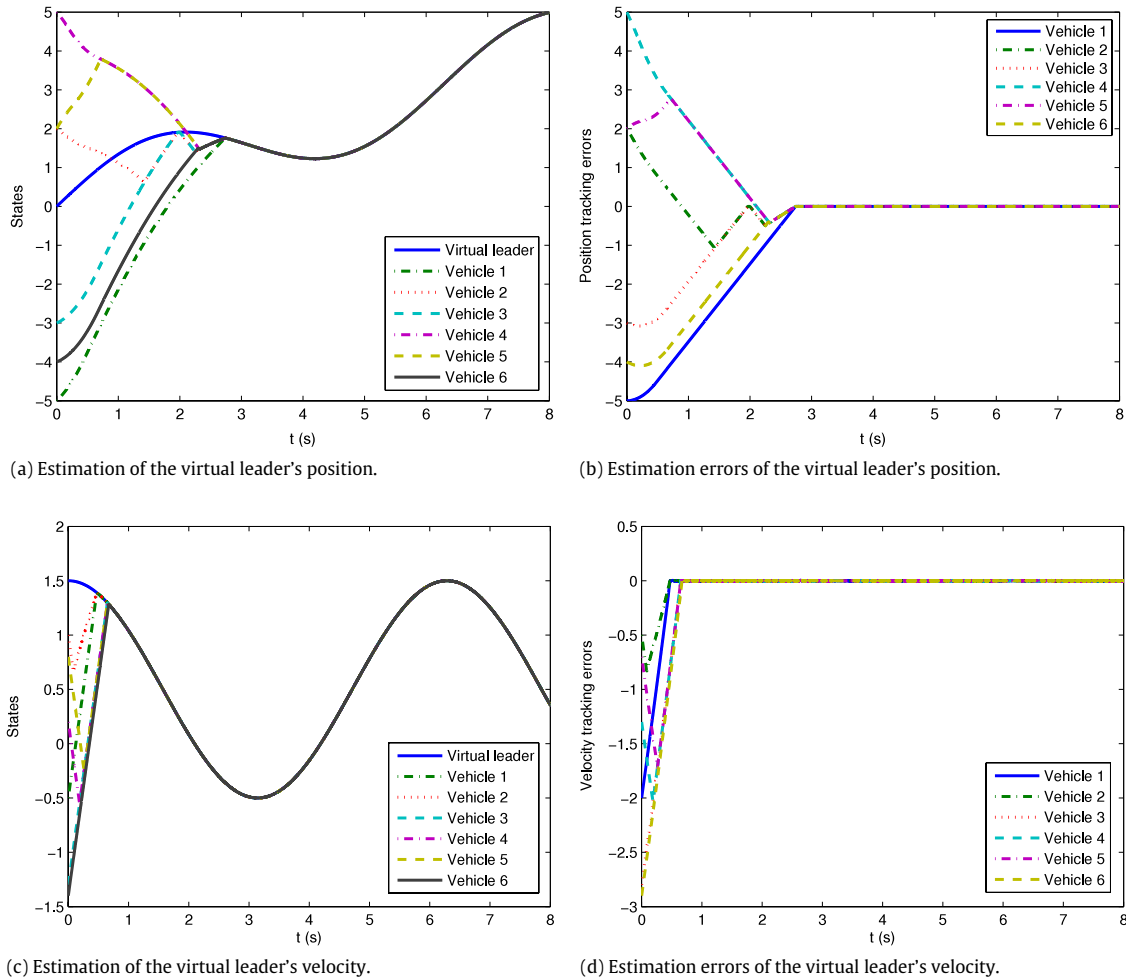


Fig. 3. Simulation results using (8).

and velocity in finite time under a directed switching network topology.

For the formation tracking algorithms (13) and (15), we present several simulation results in the two-dimensional space. Here $r_i \in \mathbb{R}^2, v_i \in \mathbb{R}^2$, and $\delta_i \in \mathbb{R}^2$. We have chosen $r(0) = -[5, 8, 7, 9, 4, 5; 8, 6, 10, 8, 8, 7]^T$ for (13) and $r(0) = -[5, 8, 7, 9, 4, 5; 8, 6, 10, 8, 8, 7]^T, v(0) = \frac{1}{10}[-3, 1, 13, -2, -8, 4; 3, -1, -13, 2, 8, -4]^T, \alpha = 2, \beta = 4$, and $\gamma = 0.8$ for

(15). We also choose $\delta_1 = [2, 0]^T, \delta_2 = [1, -1]^T, \delta_3 = [-1, -1]^T, \delta_4 = [-2, 0]^T, \delta_5 = [-1, 1]^T$, and $\delta_6 = [1, 1]^T$. Fig. 4(a) shows the positions of the six vehicles and the virtual leader using (13) when $r_0(t) = [10 \sin(\frac{t}{10}) - 5, \frac{t}{3} - 5]^T$. In addition, two snapshots are provided to show the formation of the six vehicles and the virtual leader at $t = 15$ s and $t = 30$ s. Fig. 4(b) shows the position differences between the six vehicles and the virtual

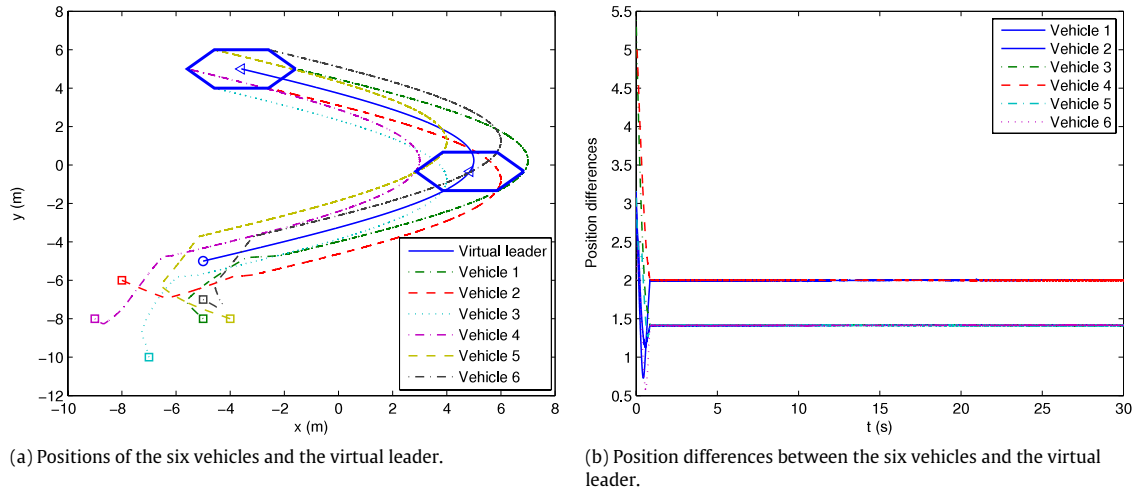


Fig. 4. Simulation results using (13). In subfigure (a), squares denote the starting positions of the six vehicles while circle denotes the starting position of the virtual leader.

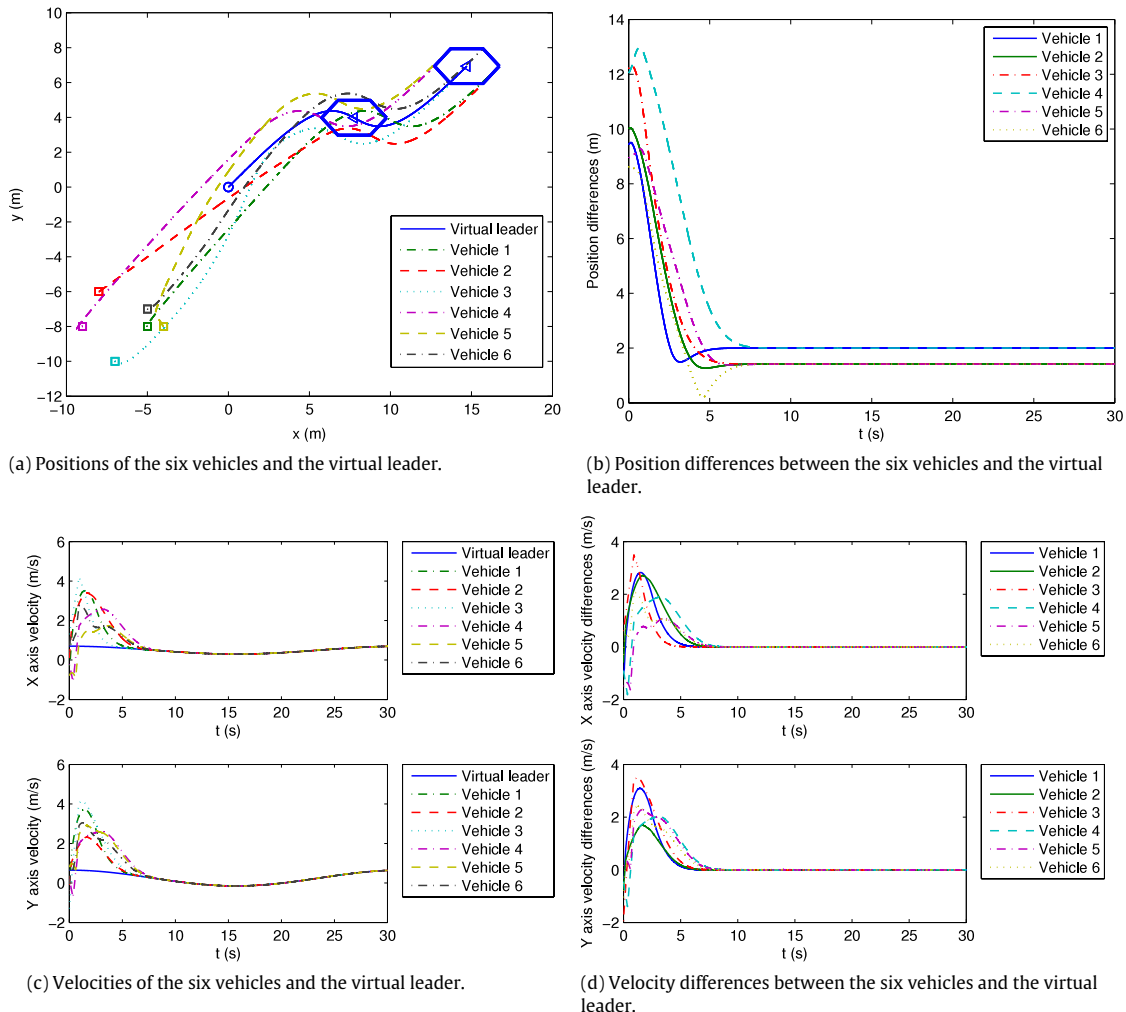


Fig. 5. Simulation results using (15). In subfigure (a), squares denote the starting positions of the six vehicles while circle denotes the starting position of the virtual leader.

leader. It can be seen that the vehicles can track the virtual leader and maintain the desired formation in finite time under a directed switching network topology. Fig. 5(a) and (c) show, respectively, the positions and velocities of the six vehicles and the virtual leader using (15) when $r_0(t) = [\sin(\frac{t}{5}) + \frac{t}{2}, 2 \sin(\frac{t}{5}) + \frac{t}{4}]^T$. In addition, two snapshots are also provided to show the formation of the six

vehicles and the virtual leader at $t = 15$ s and $t = 30$ s. Fig. 5(b) and (d) show, respectively, the position and velocity differences between the six vehicles and the virtual leader using (15). It can be seen that the vehicles can also track the virtual leader and maintain the desired formation in finite time under a directed switching network topology.

7. Conclusion

This paper proposed a simple but efficient framework to achieve finite-time decentralized formation tracking of multiple autonomous vehicles by introducing the decentralized sliding mode estimators. First, we proposed and studied both first-order and second-order decentralized sliding mode estimators. In particular, the proposed decentralized sliding mode estimators were shown to guarantee accurate estimation in finite time. Then the decentralized sliding mode estimators were employed to achieve decentralized formation tracking of multiple autonomous vehicles in finite time. Because accurate estimation can be achieved in finite time when utilizing the proposed sliding mode estimators, many formation tracking/flying scenarios can be decoupled into two simple subtasks, that is, decentralized sliding mode estimation and vehicle desired state tracking, without imposing a stringent condition on the information flow.

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