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Brief paper

Distributed discrete-time coordinated tracking with a time-varying reference state and limited communication[☆]

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ABSTRACT

This paper studies a distributed discrete-time coordinated tracking problem where a team of vehicles communicating with their local neighbors at discrete-time instants tracks a time-varying reference state available to only a subset of the team members. We propose a PD-like discrete-time consensus algorithm to address the problem under a fixed communication graph. We then study the condition on the communication graph, the sampling period, and the control gain to ensure stability and give the quantitative bound of the tracking errors. It is shown that the ultimate bound of the tracking errors is proportional to the sampling period. The benefit of the proposed PD-like discrete-time consensus algorithm is also demonstrated through comparison with an existing P-like discrete-time consensus algorithm. Simulation results are presented as a proof of concept.

1. Introduction

Distributed multi-vehicle cooperative control, including consensus (Jadbabaie, Lin, & Morse, 2003; Olfati-Saber & Murray, 2004; Ren & Beard, 2005; Xiao & Boyd, 2004), rendezvous (Dimarogonas & Kyriakopoulos, 2007; Lin, Morse, & Anderson, 2003), and formation control (Fax & Murray, 2004; Lafferriere, Williams, Caughman, & Veerman, 2005), has become an active research direction in the control community due to its potential applications in both civilian and military sectors. By having a group of locally communicating vehicles working cooperatively, many benefits can be achieved such as high adaptation, simple design and maintenance, and low cost and complexity.

As an important approach in distributed multi-vehicle cooperative control, consensus has been studied extensively, see Ren, Beard, and Atkins (2007) and references therein. The basic idea of consensus is the agreement of all vehicles on some common features by negotiating with their local (time-varying) neighbors. Examples of the features include positions, phases, velocities, and attitudes. Inspired by Jadbabaie et al. (2003) and Vicsek, Czirok, Jacob, Cohen, and Schochet (1995) shows that consensus can be achieved if the undirected communication graph is jointly connected. Fang and Antsaklis (2005), Olfati-Saber and Murray (2004) and Ren and Beard (2005) take into account the fact that the communication graph may be unidirectional/directed. In particular, Olfati-Saber and Murray (2004) shows that average consensus is achieved if the communication graph is strongly connected and balanced at each time, while Ren and Beard (2005) shows that consensus can be achieved if the communication graph has a directed spanning tree jointly. In all these references, the consensus algorithms studied are *proportional like* (P-like) control strategies that employ only the states from local neighbors. It is shown in Ren (2007) that these P-like control strategies cannot be used to track a time-varying reference state that is available to only a subset of the team members. Instead, *proportional and derivative like* (PD-like) continuous-time consensus algorithms are proposed in Ren (2007) to track a time-varying reference state that is available to only a subset of the team members. These PD-like continuous-time consensus algorithms employ both the local neighbors' states and their derivatives. However, the requirement for instantaneous measurements of the derivatives of the local neighbors' states may not be realistic in applications. It will be more meaningful to derive and study the PD-like consensus algorithms in a discrete-time formulation where the requirement for instantaneous measurements of state derivatives is removed.

In this paper, we study a distributed discrete-time coordinated tracking problem where a team of vehicles communicating with their local neighbors at discrete-time instants tracks a time-varying reference state available to only a subset of the

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team members by expanding on our preliminary work reported in Cao, Ren, and Li (2008). We address the problem under a fixed communication graph by proposing a PD-like discrete-time consensus algorithm. When the sampling period is small, it is shown that the tracking errors are ultimately bounded if the changing rate of the reference state is bounded, the virtual leader whose state is the reference state has a directed path to all team members, and the sampling period and control gain satisfy certain conditions. In particular, it is shown that the ultimate bound of the tracking errors is proportional to the sampling period.

In contrast to the PD-like continuous-time consensus algorithms in Ren (2007), all factors, namely, the communication graph, the sampling period, and the control gain play an important role in the stability of the PD-like discrete-time consensus algorithm. To demonstrate the benefit of the PD-like discrete-time consensus algorithm, we also compare the algorithm with an existing P-like discrete-time consensus algorithm. It is shown that the tracking errors using the PD-like discrete-time consensus algorithm with a time-varying reference state that is available to only a subset of the team members will go to zero if the sampling period tends to zero. However, under the same condition, the tracking errors using the P-like discrete-time consensus algorithm with a time-varying reference state that is available to only a subset of the team members are not guaranteed to go to zero even if the sampling period tends to zero.

2. Background and preliminaries

2.1. Graph theory notions

For a system with n vehicles, the communication graph among these vehicles is modeled by a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{W})$, where $\mathcal{V} = \{1, 2, \dots, n\}$ and $\mathcal{W} \subseteq \mathcal{V}^2$ represent, respectively, the vehicle set and the edge set. An edge denoted as (i, j) means that the jth vehicle can access the information of the ith vehicle. If $(i, j) \in \mathcal{W}$, the ith vehicle is a neighbor of the jth vehicle. If $(i, j) \in \mathcal{W}$, then vehicle i is the parent node and vehicle j is the child node. A directed path is a sequence of edges in a directed graph in the form of $(i_1, i_2), (i_2, i_3), \dots$, where $i_k \in \mathcal{V}$. A directed graph has a directed spanning tree if there exists at least one vehicle that has a directed path to all other vehicles.

The communication graph for an n-vehicle system can be mathematically represented by two types of matrices: the adjacency matrix $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{n\times n}$ where $a_{ij} > 0$ if $(j, i) \in \mathcal{W}$ and $a_{ij} = 0$ otherwise, and the (nonsymmetric) Laplacian matrix $\mathcal{L} = [\ell_{ij}] \in \mathbb{R}^{n\times n}$ where $\ell_{ii} = \sum_{j=1,j\neq i}^{n} a_{ij}$ and $\ell_{ij} = -a_{ij}$ for $i \neq j$. We assume that $a_{ii} = 0, i = 1, \dots, n$. It is straightforward to verify that zero is an eigenvalue of $\mathcal{L}$ with a corresponding eigenvector $\mathbf{1}_n$, where $\mathbf{1}_n \in \mathbb{R}^n$ is an all-one column vector.

2.2. PD-like continuous-time consensus algorithm with a time-varying reference state

Consider vehicles with single-integrator kinematics given by

$$\dot{\xi}_i(t) = u_i(t), \quad i = 1, \dots, n \tag{1}$$

where $\xi_i(t) \in \mathbb{R}$ and $u_i(t) \in \mathbb{R}$ represent, respectively, the state and the control input of the ith vehicle. Suppose that there exists a *virtual leader*, labeled as vehicle $n + 1$, whose state is $\xi^r(t)$. A PD-like continuous-time consensus algorithm with a time-varying reference state is proposed in Ren (2007) as

$$\begin{aligned} u_i(t) = {} & \frac{1}{\sum_{j=1}^{n+1} a_{ij}} \sum_{j=1}^{n} a_{ij} \left\{\dot{\xi}_j(t) - \gamma[\xi_i(t) - \xi_j(t)]\right\} \\ & + \frac{a_{i(n+1)}}{\sum_{j=1}^{n+1} a_{ij}} \left\{\dot{\xi}^r(t) - \gamma[\xi_i(t) - \xi^r(t)]\right\}, \end{aligned} \tag{2}$$

where a_{ij} is the (i, j)th entry of the adjacency matrix $\mathcal{A}$, $i, j = 1, 2, \dots, n$, γ is a positive gain, $\xi^r(t) \in \mathbb{R}$ is the time-varying reference state, and $a_{i(n+1)} > 0$ if the ith vehicle can access the virtual leader's state[1] and $a_{i(n+1)} = 0$ otherwise. The objective of (2) is to guarantee that $\xi_i(t) \to \xi^r(t)$, $i = 1, \dots, n$, as $t \to \infty$.

2.3. PD-like discrete-time consensus algorithm with a time-varying reference state

Note that (2) requires each vehicle to obtain instantaneous measurements of the derivatives of its local neighbors' states and the derivative of the reference state if the virtual leader is a neighbor of the vehicle. This requirement may not be realistic in real applications. We next propose a PD-like discrete-time consensus algorithm with a time-varying reference state. In discrete-time formulation, the single-integrator kinematics (1) can be approximated by

$$\frac{\xi_i[k+1] - \xi_i[k]}{T} = u_i[k], \tag{3}$$

where k is the discrete-time index, T is the sampling period, and $\xi_i[k]$ and $u_i[k]$ represent, respectively, the state and the control input of the ith vehicle at $t = kT$. We sample (2) to obtain

$$\begin{aligned} u_i[k] = {} & \frac{1}{\sum_{j=1}^{n+1} a_{ij}} \sum_{j=1}^{n} a_{ij} \left(\frac{\xi_j[k] - \xi_j[k-1]}{T} - \gamma\{\xi_i[k] - \xi_j[k]\}\right) \\ & + \frac{a_{i(n+1)}}{\sum_{j=1}^{n+1} a_{ij}} \left(\frac{\xi^r[k] - \xi^r[k-1]}{T} - \gamma\{\xi_i[k] - \xi^r[k]\}\right), \end{aligned} \tag{4}$$

where $\xi^r[k]$ represents the reference state at $t = kT$, and $\frac{\xi_j[k]-\xi_j[k-1]}{T}$ and $\frac{\xi^r[k]-\xi^r[k-1]}{T}$ are used to approximate, respectively, $\dot{\xi}_j(t)$ and $\dot{\xi}^r(t)$ in (2) because $\xi_j[k + 1]$ and $\xi^r[k + 1]$ cannot be accessed at $t = kT$. Using (4) for (3), we get the PD-like discrete-time consensus algorithm with a time-varying reference state as

$$\begin{aligned} \xi_i[k+1] = {} & \xi_i[k] \\ & + \frac{T}{\sum_{j=1}^{n+1} a_{ij}} \sum_{j=1}^{n} a_{ij} \left(\frac{\xi_j[k] - \xi_j[k-1]}{T} - \gamma\{\xi_i[k] - \xi_j[k]\}\right) \\ & + \frac{T a_{i(n+1)}}{\sum_{j=1}^{n+1} a_{ij}} \left(\frac{\xi^r[k] - \xi^r[k-1]}{T} - \gamma\{\xi_i[k] - \xi^r[k]\}\right). \end{aligned} \tag{5}$$

Note that using algorithm (5), each vehicle essentially updates its next state based on its current state and its neighbors' current and previous states as well as the virtual leader's current and previous states if the virtual leader is a neighbor of the vehicle. As a result, (5) can be easily implemented in practice.

3. Convergence analysis of the PD-like discrete-time consensus algorithm with a time-varying reference state

In this section, we analyze algorithm (5). Before moving on, we let I_n denote the $n \times n$ identity matrix and $\mathbf{0}_{n\times n} \in \mathbb{R}^{n\times n}$ be the all-zero matrix. Also let $\mathrm{diag}\{c_1, \dots, c_n\}$ denote a diagonal matrix with diagonal entries c_i. A matrix is nonnegative if all of its entries are nonnegative.

[1] That is, the virtual leader is a neighbor of vehicle i.

Define the tracking error for vehicle i as $\delta_i[k] \triangleq \xi_i[k] - \xi^r[k]$. It follows that (5) can be written as

$$\delta_i[k+1] = \delta_i[k] + \frac{T}{\sum_{j=1}^{n+1} a_{ij}} \sum_{j=1}^{n} a_{ij}\left(\frac{\delta_j[k]-\delta_j[k-1]}{T} - \gamma\{\delta_i[k]-\delta_j[k]\}\right) + \frac{Ta_{i(n+1)}}{\sum_{j=1}^{n+1} a_{ij}}\left\{\frac{\xi^r[k]-\xi^r[k-1]}{T} - \gamma\delta_i[k]\right\} - \{\xi^r[k+1]-\xi^r[k]\} + \frac{1}{\sum_{j=1}^{n+1} a_{ij}}\sum_{j=1}^{n} a_{ij}\{\xi^r[k]-\xi^r[k-1]\},$$

which can then be written in matrix form as

$$\begin{aligned}\Delta[k+1] &= [(1-T\gamma)I_n + (1+T\gamma)D^{-1}\mathcal{A}]\Delta[k] \\ &\quad - D^{-1}\mathcal{A}\Delta[k-1] + X^r[k], \end{aligned} \tag{6}$$

where $D = \operatorname{diag}\{\sum_{j=1}^{n+1} a_{1j}, \ldots, \sum_{j=1}^{n+1} a_{nj}\}$, $\Delta[k] = [\delta_1[k], \ldots, \delta_n[k]]^{\mathrm{T}}$, $\mathcal{A}$ is the adjacency matrix, and $X^r[k] = \{2\xi^r[k] - \xi^r[k-1] - \xi^r[k+1]\}\mathbf{1}_n$. By defining $Y[k+1] \triangleq \begin{bmatrix}\Delta[k+1] \\ \Delta[k]\end{bmatrix}$, it follows from (6) that

$$Y[k+1] = \tilde{A}Y[k] + \tilde{B}X^r[k], \tag{7}$$

where

$$\tilde{A} = \begin{bmatrix}(1-T\gamma)I_n + (1+T\gamma)D^{-1}\mathcal{A} & -D^{-1}\mathcal{A} \\ I_n & \mathbf{0}_{n\times n}\end{bmatrix} \tag{8}$$

and $\tilde{B} = \begin{bmatrix} I_n \\ \mathbf{0}_{n\times n}\end{bmatrix}$. It follows that the solution of (7) is

$$Y[k] = \tilde{A}^k Y[0] + \sum_{i=1}^{k} \tilde{A}^{k-i}\tilde{B}X^r[i-1]. \tag{9}$$

Note that the eigenvalues of $\tilde{A}$ play an important role in determining the value of $Y[k]$ as $k \to \infty$. In the following, we will study the eigenvalues of $\tilde{A}$. Before moving on, we first study the eigenvalues of $D^{-1}\mathcal{A}$.

Lemma 3.1. *Suppose that the virtual leader has a directed path to all vehicles 1 to n. Then $D^{-1}\mathcal{A}$ satisfies $\left\|(D^{-1}\mathcal{A})^n\right\|_\infty < 1$, where D is defined right after (6) and $\mathcal{A}$ is the adjacency matrix. If $\left\|(D^{-1}\mathcal{A})^n\right\|_\infty < 1$, $D^{-1}\mathcal{A}$ has all eigenvalues within the unit circle.*

Proof. For the first statement, denote $\bar{i}_1$ as the set of vehicles that are the children of the virtual leader, and $\bar{i}_j, j = 2, 3, \ldots, m$, as the set of vehicles that are the children of $\bar{i}_{j-1}$ that are not in the set $\bar{i}_r, r = 1, \ldots, j-1$. Because the virtual leader has a directed path to all vehicles 1 to n, there are at most n edges from the virtual leader to all vehicles 1 to n, which implies $m \le n$. Let p_i and q_i^{T} denote, respectively, the ith column and row of $D^{-1}\mathcal{A}$. When the virtual leader has a directed path to all vehicles 1 to n, without loss of generality, assume that the kth vehicle is a child of the virtual leader, i.e., $a_{k(n+1)} > 0$. It follows that $q_k\mathbf{1}_n = 1 - \frac{a_{k(n+1)}}{\sum_{j=1}^{n+1} a_{kj}} < 1$. The same property also applies to other elements in set $\bar{i}_1$. Similarly, assume that the lth vehicle (one node in set $\bar{i}_2$) is a child of the kth vehicle (one node in set $\bar{i}_1$), which implies $a_{lk} > 0$. It follows that the sum of the lth row of $(D^{-1}\mathcal{A})^2$ can be written as $q_l^{\mathrm{T}}\sum_{i=1}^{n} p_i \le q_l^{\mathrm{T}}\mathbf{1}_n = 1 - \frac{a_{lk}}{\sum_{j=1}^{n+1} a_{lj}} < 1$. Meanwhile, the sum of the kth row of $(D^{-1}\mathcal{A})^2$ is also less than 1. By following a similar analysis, every row of $(D^{-1}\mathcal{A})^m$ has a sum less than one when the virtual leader has a directed path to all vehicles 1 to n. Because $m \le n$ and $D^{-1}\mathcal{A}$ is nonnegative, $\left\|(D^{-1}\mathcal{A})^n\right\|_\infty < 1$ holds.

For the second statement, when $\left\|(D^{-1}\mathcal{A})^n\right\|_\infty < 1$, $\lim_{s\to\infty}\left\|[(D^{-1}\mathcal{A})^n]^s\right\|_\infty \le \lim_{s\to\infty}\left\|(D^{-1}\mathcal{A})^n\right\|_\infty^s = 0$, which implies that $\lim_{s\to\infty}\left\|[(D^{-1}\mathcal{A})^n]^s\right\|_\infty = 0$. Assume that some eigenvalues of $D^{-1}\mathcal{A}$ are not within the unit circle. By writing $D^{-1}\mathcal{A}$ in a Jordan canonical form, it can be computed that $\lim_{s\to\infty}[(D^{-1}\mathcal{A})^n]^s \ne \mathbf{0}_{n\times n}$, which results in a contradiction. Therefore, $D^{-1}\mathcal{A}$ has all eigenvalues within the unit circle. ∎

It can be noted from Lemma 3.1 that all eigenvalues of $D^{-1}\mathcal{A}$ are within the unit circle if the virtual leader has a directed path to all vehicles 1 to n. We next study the conditions under which all eigenvalues of $\tilde{A}$ are within the unit circle. Before moving on, we need the following Schur's formula.

Lemma 3.2 (*Schur's Formula*). *Let $A_{11}, A_{12}, A_{21}, A_{22} \in \mathbb{R}^{n\times n}$ and $M = \begin{bmatrix}A_{11} & A_{12} \\ A_{21} & A_{22}\end{bmatrix}$. Then $\det(M) = \det(A_{11}A_{22} - A_{12}A_{21})$, where $\det(\cdot)$ denotes the determinant of a matrix, if A_{11}, A_{12}, A_{21}, and A_{22} commute pairwise.*

Lemma 3.3. *Assume that the virtual leader has a directed path to all vehicles 1 to n. Let λ_i be the ith eigenvalue of $D^{-1}\mathcal{A}$, where D is defined right after (6) and $\mathcal{A}$ is the adjacency matrix. Then $\tau_i > 0$ holds, where $\tau_i \triangleq \frac{2|1-\lambda_i|^2\{2[1-\mathrm{Re}(\lambda_i)]-|1-\lambda_i|^2\}}{|1-\lambda_i|^4 + 4[\mathrm{Im}(\lambda_i)]^2}$, and $\mathrm{Re}(\cdot)$ and $\mathrm{Im}(\cdot)$ denote, respectively, the real and imaginary parts of a number. If positive scalars T and γ satisfy*

$$T\gamma < \min\{1, \min_{i=1,\ldots,n} \tau_i\}, \tag{10}$$

then $\tilde{A}$, defined in (8), has all eigenvalues within the unit circle.

Proof. For the first statement, when the virtual leader has a directed path to all vehicles 1 to n, it follows from the second statement in Lemma 3.1 that $|\lambda_i| < 1$. It then follows that $|1-\lambda_i|^2 > 0$ and $|1-\lambda_i|^2 = 1 - 2\,\mathrm{Re}(\lambda_i) + [\mathrm{Re}(\lambda_i)]^2 + [\mathrm{Im}(\lambda_i)]^2 < 2[1-\mathrm{Re}(\lambda_i)]$, which implies $\tau_i > 0$.

For the second statement, note that the characteristic polynomial of $\tilde{A}$ is given by

$$\begin{aligned}\det(sI_{2n} - \tilde{A}) &= \det\left(\begin{bmatrix} sI_n - [(1-T\gamma)I_n + (1+T\gamma)D^{-1}\mathcal{A}] & D^{-1}\mathcal{A} \\ -I_n & sI_n\end{bmatrix}\right) \\ &= \det\left([sI_n - (1-T\gamma)I_n - (1+T\gamma)D^{-1}\mathcal{A}]sI_n + D^{-1}\mathcal{A}\right) \\ &= \det\left([s^2 + (T\gamma - 1)s]I_n + [1-(1+T\gamma)s]D^{-1}\mathcal{A}\right),\end{aligned}$$

where we have used Lemma 3.2 to obtain the second to the last equality because $sI_n - [(1-T\gamma)I_n + (1+T\gamma)D^{-1}\mathcal{A}]$, $D^{-1}\mathcal{A}$, $-I_n$ and sI_n commute pairwise. Noting that λ_i is the ith eigenvalue of $D^{-1}\mathcal{A}$, we can get $\det(sI_n + D^{-1}\mathcal{A}) = \prod_{i=1}^{n}(s+\lambda_i)$. It thus follows that $\det(sI_{2n} - \tilde{A}) = \prod_{i=1}^{n}\{s^2 + (T\gamma-1)s + [1-(1+T\gamma)s]\lambda_i\}$. Therefore, the roots of $\det(sI_{2n} - \tilde{A}) = 0$ satisfy

$$s^2 + [T\gamma - 1 - (1+T\gamma)\lambda_i]s + \lambda_i = 0. \tag{11}$$

It can be noted that each eigenvalue of $D^{-1}\mathcal{A}$, λ_i, corresponds to two eigenvalues of $\tilde{A}$. Instead of computing the roots of (11) directly, we apply the bilinear transformation $s = \frac{z+1}{z-1}$ to (11) to get

$$T\gamma(1-\lambda_i)z^2 + 2(1-\lambda_i)z + (2+T\gamma)\lambda_i + 2 - T\gamma = 0. \tag{12}$$

Because the bilinear transformation maps the left half of the complex s-plane to the interior of the unit circle in the z-plane, it follows that (11) has all roots within the unit circle if and only if (12) has all roots in the open left half plane (LHP).

In the following, we will study the condition on T and γ under which (12) has all roots in the open LHP. Letting z_1 and z_2 denote the roots of (12), it follows from (12) that

$$z_1 + z_2 = -\frac{2}{T\gamma} \tag{13}$$

$$z_1 z_2 = \frac{(2+T\gamma)\lambda_i + 2 - T\gamma}{T\gamma(1-\lambda_i)}. \tag{14}$$

Noting that (13) implies that $\mathrm{Im}(z_1) + \mathrm{Im}(z_2) = 0$, we define $z_1 = a_1 + \mathbf{j}b$ and $z_2 = a_2 - \mathbf{j}b$, where $\mathbf{j}$ is the imaginary unit. It can be noted that z_1 and z_2 have negative real parts if and only if $a_1 a_2 > 0$ and $a_1 + a_2 < 0$. Note that (13) implies $a_1 + a_2 < 0$ because $T\gamma > 0$. We next show the sufficient condition on T and γ such that $a_1 a_2 > 0$ holds. By substituting the definitions of z_1 and z_2 into (14), we have $a_1a_2 + b^2 + \mathbf{j}(a_2 - a_1)b = \frac{(2+T\gamma)\lambda_i+2-T\gamma}{T\gamma(1-\lambda_i)}$, which implies

$$a_1 a_2 + b^2 = -\frac{2+T\gamma}{T\gamma} + \frac{4[1-\mathrm{Re}(\lambda_i)]}{T\gamma|1-\lambda_i|^2} \tag{15}$$

$$(a_2 - a_1)b = \frac{4\,\mathrm{Im}(\lambda_i)}{T\gamma|1-\lambda_i|^2}. \tag{16}$$

It follows from (16) that $b = \frac{4\,\mathrm{Im}(\lambda_i)}{T\gamma(a_2-a_1)|1-\lambda_i|^2}$. Considering also the fact that $(a_2 - a_1)^2 = (a_1 + a_2)^2 - 4a_1a_2 = \frac{4}{T^2\gamma^2} - 4a_1a_2$. After some manipulation, (15) can be written as

$$K_1(a_1a_2)^2 + K_2 a_1 a_2 + K_3 = 0, \tag{17}$$

where $K_1 = T^2\gamma^2|1-\lambda_i|^4$, $K_2 = -|1-\lambda_i|^4 + (2+T\gamma)T\gamma|1-\lambda_i|^4 - 4[1-\mathrm{Re}(\lambda_i)]T\gamma|1-\lambda_i|^2$ and $K_3 = \frac{1}{T\gamma}\{4[1-\mathrm{Re}(\lambda_i)]|1-\lambda_i|^2 - (2+T\gamma)|1-\lambda_i|^4\} - 4[\mathrm{Im}(\lambda_i)]^2$. It can be computed that $K_2^2 - 4K_1K_3 = \{|1-\lambda_i|^4 + (2+T\gamma)T\gamma|1-\lambda_i|^4 - 4[1-\mathrm{Re}(\lambda_i)]T\gamma|1-\lambda_i|^2\}^2 + 16T^2\gamma^2|1-\lambda_i|^4[\mathrm{Im}(\lambda_i)]^2 \geq 0$, which implies that (17) has two real roots. Because $|\lambda_i| < 1$, it is straightforward to show that $K_1 > 0$. Therefore, a sufficient condition for $a_1a_2 > 0$ is that $K_2 < 0$ and $K_3 > 0$. When $0 < T\gamma \leq 1$, because $|1-\lambda_i|^2 < 2[1-\mathrm{Re}(\lambda_i)]$ as shown in the proof of the first statement, it follows that $K_2 < -|1-\lambda_i|^4 + (2+T\gamma)T\gamma|1-\lambda_i|^4 - 2|1-\lambda_i|^2 T\gamma|1-\lambda_i|^2 = |1-\lambda_i|^4[-1+(T\gamma)^2] \leq 0$. Similarly, when $0 < T\gamma < \tau_i$, it follows that $K_3 > 0$. Therefore, if positive scalars γ and T satisfy (10), all eigenvalues of $\tilde{A}$ are within the unit circle. ■

In the following, we apply Lemma 3.3 to derive our main result.

Theorem 3.1. *Assume that the reference state $\xi^r[k]$ satisfies $|\frac{\xi^r[k]-\xi^r[k-1]}{T}| \leq \bar{\xi}$ (i.e., the changing rate of $\xi^r[k]$ is bounded), and the virtual leader has a directed path to all vehicles 1 to n. When positive scalars γ and T satisfy (10), using algorithm (5), the maximum tracking error among the n vehicles is ultimately bounded by $2T\bar{\xi}\left\|(I_{2n}-\tilde{A})^{-1}\right\|_\infty$, where $\tilde{A}$ is defined in (8).*

Proof. It follows from (9) that

$$\begin{aligned}\|Y[k]\|_\infty &\leq \left\|\tilde{A}^k Y[0]\right\|_\infty + \left\|\sum_{i=1}^{k}\tilde{A}^{k-i}\tilde{B}X^r[i-1]\right\|_\infty \\ &\leq \left\|\tilde{A}^k\right\|_\infty \|Y[0]\|_\infty + 2T\bar{\xi}\left\|\sum_{i=0}^{k-1}\tilde{A}^i\right\|_\infty \left\|\tilde{B}\right\|_\infty,\end{aligned}$$

where we have used the fact that

$$\left\|X^r[i]\right\|_\infty = \left\|\{2\xi^r[i] - \xi^r[i-1] - \xi^r[i+1]\}\mathbf{1}_n\right\|_\infty \leq 2T\bar{\xi}$$

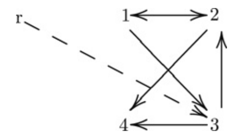


Fig. 1. Directed graph for four vehicles. A solid arrow from *j* to *i* denotes that vehicle *i* can receive information from vehicle *j*. A dashed arrow from r to *l* denotes that vehicle *l* can receive information from the virtual leader.

for all i because $|\frac{\xi^r[k]-\xi^r[k-1]}{T}| \leq \bar{\xi}$. When the virtual leader has a directed path to all vehicles 1 to n, it follows from Lemma 3.3 that $\tilde{A}$ has all eigenvalues within the unit circle if positive scalars T and γ satisfy (10). Therefore, $\lim_{k\to\infty}\tilde{A}^k = \mathbf{0}_{2n\times 2n}$. It thus follows that $\lim_{k\to\infty}\|Y[k]\|_\infty \leq \lim_{k\to\infty} 2T\bar{\xi}\left\|\sum_{i=0}^{k-1}\tilde{A}^i\right\|_\infty \left\|\tilde{B}\right\|_\infty$. Because all eigenvalues of $\tilde{A}$ are within the unit circle, it follows from Lemma 5.6.10 in Horn and Johnson (1985) that there exists a matrix norm $\|\!|\cdot|\!\|$ such that $\|\!|\tilde{A}|\!\| < 1$. It then follows from Theorem 4.3 in Moon and Stirling (2000) that $\lim_{k\to\infty}\left\|\sum_{i=0}^{k-1}\tilde{A}^i\right\|_\infty \leq \left\|(I_{2n}-\tilde{A})^{-1}\right\|_\infty$. Also note that $\left\|\tilde{B}\right\|_\infty = 1$. The theorem follows directly by noting that $\|Y[k]\|_\infty$ denotes the maximum tracking error among the n vehicles. ■

Remark 3.2. From Theorem 3.1, it can be noted that the ultimate bound of the tracking errors using PD-like discrete-time consensus algorithm (5) with a time-varying reference state is proportional to the sampling period T. As $T \to 0$, the tracking errors will go to zero ultimately when the changing rate of the reference state is bounded and the virtual leader has a directed path to all vehicles 1 to n.

4. Comparison between P-like and PD-like discrete-time consensus algorithms with a time-varying reference state

A P-like continuous-time consensus algorithm without a reference state is studied for (1) in Jadbabaie et al. (2003), Olfati-Saber and Murray (2004) and Ren and Beard (2005) as $u_i(t) = -\sum_{j=1}^n a_{ij}[\xi_i(t) - \xi_j(t)]$. When there exists a virtual leader whose state is the reference state $\xi^r(t)$, a P-like continuous-time consensus algorithm is given as

$$u_i(t) = -\sum_{j=1}^{n} a_{ij}[\xi_i(t) - \xi_j(t)] - a_{i(n+1)}[\xi_i(t) - \xi^r(t)], \tag{18}$$

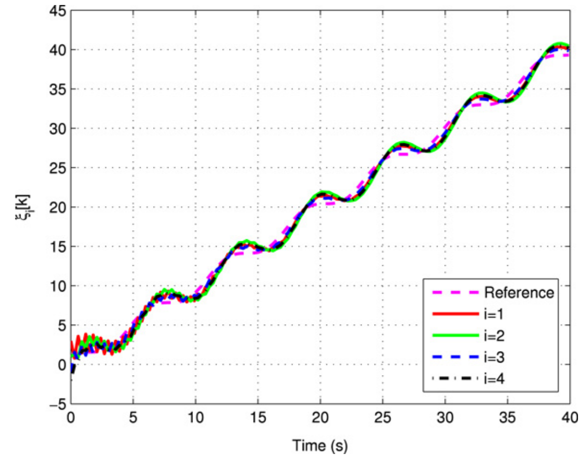
where a_{ij} and $a_{i(n+1)}$ are defined as in (2). By sampling (18) and using the sampled algorithm for (3), we get the P-like discrete-time consensus algorithm with a time-varying reference state as

$$\begin{aligned}\xi_i[k+1] = \xi_i[k] &- T\sum_{j=1}^{n} a_{ij}(\xi_i[k] - \xi_j[k]) \\ &- Ta_{i(n+1)}(\xi_i[k] - \xi^r[k]).\end{aligned} \tag{19}$$

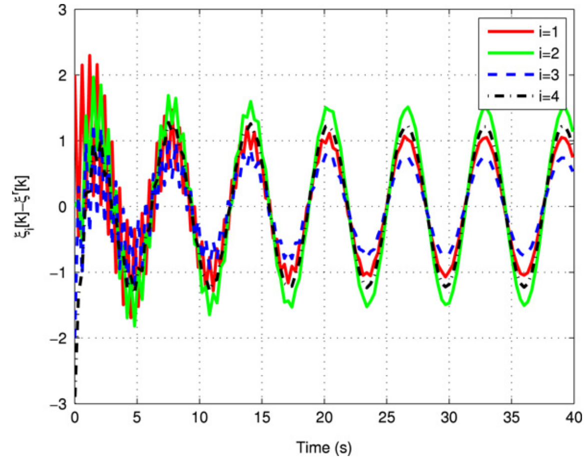
Letting δ_i be defined as in Section 3, we rewrite (19) as $\delta_i[k+1] = \delta_i[k] - T\sum_{j=1}^n a_{ij}(\delta_i[k] - \delta_j[k]) - Ta_{i(n+1)}\delta_i[k] - (\xi^r[k+1] - \xi^r[k])$, which can then be written in matrix form as

$$\Delta[k+1] = Q\Delta[k] - (\xi^r[k+1] - \xi^r[k])\mathbf{1}_n, \tag{20}$$

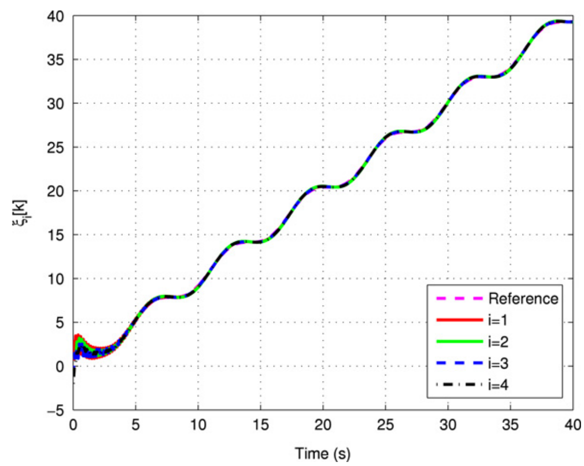
where $\Delta[k] = [\delta_1[k], \ldots, \delta_n[k]]^{\mathrm{T}}$ and $Q = I_n - T\mathcal{L} - T\mathrm{diag}\{a_{1(n+1)}, \ldots, a_{n(n+1)}\}$ with $\mathcal{L}$ being the (nonsymmetric) Laplacian matrix. It follows that Q is nonnegative when $T < \min_{i=1,\ldots,n}\frac{1}{\sum_{j=1}^{n+1} a_{ij}}$.



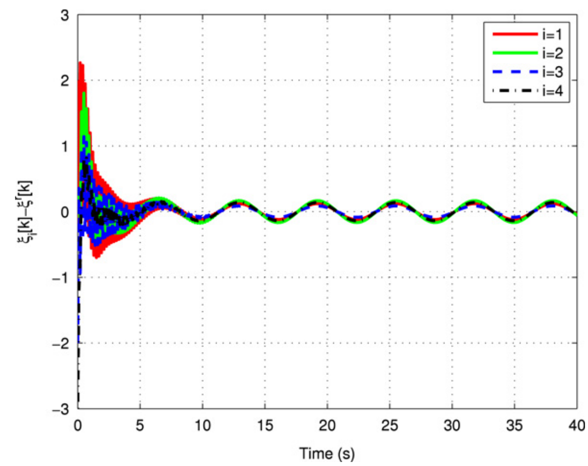
(a) States ($T = 0.3$ s and $\gamma = 1$).



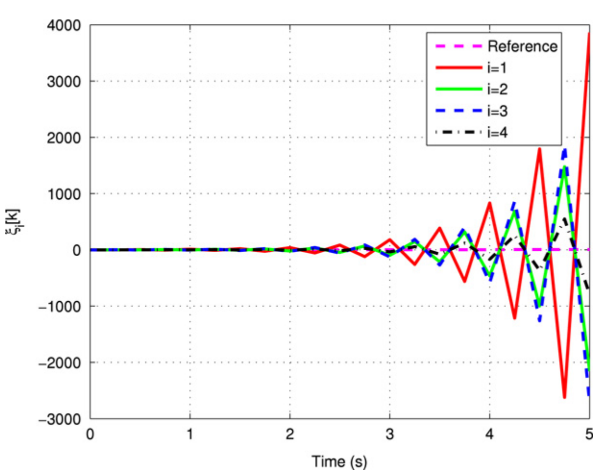
(b) Tracking errors ($T = 0.3$ s and $\gamma = 1$).



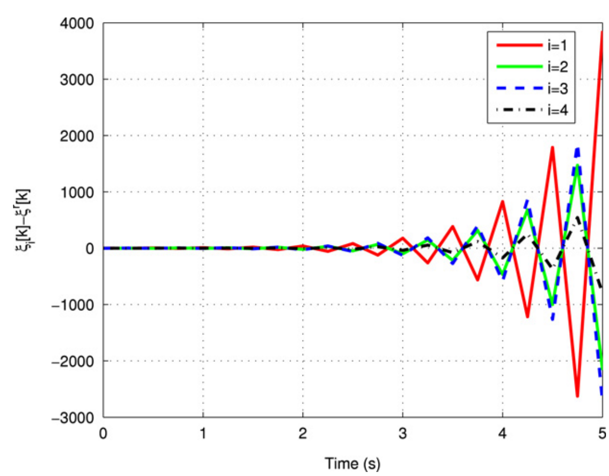
(c) States ($T = 0.1$ s and $\gamma = 3$).



(d) Tracking errors ($T = 0.1$ s and $\gamma = 3$).



(e) States ($T = 0.25$ s and $\gamma = 3$).



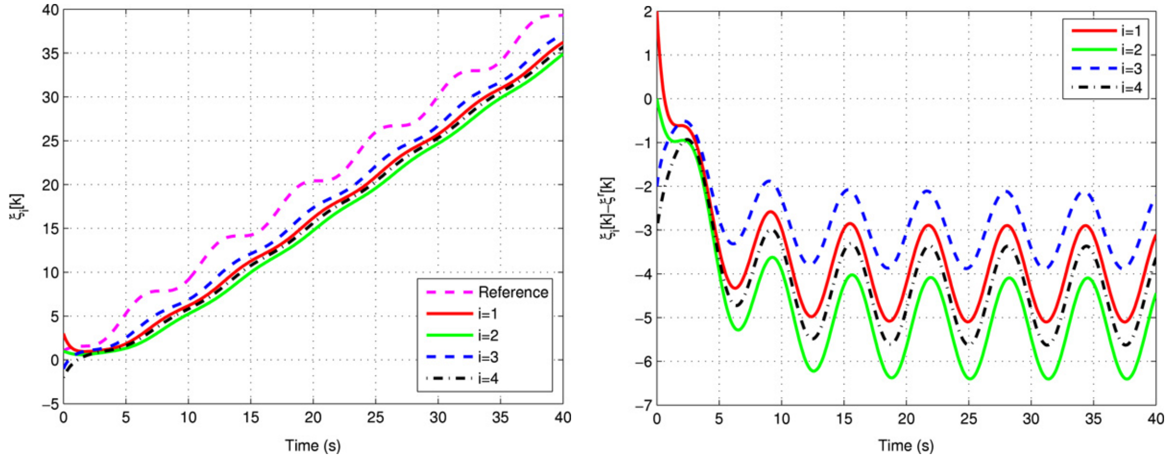
(f) Tracking errors ($T = 0.25$ s and $\gamma = 3$).

Fig. 2. Distributed discrete-time coordinated tracking using PD-like discrete-time consensus algorithm (5) under different T and γ.

Lemma 4.1. *Assume that the virtual leader has a directed path to all vehicles* 1 *to n. When* $T < \min_{i=1,\ldots,n} \frac{1}{\sum_{j=1}^{n+1} a_{ij}}$, *Q satisfies* $\|Q^n\|_\infty < 1$, *where Q is defined right after* (20). *Furthermore, if* $\|Q^n\|_\infty < 1$, *Q has all eigenvalues within the unit circle.*

Proof. The proof is similar to that of Lemma 3.1 and is omitted here. ■

Theorem 4.1. *Assume that the reference state* $\xi^r[k]$ *satisfies* $\left|\frac{\xi^r[k]-\xi^r[k-1]}{T}\right| \leq \bar{\xi}$, *and the virtual leader has a directed path to all vehicles* 1 *to n. When* $T < \min_{i=1,\ldots,n} \frac{1}{\sum_{j=1}^{n+1} a_{ij}}$, *using algorithm* (19), *the maximum tracking error among the n vehicles is ultimately bounded by* $\bar{\xi} \left\|(\mathcal{L} + \mathrm{diag}\{a_{1(n+1)}, \ldots, a_{n(n+1)}\})^{-1}\right\|_\infty$, *where Q is defined after* (20).



(a) States ($T = 0.1$ s and $\gamma = 3$). (b) Tracking errors ($T = 0.1$ s and $\gamma = 3$).

Fig. 3. Distributed discrete-time coordinated tracking using P-like discrete-time consensus algorithm (19).

Proof. The solution of (20) is

$$\Delta[k] = Q^k \Delta[0] - \sum_{i=1}^{k} Q^{k-i}(\xi^r[k] - \xi^r[k-1])\mathbf{1}_n. \qquad (21)$$

The proof then follows a similar line to that of Theorem 3.1 by noting that $\|\Delta[k]\|_\infty$ denotes the maximum tracking error among the n vehicles. ■

Remark 4.2. In contrast to the results in Theorem 3.1, the ultimate bound of the tracking errors using P-like discrete-time consensus algorithm (19) with a time-varying reference state is not proportional to the sampling period T. In fact, as shown in Ren (2007), even when $T \to 0$, the tracking errors using (19) are not guaranteed to go to zero ultimately. As a special case, when the reference state is constant (i.e., $\dot{\xi} = 0$), it follows from Theorems 3.1 and 4.1 that the tracking error will go to zero ultimately for both the P-like and PD-like discrete-time consensus algorithms.

The comparison between Theorems 3.1 and 4.1 shows the benefit of the PD-like discrete-time consensus algorithm over the P-like discrete-time consensus algorithm when there exists a time-varying reference state that is available to only a subset of the team members.

5. Simulations

In this section, a simulation example is presented to illustrate the PD-like discrete-time consensus algorithm proposed in Section 2. To show the benefit of the PD-like discrete-time consensus algorithm, the related simulation result obtained by applying the P-like discrete-time consensus algorithm is also presented.

We consider a team of four vehicles with a directed communication graph given by Fig. 1 and let the third vehicle have access to the time-varying reference state. It can be noted that the virtual leader has a directed path to all four vehicles. We let the nonzero a_{ij} (respectively, $a_{i(n+1)}$) to be one if j (respectively, the virtual leader) is a neighbor of vehicle i.

For both the PD-like and P-like discrete-time consensus algorithms with a time-varying reference state, we let the initial states of the four vehicles be $[\xi_1[0], \xi_2[0], \xi_3[0], \xi_4[0]] = [3, 1, -1, -2]$. For the PD-like discrete-time consensus algorithm, we also let $[\xi_1[-1], \xi_2[-1], \xi_3[-1], \xi_4[-1]] = [0, 0, 0, 0]$. The time-varying reference state is chosen as $\xi^r[k] = \sin(kT) + kT$.

Fig. 2(a) and (b) show, respectively, the states $\xi_i[k]$ and tracking errors $\xi_i[k] - \xi^r[k]$ by using PD-like discrete-time consensus algorithm (5) with a time-varying reference state when $T = 0.3$ s and $\gamma = 1$. From Fig. 2(b), it can be seen that the four vehicles track the reference state with relatively large tracking errors. Fig. 2(c) and (d) show, respectively, the states $\xi_i[k]$ and tracking errors $\xi_i[k] - \xi^r[k]$ by using the same algorithm with the same time-varying reference state when $T = 0.1$ s and $\gamma = 3$. From Fig. 2(d), it can be seen that the four vehicles track the reference state with very small tracking errors ultimately. We can see that the tracking errors will become smaller if the sampling period becomes smaller. Fig. 2(e) and (f) show, respectively, the states $\xi_i[k]$ and tracking errors $\xi_i[k] - \xi^r[k]$ obtained by using PD-like discrete-time consensus algorithm (5) with the same time-varying reference state when $T = 0.25$ s and $\gamma = 3$. Note that the product $T\gamma$ is larger than the positive upper bound derived in Theorem 3.1. It can be noted that the tracking errors become unbounded in this case. Fig. 3(a) and (b) show, respectively, the states $\xi_i[k]$ and tracking errors $\xi_i[k] - \xi^r[k]$ obtained by using P-like discrete-time consensus algorithm (19) with the same time-varying reference state when $T = 0.1$ s and $\gamma = 3$. It can be seen from Fig. 3(a) and (b) that the tracking errors using P-like discrete-time consensus algorithm (19) are much larger than those using PD-like discrete-time consensus algorithm (5) under the same condition. This shows the benefit of the PD-like discrete-time consensus algorithm over the P-like discrete-time consensus algorithm when there exists a time-varying reference state that is available to only a subset of the team members.

6. Conclusion and future work

In this paper, we studied the PD-like consensus algorithm for multi-vehicle systems in a discrete formulation when there exists a time-varying reference state that is available to only a subset of the team members. We analyzed the condition on the communication graph, the sampling period, and the control gain to ensure stability and showed the quantitative bound of the tracking errors. We also compared the PD-like discrete-time consensus algorithm with an existing P-like discrete-time consensus algorithm. The comparison shows the benefit of the PD-like discrete-time consensus algorithm over the P-like discrete-time consensus algorithm when there exists a time-varying reference state that is available to only a subset of the team members. Although this paper focuses on studying the PD-like discrete-time consensus algorithm over a directed fixed communication graph, a similar analysis may be extended to account for the case of a directed switching communication graph. This will be one of our future research directions.

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