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Distributed containment control with multiple stationary or dynamic leaders in fixed and switching directed networks☆

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ABSTRACT

In this paper, we study the problem of distributed containment control of a group of mobile autonomous agents with multiple stationary or dynamic leaders under both fixed and switching directed network topologies. First, when the leaders are stationary and all followers share an inertial coordinate frame, we present necessary and sufficient conditions on the fixed or switching directed network topology such that all followers will ultimately converge to the stationary convex hull formed by the stationary leaders for arbitrary initial states in a space of any finite dimension. When the directed network topology is fixed, we partition the (nonsymmetric) Laplacian matrix and explore its properties to derive the convergence results. When the directed network topology is switching, the commonly adopted decoupling technique based on the Kronecker product in a high-dimensional space can no longer be applied and we hence present an important *coordinate transformation technique* to derive the convergence results. The proposed coordinate transformation technique also has potential applications in other high-dimensional distributed control scenarios and might be used to simplify the analysis of a high-dimensional system to that of a one-dimensional system when the decoupling technique based on the Kronecker product cannot be applied. Second, when the leaders are dynamic and all followers share an inertial coordinate frame, we propose a distributed tracking control algorithm without velocity measurements. When the directed network topology is fixed, we derive conditions on the network topology and the control gain to guarantee that all followers will ultimately converge to the dynamic convex hull formed by the dynamic leaders for arbitrary initial states in a space of any finite dimension. When the directed network topology is switching, we derive conditions on the network topology and the control gain such that all followers will ultimately converge to the minimal hyperrectangle that contains the dynamic leaders and each of its hyperplanes is normal to one axis of the inertial coordinate frame in any high-dimensional space. We also show via some counterexamples that it is, in general, impossible to find distribute containment control algorithms without velocity measurements to guarantee that all followers will ultimately converge to the convex hull formed by the dynamic leaders under a switching network topology in a high-dimensional space. Simulation results are presented as a proof of concept.

1. Introduction

The past decade has witnessed the increasing research interest in distributed cooperative control of a group of mobile autonomous agents. Although individual agents can be employed to accomplish various tasks, great benefits including low cost, high adaptivity, and easy maintenance can be achieved by having a group of agents work cooperatively. An important feature in distributed cooperative control of multiple agents is that each agent updates its own state based on the information from itself and its local (time-varying) neighbors. Therefore, it is fundamental to study the *consensus* problem. Consensus means the agreement of a group of agents on a common state. By specifying desired separations among these agents, a geometric formation can be achieved accordingly. The study of consensus algorithms can be dated back to Winkler (1968) and DeGroot (1974). Detailed information about the recent study of consensus algorithms in cooperative control can be found in Murray (2007) and Ren, Beard, and Atkins (2007).

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Existing consensus algorithms often focus on leaderless coordination for a group of agents. However, in real applications, there might exist either one or multiple leaders for these agents. Consensus with a constant leader was addressed in the leader-following case of Jadbabaie, Lin, and Morse (2003) under a switching undirected network topology while in Jin and Murray (2006) and Moore and Lucarelli (2007) under a fixed directed network topology. Ref. (Hong, Hu, & Gao, 2006) solved the problem of consensus with a time-varying leader under a variable undirected network topology by assuming that the leader's acceleration input is available to each follower in the group. PD-like continuous-time and discrete-time consensus algorithms were proposed in, respectively, Ren (2007) and Cao, Ren, and Li (2009) for a group of agents to track a time-varying leader. Note that all the above references considered only one leader. In contrast, multiple leaders were introduced in Ji, Ferrari-Trecate, Egerstedt, and Buffa (2008) to solve the containment control problem, where a team of followers is guided by multiple leaders. In particular, Ji et al. (2008) proposed a stop-and-go strategy to drive a collection of mobile agents to the convex polytope spanned by multiple stationary/moving leaders. Note that the study in Ji et al. (2008) focused on *fixed undirected* interaction. However, the interaction among different agents in physical systems may be directed and/or switching due to heterogeneity, nonuniform communication/sensing powers, unreliable communication/sensing, limited communication/sensing range, and/or sensing with a limited field of view. Therefore, it is important and meaningful to consider *fixed/switching directed* network topologies.

This paper studies the problem of distributed containment control of a group of mobile autonomous agents with multiple stationary or dynamic leaders under fixed and switching directed network topologies by expanding on our preliminary work reported in Cao and Ren (2009). The contribution of this paper is twofold. First, when the leaders are stationary and all agents share an inertial coordinate frame, necessary and sufficient conditions on the fixed or switching directed network topology are given such that all followers will ultimately converge to the stationary convex hull formed by the stationary leaders for arbitrary initial states in a space of any finite dimension. In particular, with a fixed directed network topology, the final states of the followers are constant. With a switching directed network topology, the final states of the followers might be changing depending on the switching graphs. An important *coordinate transformation technique* is presented to deal with the challenge that the commonly adopted decoupling technique based on Kronecker product in a high-dimensional space can no longer be used when the directed network topology is switching. We should emphasize that the proposed coordinate transformation technique has potential applications in other high-dimensional distributed control scenarios and might be used to simplify the analysis of a high-dimensional system to that of a one-dimensional system when the decoupling technique based on Kronecker product cannot be applied. Second, when the leaders are dynamic and all followers share an inertial coordinate frame, we propose a distributed tracking control algorithm without velocity measurements. When the directed network topology is fixed, we derive conditions on the network topology and the control gain to guarantee that all followers will ultimately converge to the dynamic convex hull formed by the dynamic leaders for arbitrary initial states in a space of any finite dimension. When the directed network topology is switching, we derive conditions on the network topology and the control gain such that all followers will ultimately converge to the minimal hyperrectangle that contains the dynamic leaders and each of the hyperplanes is normal to one axis of the inertial coordinate frame in any high-dimensional space. We also show through some counterexamples that it is, in general, impossible to find distribute containment control algorithms without velocity measurements to guarantee that all followers will ultimately converge to the convex hull formed by the dynamic leaders under a switching network topology in a high-dimensional space.

In contrast to Ji et al. (2008), which used partial difference equations, we use a Lyapunov-based approach. Our results generalize the results in Ji et al. (2008) to the case of fixed/switching directed network topologies. In addition, in the case of dynamic leaders under a directed network topology, we show ultimate containment of the followers in the dynamic convex hull formed by the dynamic leaders without the need to impose stop-and-go motions for the leaders to ensure containment. Note that we adopt a discontinuous distributed control algorithm in the case of multiple dynamic leaders. Although discontinuous control laws were also employed in Gazi (2005) and Ferrara and Vecchio (2007), the problems studied in Gazi (2005) and Ferrara and Vecchio (2007) are different from the problem studied in this paper.

The remainder of this paper is organized as follows: Section 2 provides basic graph theory notions, definitions, and notations used in this paper. Section 3 focuses on the stability analysis of containment control algorithms with multiple stationary leaders. Section 4 focuses on the stability analysis of containment control algorithms with multiple dynamic leaders. In Section 5, several simulation examples are presented to illustrate the theoretical results. Finally, a short conclusion is given in Section 6.

2. Preliminaries

2.1. Graph theory notions

For an n-agent system, the interaction among all agents can be modeled by a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{W})$, where $\mathcal{V} = \{v_1, v_2, \ldots, v_n\}$ and $\mathcal{W} \subseteq \mathcal{V}^2$ represent, respectively, the agent set and the edge set. Each edge denoted as (v_i, v_j) means that agent j can access the state information of agent i. Accordingly, agent i is a neighbor of agent j. We use $\mathcal{N}_j$ to denote the neighbor set of agent j. A directed path is a sequence of edges in a directed graph of the form $(v_1, v_2), (v_2, v_3), \ldots$, where $v_i \in \mathcal{V}$. A directed graph has a directed spanning tree if there exists at least one agent that has a directed path to every other agent. The union of a set of directed graphs $\mathcal{G}_{i_1}, \ldots, \mathcal{G}_{i_m}$ is a directed graph with the edge set given by the union of the edge sets of the directed graphs $\mathcal{G}_{i_j}, j = 1, \ldots, m$, where $\mathcal{G}_{i_j}, j = 1, \ldots, m$, have the same agent set.

Two matrices are frequently used to represent the interaction graph: the adjacency matrix $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{n\times n}$ with $a_{ij} > 0$ if $(v_j, v_i) \in \mathcal{W}$ and $a_{ij} = 0$ otherwise, and the (nonsymmetric) Laplacian matrix $\mathcal{L} = [\ell_{ij}] \in \mathbb{R}^{n\times n}$ with $\ell_{ii} = \sum_{j=1, j\neq i}^{n} a_{ij}$ and $\ell_{ij} = -a_{ij}, i \neq j$. It is straightforward to verify that $\mathcal{L}$ has at least one zero eigenvalue with a corresponding eigenvector $\mathbf{1}_n$, where $\mathbf{1}_n$ is an $n \times 1$ all-one column vector.

2.2. Definitions and notations

Definition 2.1. For the n-agent system, an agent is called a *leader* if the agent has no neighbor. An agent is called a *follower* if the agent has a neighbor. Assume that there are m leaders, where $m < n$, and $n - m$ followers. We use $\mathcal{R}$ and $\mathcal{F}$ to denote, respectively, the leader set and the follower set.

Condition 2.2. *For each follower, there exists at least one leader that has a directed path to the follower.*

Definition 2.3. Let $\mathcal{C}$ be a set in a real vector space $V \subseteq \mathbb{R}^p$. The set $\mathcal{C}$ is called *convex* if, for any x and y in $\mathcal{C}$, the point $(1-z)x + zy$ is in $\mathcal{C}$ for any $z \in [0, 1]$. The convex hull for a set of points $X = \{x_1, \ldots, x_n\}$ in V is the minimal convex set containing all points in X. We use $\mathbf{Co}\{X\}$ to denote the convex hull of X. In particular, $\mathbf{Co}\{X\} = \{\sum_{i=1}^{n} \alpha_i x_i | x_i \in X, \alpha_i \in \mathbb{R} \geq 0, \sum_{i=1}^{n} \alpha_i = 1\}$. When $V \subseteq \mathbb{R}$, $\mathbf{Co}\{X\} = \{x | x \in [\min_i x_i, \max_i x_i]\}$.

Definition 2.4. A matrix $E \in \mathbb{R}^{m\times n}$ is said positive (nonnegative), i.e., $E > (\geq)0$, if each entry of E is positive (nonnegative). A matrix $F \in \mathbb{R}^{m\times n}$ is said negative (nonpositive), i.e., $F < (\leq)0$, if each entry of F is negative (nonpositive). A square nonnegative matrix is (row) stochastic if all of its row sums are 1. We use I_p to denote the $p \times p$ identity matrix, $\mathbf{0}_{p\times q}$ to denote the $p \times q$ all-zero matrix, and $\mathbf{0}_p$ to denote the $p \times 1$ all-zero vector.

Definition 2.5 (*Alefeld and Schneider (1982)*)**.** A real matrix $B = [b_{ij}] \in \mathbb{R}^{n\times n}$ is called an M-matrix if it can be written as $B = sI_n - C$, where $s > 0$ and $C \in \mathbb{R}^{n\times n} \geq 0$ satisfies $\rho(C) \leq s$, where $\rho(C)$ is the spectral radius of the matrix C. The matrix B is called a nonsingular M-matrix if $\rho(C) < s$.

Lemma 2.1 (*Alefeld and Schneider (1982)*)**.** *Define $Z^{n\times n} := \{B = [b_{ij}] \in \mathbb{R}^{n\times n} | b_{ij} \leq 0, i \neq j\}$. $B \in Z^{n\times n}$ is a nonsingular M-matrix if and only if B^{-1} exists and $B^{-1} \geq 0$.*

3. Stability analysis with multiple stationary leaders

In this section, we study the conditions on, respectively, the fixed and switching directed network topology such that all followers will ultimately converge to the stationary convex hull formed by the stationary leaders.

Consider a group of n autonomous agents with single-integrator kinematics given by

$$\dot{x}_i^0(t) = u_i^0(t), \quad i = 1, \ldots, n, \tag{1}$$

where $x_i^0 \in \mathbb{R}^p$ and $u_i^0 \in \mathbb{R}^p$ are, respectively, the position and the control input of the ith agent represented in a common inertial coordinate frame $\mathcal{C}_0$, and $\dot{x}_i^0$ is the velocity of the ith agent relative to $\mathcal{C}_0$ represented in $\mathcal{C}_0$. A common consensus algorithm for (1) was studied in Jadbabaie et al. (2003), Olfati-Saber and Murray (2004), Ren and Beard (2005), Moreau (2004a) and Lin, Broucke, and Francis (2004) as

$$u_i^0(t) = -\sum_{j=1}^{n} a_{ij}(t)[x_i^0(t) - x_j^0(t)], \quad i = 1, \ldots, n, \tag{2}$$

where $a_{ij}(t)$ is the (i,j)th entry of the adjacency matrix $\mathcal{A}(t)$ associated with the directed graph $\mathcal{G}(t)$ at time t. The objective of (2) is to guarantee consensus, i.e., $x_i^0(t) \to x_j^0(t)$ for arbitrary initial conditions $x_i^0(0), i = 1, \ldots, n$. Conditions on the network topology to ensure consensus were studied in Jadbabaie et al. (2003), Olfati-Saber and Murray (2004), Ren and Beard (2005), Moreau (2004a) and Lin et al. (2004), but these references only considered the case when there exists at most one leader in the group.

Suppose that there are m, $m < n$, stationary leaders and $n - m$ followers. Eq. (2) becomes

$$u_i^0(t) = 0, \quad i \in \mathcal{R}$$
$$u_i^0(t) = -\sum_{j\in\mathcal{F}\bigcup\mathcal{R}} a_{ij}(t)[x_i^0(t) - x_j^0(t)], \quad i \in \mathcal{F}, \tag{3}$$

where $\mathcal{R}$ and $\mathcal{F}$ are defined in Definition 2.1. Note that $x_j^0, j \in \mathcal{R}$, is constant because the leaders are stationary.

3.1. Fixed directed interaction

We first study the case when the network topology is fixed (i.e., $a_{ij}(t)$ in (3) is constant). Let $x^0(t)$ denote the column stack vector of all $x_i^0(t)$. Then the closed-loop system using (3) for (1) can be written as

$$\dot{x}^0(t) = -(\mathcal{L} \otimes I_p)x^0(t), \tag{4}$$

where $\otimes$ represents the Kronecker product and $\mathcal{L}$ is the (nonsymmetric) Laplacian matrix. Therefore, it is important to study the property of $\mathcal{L}$ in (4). Without loss of generality, we assume that agents 1 to $n - m$, $m < n$, are followers and agents $n - m + 1$ to n are leaders. Accordingly, $\mathcal{L}$ can be partitioned as

$$\begin{bmatrix} \mathcal{L}_1 & \mathcal{L}_2 \\ \mathbf{0}_{m\times(n-m)} & \mathbf{0}_{m\times m} \end{bmatrix}, \tag{5}$$

where $\mathcal{L}_1 \in \mathbb{R}^{(n-m)\times(n-m)}$ and $\mathcal{L}_2 \in \mathbb{R}^{(n-m)\times m}$. Note that the last m rows of $\mathcal{L}$ are all equal to zero because the last m agents are the leaders, who do not receive information from any other agent.

Lemma 3.1. *$\mathcal{L}_1$ is invertible if and only if Condition 2.2 is satisfied in the directed graph $\mathcal{G}$.*

Proof. Consider the following new Laplacian matrix given by

$$\overline{\mathcal{L}} = \begin{bmatrix} \mathcal{L}_1 & \mathcal{L}_2\mathbf{1}_m \\ \mathbf{0}_{1\times(n-m)} & 0 \end{bmatrix}. \tag{6}$$

According to the definition of the directed spanning tree and Condition 2.2, the graph associated with $\overline{\mathcal{L}}$, denoted as $\overline{\mathcal{G}}$, has a directed spanning tree if and only if Condition 2.2 is satisfied in the graph associated with $\mathcal{L}$ (i.e., $\mathcal{G}$). From Lemma 3.3 in Ren and Beard (2005), $\overline{\mathcal{L}}$ has exactly one zero eigenvalue if and only if $\overline{\mathcal{G}}$ has a directed spanning tree. Note that $\det(\mathcal{L}_1) \neq 0$, where $\det(\cdot)$ denotes the determinant of a matrix, if and only if $\overline{\mathcal{L}}$ has exactly one zero eigenvalue. Combining the previous arguments shows that $\mathcal{L}_1$ is invertible, that is, $\det(\mathcal{L}_1) \neq 0$, if and only if Condition 2.2 is satisfied in $\mathcal{G}$. □

We next state the result in the case of a fixed directed network topology. We use $x_F^0(t)$ and x_L^0 to denote the column stack vector of, respectively, the followers' positions and the leaders' positions. Note that x_L^0 is constant.

Theorem 3.1. *Suppose that the directed network topology is fixed. Using (3) for (1), all followers will converge to the stationary convex hull formed by the stationary leaders for arbitrary initial conditions if and only if Condition 2.2 is satisfied in the directed graph $\mathcal{G}$. In addition, the final positions of the followers are given by $-(\mathcal{L}_1^{-1}\mathcal{L}_2 \otimes I_p)x_L^0$.*

Proof (*Necessity*)**.** When Condition 2.2 is not satisfied in the directed graph $\mathcal{G}$, there exists at least one follower, labeled as k, such that all leaders do not have directed paths to follower k for $t > 0$. It follows that the position of follower k is independent of the positions of the leaders for $t > 0$. Therefore, follower k cannot always converge to the stationary convex hull formed by the stationary leaders for arbitrary initial conditions.

(Sufficiency) Note that (4) is equivalent to the following equation

$$\dot{x}_F^0(t) = -(\mathcal{L}_1 \otimes I_p)x_F^0(t) - (\mathcal{L}_2 \otimes I_p)x_L^0. \tag{7}$$

Taking the Laplace transform to (7) gives that

$$sX_F^0(s) - x_F^0(0) = -(\mathcal{L}_1 \otimes I_p)X_F^0(s) - \frac{1}{s}(\mathcal{L}_2 \otimes I_p)x_L^0, \tag{8}$$

where $X_F^0(s)$ is the Laplacian transform of $x_F^0(t)$. After some simplification, (8) can be written as $X_F^0(s) = (sI_{n-m} \otimes I_p + \mathcal{L}_1 \otimes I_p)^{-1}[x_F^0(0) - \frac{1}{s}(\mathcal{L}_2 \otimes I_p)x_L^0]$. When Condition 2.2 is satisfied in $\mathcal{G}$, it follows from Lemma 3.1 that $\mathcal{L}_1$ is invertible. According to the Gershgorin disc theorem, it follows that the eigenvalues of $\mathcal{L}_1$ are either on the open right half plane or at the origin. Combining with the fact that $\mathcal{L}_1$ is invertible shows that $\mathcal{L}_1$ has all eigenvalues on the open right half plane. It thus follows that $-(\mathcal{L}_1 \otimes I_p)$ is Hurwitz. Based on the final value theorem, we have that

$$\begin{aligned}\lim_{t\to\infty} x_F^0(t) &= \lim_{s\to 0} sX_F^0(s)\\ &= \lim_{s\to 0} s(sI_n + \mathcal{L}_1 \otimes I_p)^{-1}\left[x_F^0(0) - \frac{1}{s}(\mathcal{L}_2 \otimes I_p)x_L^0\right]\\ &= -(\mathcal{L}_1 \otimes I_p)^{-1}(\mathcal{L}_2 \otimes I_p)x_L^0\\ &= -(\mathcal{L}_1^{-1}\mathcal{L}_2 \otimes I_p)x_L^0.\end{aligned}$$

We next study the property of $\mathcal{L}_1^{-1}\mathcal{L}_2$. Because $\mathcal{L}_1$ is a nonsingular M-matrix, it follows from Lemma 2.1 that $\mathcal{L}_1^{-1} \geq 0$. Note also that $\begin{bmatrix}\mathcal{L}_1 & \mathcal{L}_2\\ \mathbf{0}_{m\times(n-m)} & \mathbf{0}_{m\times m}\end{bmatrix}\mathbf{1}_n = \mathbf{0}_n$, which implies that $\mathcal{L}_1\mathbf{1}_{n-m} + \mathcal{L}_2\mathbf{1}_m = \mathbf{0}_{n-m}$. That is, $-\mathcal{L}_1^{-1}\mathcal{L}_2\mathbf{1}_m = \mathbf{1}_{n-m}$. Combining with the fact that $\mathcal{L}_1^{-1} \geq 0$ and $\mathcal{L}_2 \leq 0$ shows that each row of $-\mathcal{L}_1^{-1}\mathcal{L}_2$ has a sum equal to 1. According to Definition 2.3, the final positions of the followers, $-\mathcal{L}_1^{-1}\mathcal{L}_2 \otimes I_p x_L^0$, are within the convex hull formed by the stationary leaders. □

3.2. Switching direct interaction

We next study the case when the directed network topology is switching. Here we assume that $\mathcal{A}(t)$ (i.e., the interaction among the n agents) is constant over time intervals $[\sum_{j=1}^{k}\Delta_j, \sum_{j=1}^{k+1}\Delta_j)$[2] and switches at time $t = \sum_{j=1}^{k}\Delta_j$ with $k = 0, 1, \ldots$, where $\Delta_j > 0, j = 1, \ldots$. Let $\mathcal{G}_k$ and $\mathcal{A}_k$ denote, respectively, the directed graph and the adjacency matrix for the n agents when $t \in [\sum_{j=1}^{k}\Delta_j, \sum_{j=1}^{k+1}\Delta_j)$.

Lemma 3.2. *Using (3) for (1), all followers will always converge to the minimal hyperrectangle that contains the stationary leaders and each of whose hyperplanes is normal to one axis of $\mathcal{C}_0$, for arbitrary initial conditions $x_i^0(0), i \in \mathcal{F}$, if and only if there exists N_2 such that Condition 2.2 is satisfied in the union of $\mathcal{G}_i, i = N_1, \ldots, N_1+N_2$, for any finite N_1.*

Proof (*Necessity*). When there does not exist N_2 such that Condition 2.2 is satisfied in the union of $\mathcal{G}_i, i = N_1, \ldots, N_1 + N_2$, for any finite N_1, there exists at least one follower, labeled as k, such that all leaders do not have directed paths to follower k for $t \in [\sum_{j=1}^{N_1}\Delta_j, \infty)$. It follows that the position of follower k is independent of the positions of the leaders for $t \geq \sum_{j=1}^{N_1}\Delta_j$. Therefore, follower k cannot always converge to the minimal hyperrectangle that contains the stationary leaders and each of whose hyperplanes is normal to one axis of $\mathcal{C}_0$ for arbitrary initial conditions.

(Sufficiency) Let $x_{i(k)}^0$ denote the kth, $k = 1, \ldots, p$, component of x_i^0 (i.e., the projection of the position of agent i to the kth axis of the coordinate frame $\mathcal{C}_0$). Define $x_{L(k)}^+ \triangleq \max_{j\in\mathcal{R}} x_{j(k)}^0$, $x_{L(k)}^- \triangleq \min_{j\in\mathcal{R}} x_{j(k)}^0$, $x_{F(k)}^+ \triangleq \max_{j\in\mathcal{F}} x_{j(k)}^0$, and $x_{F(k)}^- \triangleq \min_{j\in\mathcal{F}} x_{j(k)}^0$.

To prove the sufficiency part, it is sufficient to show that $x_{i(k)}^0(t), i \in \mathcal{F}$, will always converge to the set $\mathcal{S}_k \triangleq [\min_{j\in\mathcal{R}} x_{j(k)}^0, \max_{j\in\mathcal{R}} x_{j(k)}^0]$ for each $k = 1, \ldots, p$, as $t \to \infty$. We study the following four cases:

Case 1: $x_{i(k)}^0(0) \in \mathcal{S}_k, i \in \mathcal{F}$. We next show that $x_{i(k)}^0(t) \in \mathcal{S}_k, i \in \mathcal{F}$, for any $t > 0$. We prove this by contradiction. Assume that there exists some follower, labeled as s, satisfying that $x_{s(k)}^0(\tilde{t}) > x_{L(k)}^+$ (respectively, $x_{s(k)}^0(\tilde{t}) < x_{L(k)}^-$) for some $\tilde{t} > 0$. Because $x_{s(k)}^0(0) \in \mathcal{S}_k$, it follows that $x_{s(k)}^0(0) \leq x_{L(k)}^+$ (respectively, $x_{s(k)}^0(0) \geq x_{L(k)}^-$). In order to guarantee that $x_{s(k)}^0(\tilde{t}) > x_{L(k)}^+$ (respectively, $x_{s(k)}^0(\tilde{t}) < x_{L(k)}^-$), there must exist at least one neighbor of s, labeled as r, satisfying that for some $0 \leq t_1 \leq t_2 \leq \tilde{t}$, $x_{r(k)}^0(t) > x_{L(k)}^+$ (respectively, $x_{r(k)}^0(t) < x_{L(k)}^-$) when $t \in [t_1, t_2]$ because otherwise $x_{s(k)}^0(t)$ cannot increase to be larger than $x_{L(k)}^+$ (respectively, $x_{s(k)}^0(t)$ cannot decrease to be smaller than $x_{L(k)}^-$) by noting that $\dot{x}_{s(k)}^0(t) < 0$ once $x_{s(k)}^0(t) > x_{L(k)}^+$ (respectively, $\dot{x}_{s(k)}^0(t) > 0$ once $x_{s(k)}^0(t) < x_{L(k)}^-$).[3] Repeating the previous analysis shows that there must exist at least one follower, labeled as o, satisfying that $x_{o(k)}^0(0) > x_{L(k)}^+$ (respectively, $x_{o(k)}^0(0) < x_{L(k)}^-$) in order to guarantee that $x_{s(k)}^0(\tilde{t}) > x_{L(k)}^+$ (respectively, $x_{s(k)}^0(\tilde{t}) < x_{L(k)}^-$) for some $\tilde{t} > 0$, which contradicts the assumption that $x_{i(k)}^0(0) \in \mathcal{S}_k, i \in \mathcal{F}$. This implies that $x_{i(k)}^0(t) \in \mathcal{S}_k, i \in \mathcal{F}$, for any $t > 0$.

Case 2: $x_{i(k)}^0(0) > x_{L(k)}^+, i \in \mathcal{F}_1 \subseteq \mathcal{F}$, while $x_{i(k)}^0(0) \in \mathcal{S}_k, i \in \mathcal{F} \setminus \mathcal{F}_1$, where $\mathcal{F}_1$ is a nonempty set. We next show that $x_{i(k)}^0(t) \in \mathcal{S}_k, i \in \mathcal{F}$, as $t \to \infty$.

Step 1: $x_{F(k)}^+$ is a non-increasing function if $x_{F(k)}^+ > x_{L(k)}^+$. We also prove this step by contradiction. Assume that $x_{F(k)}^+(t_2) > x_{F(k)}^+(t_1)$ for some $t_2 > t_1 \geq 0$. There exists at least one follower, labeled as s, satisfying that $x_{s(k)}^0(t_2) > x_{F(k)}^+(t_1)$. Based on the argument in Case 1, there must exist $t_1 \leq t_3 < t_4 \leq t_2$ such that at least one neighbor of follower s, labeled as r, satisfies that $x_{r(k)}^0(t) > x_{F(k)}^+(t_1)$ for $t \in [t_3, t_4]$. Repeating the previous analysis shows that there exists at least one follower, labeled as o, satisfying $x_{o(k)}^0(t_1) > x_{F(k)}^+(t_1)$, which contradicts the fact that $x_{F(k)}^+(t_1) = \max_{i\in\mathcal{F}} x_{i(k)}^0(t_1)$. Therefore, $x_{F(k)}^+$ is a non-increasing function.

Step 2: $x_{F(k)}^+$ will decrease after a finite period of time if $x_{F(k)}^+ > x_{L(k)}^+$ and there exists N_2 such that Condition 2.2 is satisfied in the union of $\mathcal{G}_i, i = N_1, \ldots, N_1 + N_2$, for any finite N_1. The proof of this step is based on Theorem 1 in Moreau (2004b) where a similar control algorithm to (3) (see Section 3.3 in Moreau (2004b)) was used. From (3), we have $\dot{x}_{F(k)}^+ = \max_{j\in\arg\max x_{j(k)}}\left[-\sum_{i\in\mathcal{F}\bigcup\mathcal{R}} a_{ji}(t)(x_j^0 - x_i^0)\right]$. Next we consider a special case when $x_{i(k)} = x_{L(k)}^+, \forall i \in \mathcal{R}$. Apparently, $\dot{x}_{F(k)}^+$ under the special case will be no larger than that under any initial $x_{i(k)}, \forall i \in \mathcal{R}$. Therefore, to prove that $x_{F(k)}^+$ will decrease after a finite period of time, it is sufficient to show that $x_{F(k)}^+$ will decrease under the special case when $x_{i(k)} = x_{L(k)}^+, \forall i \in \mathcal{R}$. We next show that $x_{F(k)}^+$ will decrease after a finite period of time under the special case. Under the special case when $x_{i(k)} = x_{L(k)}^+, \forall i \in \mathcal{R}$, the stationary leaders can be considered a single stationary leader with the state $x_{L(k)}^+$. Meanwhile, when there exists N_2 such that Condition 2.2 is satisfied in the union of $\mathcal{G}_i, i = N_1, \ldots, N_1 + N_2$, for any finite N_1 for multiple stationary leaders, by considering the multiple stationary leaders a single stationary leader, the stationary leader has a directed path to every follower. From Theorem 1 in Moreau (2004b), consensus is achieved when $x_{i(k)} = x_{L(k)}^+, \forall i \in \mathcal{R}$, and the single leader with the state $x_{L(k)}^+$ has a directed path to every follower. Therefore, under the special case, $x_{F(k)}^+$ will decrease after a finite period of time because otherwise consensus cannot be achieved. This completes the proof of the step.

Combining Steps 1 and 2, we get that $x_{F(k)}^+ \leq x_{L(k)}^+$ as $t \to \infty$. In addition, when $x_{i(k)}^0(0) \in \mathcal{S}_k, i \in \mathcal{F}\setminus\mathcal{F}_1$, it follows a similar analysis in Case 1 that $x_{F(k)}^- \geq x_{L(k)}^-$ for all t. Therefore, $x_{i(k)}^0(t) \in \mathcal{S}_k, i \in \mathcal{F}$, as $t \to \infty$.

Case 3. $x_{i(k)}^0(0) < x_{L(k)}^-, i \in \mathcal{F}_2 \subseteq \mathcal{F}$, while $x_{i(k)}^0(0) \in \mathcal{S}_k, i \in \mathcal{F} \setminus \mathcal{F}_2$, where $\mathcal{F}_2$ is a nonempty set. The analysis of this case is

[2] When $k = 0$, we define $\sum_{j=1}^{k}\Delta_j \triangleq 0$.

[3] Note here r might be different for $t \in [t_1, t_2]$.

similar to that of Case 2 by showing that $x^-_{F(k)} \geq x^-_{L(k)}$ as $t \to \infty$ and $x^+_{F(k)} \leq x^+_{L(k)}$ for all $t > 0$.

Case 4. $x^0_{i(k)}(0) > x^+_{L(k)}, i \in \mathcal{F}_3 \subseteq \mathcal{F}, x^0_{i(k)}(0) < x^-_{L(k)}, i \in \mathcal{F}_4 \subseteq \mathcal{F}$, and $x^0_{i(k)}(0) \in \mathcal{S}_k, i \in \mathcal{F} \setminus (\mathcal{F}_3 \bigcup \mathcal{F}_4)$, where $\mathcal{F}_3$ and $\mathcal{F}_4$ are two nonempty sets satisfying $\mathcal{F}_3 \bigcap \mathcal{F}_4 = \varnothing$. The analysis of this case includes four steps:

Step 1: $x^+_{F(k)}$ is a non-increasing function if $x^+_{F(k)} > x^+_{L(k)}$.

Step 2: $x^+_{F(k)}$ will decrease after a finite period of time if $x^+_{F(k)} > x^+_{L(k)}$ and there exists N_2 such that Condition 2.2 is satisfied in the union of $\mathcal{G}_i, i = N_1, \ldots, N_1 + N_2$, for any finite N_1.

Step 3: $x^-_{F(k)}$ is a non-decreasing function if $x^-_{F(k)} < x^-_{L(k)}$.

Step 4: $x^-_{F(k)}$ will increase after a finite period of time if $x^-_{F(k)} < x^-_{L(k)}$ and there exists N_2 such that Condition 2.2 is satisfied in the union of $\mathcal{G}_i, i = N_1, \ldots, N_1 + N_2$, for any finite N_1.

The analysis of Steps 1 and 3 is similar to that of Step 1 in Case 2 and the analysis of Steps 2 and 4 is similar to that of Step 2 in Case 2. The detailed analysis of Steps 1–4 is omitted here.

Noting that the above results are valid for each $k = 1, \ldots, p$, it follows that $x^0_{i(k)}(t), i \in \mathcal{F}$, will always converge to the set $\mathcal{S}_k$ for each $k = 1, \ldots, p$, as $t \to \infty$. This completes the sufficiency part. □

Note that in Lemma 3.2, we only show that the followers will finally converge to the minimal hyperrectangle that contains the stationary leaders and each of whose hyperplanes is normal to one axis of $\mathcal{C}_0$. In order to show that the followers will finally converge to the convex hull formed by the stationary leaders, we need to introduce the following lemma.

Lemma 3.3. *Given n fixed points $x_i \in \mathbb{R}^p, i = 1, \ldots, n$, relative to the inertial coordinate frame $\mathcal{C}_0$. The convex hull formed by the n points is equivalent to the intersection of all minimal hyperrectangles containing the n points.*

Proof. For simplicity, we use $\mathbf{Co}_n$ to denote the convex hull formed by the n points and $\mathbf{Hr}_n$ to denote the set of all minimal hyperrectangles containing the n points. The lemma is equivalent to the following two statements: (i) For any point $\eta^0 \in \mathbf{Co}_n$ and any $\varpi^0 \in \mathbf{Hr}_n$, $\eta^0 \in \varpi^0$, and (ii) For any point $\zeta^0 \notin \mathbf{Co}_n$, there exists $\vartheta^0 \in \mathbf{Hr}_n$ such that $\zeta^0 \notin \vartheta^0$.

Proof of Statement (i): Noting that $\mathbf{Co}_n \subseteq \varpi^0$, it is trivial to show that $\eta^0 \in \varpi^0$.

Proof of Statement (ii): Because $\zeta^0 \notin \mathbf{Co}_n$, it follows that there exists a $(p-1)$-dimensional space which can divide the whole p-dimensional space into two sets $\overline{\mathcal{S}}$ and $\underline{\mathcal{S}}$ such that $\zeta^0 \in \overline{\mathcal{S}}$, $\mathbf{Co}_n \subseteq \underline{\mathcal{S}}$, and $\overline{\mathcal{S}} \bigcap \underline{\mathcal{S}} = \varnothing$.[4] Arbitrarily choose $p-1$ orthogonal vectors in the $(p-1)$-dimensional space and a vector normal to the $(p-1)$-dimensional space as the coordinate frame $\mathcal{C}_1$. Let ϑ^0 be the minimal hyperrectangle that contains the n points and each of whose hyperplanes is normal to one axis of $\mathcal{C}_1$. It then follows that $\zeta^0 \in \overline{\mathcal{S}}$ and $\vartheta^0 \subseteq \underline{\mathcal{S}}$. Because $\overline{\mathcal{S}} \bigcap \underline{\mathcal{S}} = \varnothing$, it follows that $\zeta^0 \notin \vartheta^0$. This completes the proof of Statement 2. □

Based on the results in Lemmas 3.2 and 3.3, we are ready to present the following result for containment control in any high-dimensional space (i.e., $p \geq 2$ in (1)).

Theorem 3.2. *Using (3) for (1), all followers will always converge to the stationary convex hull formed by the stationary leaders for arbitrary initial conditions $x^0_i(0), i \in \mathcal{F}$, if and only if there exists N_2 such that Condition 2.2 is satisfied in the union of $\mathcal{G}_i, i = N_1, \ldots, N_1 + N_2$, for any finite N_1.*

Proof. Note that both (1) and (3) are represented in the inertial coordinate frame $\mathcal{C}_0$. For the purpose of analysis, we intentionally introduce another arbitrary (nonexisting) inertial coordinate frame $\mathcal{C}_1$. Mathematically, there is a (unique and reversible) map from $\mathcal{C}_0$ to $\mathcal{C}_1$. That is, given any point q, we have that

$$q^0 = R^0_1 q^1 + \nu^0, \tag{9}$$

where q^0 and q^1 are, respectively, the coordinates of the point q with respect to $\mathcal{C}_0$ and $\mathcal{C}_1$, R^0_1 is the rotational matrix from $\mathcal{C}_1$ to $\mathcal{C}_0$, and ν^0 is the translational vector from the origin of $\mathcal{C}_0$ to the origin of $\mathcal{C}_1$ represented in $\mathcal{C}_0$. Using (3) for (1), we have that

$$\dot{x}^0_i(t) = -\sum_{j \in \mathcal{F} \bigcup \mathcal{R}} a_{ij}(t)[x^0_i(t) - x^0_j(t)], \quad i \in \mathcal{F}. \tag{10}$$

Note that $x^0_i(t) = R^0_1 x^1_i(t) + \nu^0$. It then follows from (10) that

$$\begin{aligned} R^0_1 \dot{x}^1_i(t) &= -\sum_{j \in \mathcal{F} \bigcup \mathcal{R}} a_{ij}(t)\{[R^0_1 x^1_i(t) + \nu^0] - [R^0_1 x^1_j(t) + \nu^0]\} \\ &= -R^0_1 \sum_{j \in \mathcal{F} \bigcup \mathcal{R}} a_{ij}(t)[x^1_i(t) - x^1_j(t)], \quad i \in \mathcal{F}, \end{aligned} \tag{11}$$

where we have used the fact that R^0_1 and ν^0 are constant to obtain the left hand side of (11). It thus follows that

$$\dot{x}^1_i(t) = -\sum_{j \in \mathcal{F} \bigcup \mathcal{R}} a_{ij}(t)[x^1_i(t) - x^1_j(t)], \quad i \in \mathcal{F}. \tag{12}$$

From (10) and (12), it can be noted that the closed-loop system of (1) using (3) where the positions of all agents are represented relative to one inertial coordinate frame can be equivalently transformed to the same form when the positions of all agents are represented relative to any other arbitrarily chosen inertial frame.

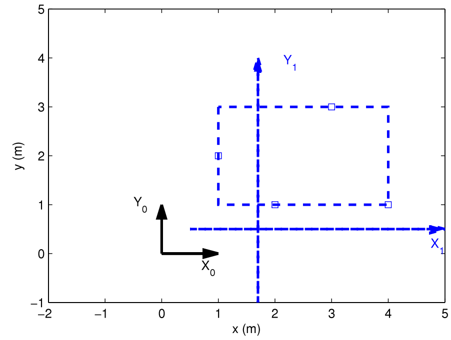
According to Lemma 3.2, we can get that the followers will converge to the minimal hyperrectangle that contains the stationary leaders and each of whose hyperplanes is normal to one axis of the coordinate frame $\mathcal{C}_1$ under the condition of the theorem.[5] Because the coordinate frame $\mathcal{C}_1$ can be arbitrary, it follows that the followers will converge to all minimal hyperrectangles that contain the stationary leaders. That is, the followers will converge to the intersection of all minimal hyperrectangles containing the stationary leaders. It then follows from Lemma 3.3 that all followers will converge to the convex hull formed by the stationary leaders under the condition of the theorem. □

Example 3.3. In order to better illustrate the results in Lemma 3.3, we simply consider a 2D example where there exists an inertial coordinate frame $\mathcal{C}_0$. We also arbitrarily choose three other inertial coordinate frames $\mathcal{C}_1$, $\mathcal{C}_2$, and $\mathcal{C}_3$ (see Fig. 1). The four squares in Fig. 1 represent the four stationary leaders. The blue rectangle (respectively, the red rectangle and the purple rectangle) is the minimal hyperrectangle that contains the stationary leaders and each of whose edges is normal to one axis of the coordinate frame $\mathcal{C}_1$ (respectively, $\mathcal{C}_2$ and $\mathcal{C}_3$). Apparently, the intersection of the three hyperrectangles[6] is equivalent to the convex hull formed by the four stationary leaders.

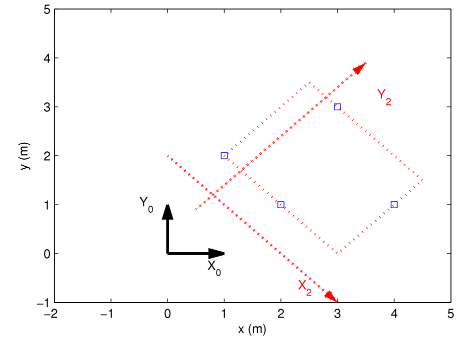
[4] That is, ζ^0 is on one side of the $(p-1)$-dimensional space and $\mathbf{Co}_n$ is on the other side of the $(p-1)$-dimensional space (possibly part of $\mathbf{Co}_n$ is on the $(p-1)$-dimensional space).

[5] We emphasize that $\mathcal{C}_1$ does not exist. We introduce $\mathcal{C}_1$ only for analysis. Although the coordinates of a point with respect to $\mathcal{C}_0$ and $\mathcal{C}_1$ are different, the physical location of the point is unique in the space.

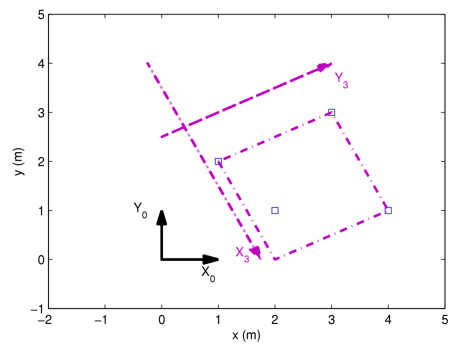
[6] Note that in general we should get the intersection of all minimal hyperrectangles containing the stationary leaders but here the three minimal hyperrectangles happen to be sufficient.



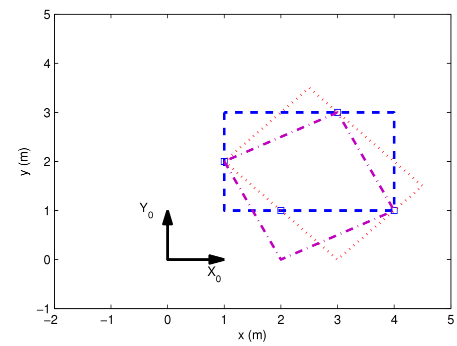
(a) The minimal rectangle containing the four leaders when the closed-loop system is represented in $\mathcal{C}_1$. Each edge of the rectangle is normal to one axis of $\mathcal{C}_1$.



(b) The minimal rectangle containing the four leaders when the closed-loop system is represented in $\mathcal{C}_2$. Each edge of the rectangle is normal to one axis of $\mathcal{C}_2$.



(c) The minimal rectangle containing the four leaders when the closed-loop system is represented in $\mathcal{C}_3$. Each edge of the rectangle is normal to one axis of $\mathcal{C}_3$.



(d) The intersection of the previous three rectangles, which is the convex hull formed by the four leaders.

Fig. 1. Illustration of Lemma 3.3 in the 2D space. The squares denote the positions of the four stationary leaders. The blue, red, and purple rectangles represent the minimal rectangles that contains the four leaders and each of whose edges is normal to one axis of, respectively, $\mathcal{C}_1$, $\mathcal{C}_2$, and $\mathcal{C}_3$. The intersection of the three rectangles is the convex hull formed by the four leaders. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

Remark 3.4. The coordinate transformation technique used in the proof of Theorem 3.2 provides a powerful tool in analyzing the group coordination behavior of a linear system with a linear algorithm in a high-dimensional space when the decoupling technique based on the Kronecker product cannot be applied. Essentially, (10) and (12) imply that the followers do not need to share a common inertial coordinate frame in the containment control problem in the case of stationary leaders. Each follower can have its own inertial coordinate frame and implement the algorithm according to its own inertial coordinate frame. Similarly, when using the traditional consensus algorithm (2) for (1), the same coordinate transformation technique in the proof of Theorem 3.2 can be used without changing the property of the closed-loop system (such as whether consensus can be achieved). That is, even if each agent has its own inertial coordinate frame, consensus can still be achieved if the directed graph has a directed spanning tree.

Remark 3.5. Existing consensus algorithms primarily studied the case where the (nonsymmetric) Laplacian matrix has exactly one zero eigenvalue. When there exist multiple leaders, the (nonsymmetric) Laplacian matrix $\mathcal{L}$ in (4) has multiple zero eigenvalues (Agaev & Chebotarev, 2000). Lemma 3.2 studied the case when the (nonsymmetric) Laplacian matrix has multiple zero eigenvalues.

Remark 3.6. Ref. (Ji et al., 2008) focus on the case where the network topology for the followers is undirected and connected. Theorem 3.2 considers a general case where the network topology for the followers is directed and not necessarily connected.

Remark 3.7. In the previous part of this section, we assume that each leader has no neighbor. However, for some network topologies, it is possible to view a subgroup of agents as one leader. For example, in the network topology given by Fig. 2, agents 1 and 2 (respectively, agents 5 and 6) can reach consensus on a constant value independent of the states of the other agents. The results in Section 3 can also be applied to this case by viewing agents 1 and 2 (respectively, agents 5 and 6) as one leader with the state being the constant consensus equilibrium of agents 1 and 2 (respectively, agents 5 and 6).

$$1 \longleftrightarrow 2 \longrightarrow 3 \longleftarrow 4 \qquad 5 \longleftrightarrow 6$$

Fig. 2. A special network topology when a subgroup of agents can be viewed as one leader.

Remark 3.8. For the discrete-time consensus algorithm (i.e., the distributed weighted averaging algorithm) with multiple stationary leaders, the convergence results are the same as those in Theorem 3.2 by following a similar analysis.

4. Stability analysis with multiple dynamic leaders

In this section, we propose a distributed tracking control algorithm without velocity measurements and then analyze the stability condition under both fixed and switching directed network topologies. We again assume that all agents share a common inertial coordinate frame $\mathcal{C}_0$. In this section, we omit the superscript 0 in the coordinate representations for the simplicity of notation.

For agents with single-integrator kinematics in (1), when there exist $m, m < n$, dynamic leaders and $n - m$ followers, we propose the following tracking control algorithm without velocity measurements as

$$u_i(t) = v_i(t), \quad i \in \mathcal{R}$$
$$u_i(t) = -\alpha \sum_{j\in\mathcal{F}\bigcup\mathcal{R}} a_{ij}(t)[x_i(t)-x_j(t)] - \beta \mathrm{sgn}\left\{\sum_{j\in\mathcal{F}\bigcup\mathcal{R}} a_{ij}(t)[x_i(t)-x_j(t)]\right\}, \quad i\in\mathcal{F}, \tag{13}$$

where $v_i(t) \in \mathbb{R}^p$ denotes the time-varying velocity of leader i, $i \in \mathcal{R}$, $a_{ij}(t)$ is defined as in (2), $\mathrm{sgn}(\cdot)$ is the signum function defined entrywise, α is a nonnegative constant scalar, and β is a positive constant scalar. We assume that $\|v_i(t)\|$, $i \in \mathcal{R}$, is bounded. Because the proposed algorithm (13) is discontinuous, we study the Filippov solutions (Filippov, 1988) of the closed-loop system of (1) using (13). Because the signum function used in (13) is measurable and locally essentially bounded, it follows from Filippov (1988) that there exist Filippov solutions of (1) using (13).

Remark 4.1. Note that in contrast to (3), a nonsmooth term (the second term of (13)) is intentionally introduced in (13) to compensate for the unknown bounded time-varying velocities of the dynamic leaders in the presence of local interaction. The effect of the nonsmooth term works in a similar way to that of the traditional variable structure control. Moreover, a smooth (nonlinear) term, in general, cannot be used to replace the nonsmooth term to achieve the same goal.

4.1. Fixed directed interaction

In this subsection, we assume that the directed interaction is fixed (i.e., all $a_{ij}(t)$ in (13) are constant). Before moving on, we need the following lemmas.

Lemma 4.1 (*Khalil (2002, Comparison Lemma)*)**.** *Consider the scalar differential equation $\dot z = f(t,z)$, where $f(t,z)$ is continuous in t and locally Lipschitz in z for all $t > 0$ and all $z \in J \subset \mathbb{R}$. Let $[t_0, T)$ (T could be infinity) be the maximal interval of existence of the solution $z(t)$, and suppose that $z(t) \in J$ for all $t \in [t_0, T)$. Let $\omega(t)$ be a continuous function whose upper right-hand derivative $D^+\omega(t)$ satisfies the differential inequality $D^+\omega(t) \le f(t,\omega(t))$ with $\omega(t_0) \le z(t_0)$ and $\omega(t) \in J$ for all $t \in [t_0, T)$. Then $\omega(t) \le z(t)$ for all $t \in [t_0, T)$.*

Lemma 4.2. *Suppose that a team consists of n followers, labeled as agents 1 to n, and a stationary leader, labeled as agent 0, whose state is given by 0. Let $\bar{\mathcal{G}}$ be the directed graph characterizing the interaction among the leader and the followers. Let $g(t) \in \mathbb{R}$ be a continuous signal satisfying that $|g(t)| \le \gamma_g$. Suppose that in $\bar{\mathcal{G}}$ the leader has directed paths to followers 1 to n. For n agents with kinematics given by (1) where $p = 1$, using*

$$u_i(t) = -\sum_{j=1}^n a_{ij} f_{i,j}[x_i(t), x_j(t), t], \quad i = 1, \ldots, n, \tag{14}$$

where a_{ij} is defined as in (13), $x_i(t) \to 0$ as $t \to \infty$ if $f_{i,j}(\cdot,\cdot,\cdot)$ satisfies that

$$f_{i,j}[x(t), y(t), t] = \epsilon[x(t)-y(t)] + \varsigma\{\mathrm{sgn}[x(t)] - \mathrm{sgn}[y(t)]\}, \tag{15}$$

for $i = 1, \ldots, n$, and $j = 1, \ldots, n$, and

$$f_{i,0}[x(t), 0, t] = \epsilon x(t) + \varsigma \mathrm{sgn}[x(t)] + g_i(t) \tag{16}$$

for any nonnegative t, where $\epsilon > 0$ and $\varsigma \ge \gamma_g$ are two positive constants.

Proof. Consider the function $V(t) \triangleq \max_i x_i(t) - \min_i x_i(t)$. It follows from the closed-loop system of (1) using (15) or (16) that $V(t)$ is non-increasing. Therefore, $|x_i - x_j| \le V(0)$. We next compute $D^+V(t)$. Note that

$$\begin{aligned} D^+V(t) &= \limsup_{h\to 0^+} \frac{1}{h}[V(t+h) - V(t)] \\ &= \lim_{h\to 0^+}\left\{\frac{1}{h}\left[\max_i x_i(t+h) - \max_i x_i(t)\right] - \frac{1}{h}\left[\min_i x_i(t+h) - \min_i x_i(t)\right]\right\} \\ &= \max_{j\in\arg\max x_j} \dot x_j(t) + \max_{k\in\arg\min x_k} [-\dot x_k(t)]. \end{aligned}$$

Base on (15) and (16), we have that

$$\begin{aligned} D^+V(t) &= \max_{j\in\arg\max x_j} \dot x_j(t) + \max_{k\in\arg\min x_k}[-\dot x_k(t)] \\ &\le \max_{j\in\arg\max x_j}\left\{-\sum_{i=1}^n a_{ji}\epsilon[x_j(t) - x_i(t)]\right\} \\ &\quad + \max_{k\in\arg\min x_k}\left\{\sum_{i=1}^n a_{ki}\epsilon[x_k(t)-x_i(t)]\right\}. \end{aligned} \tag{17}$$

Consider the closed-loop system given by

$$\dot r_i(t) = -\epsilon\sum_{j=0}^n a_{ij}[r_i(t) - r_j(t)], \quad i = 1, \ldots, n, \tag{18}$$

where $r_i(0) = x_i(0)$, $i = 0, 1, \ldots, n$. Consider the function $\widetilde V(t) \triangleq \max_i r_i(t) - \min_i r_i(t)$. It can be computed that

$$\begin{aligned} D^+\widetilde V(t) &= \max_{j\in\arg\max r_j} \dot r_j(t) + \max_{k\in\arg\min r_k}[-\dot r_k(t)] \\ &= \max_{j\in\arg\max r_j}\left\{-\sum_{i=1}^n a_{ji}\epsilon[r_j(t) - r_i(t)]\right\} \\ &\quad + \max_{k\in\arg\min r_k}\left\{\sum_{i=1}^n a_{ki}\epsilon[r_k(t) - r_i(t)]\right\}. \end{aligned} \tag{19}$$

Note that $\widetilde V(t)$ is piecewise differentiable. Without loss of generality, assume that $D^+\widetilde V(t)$ is differentiable over intervals $[t_i, t_{i+1})$, $i = 0, 1, \ldots$, where $t_0 = 0$. It follows that $D^+\widetilde V(t) = \dot{\widetilde V}(t)$ for $t \in [t_i, t_{i+1})$. For $t \in [t_0, t_1)$, because $|r_j(t) - r_i(t)| \le |\widetilde V(t)|$, we have $|\dot{\widetilde V}(t)| \le \max_{j\in\arg\max r_j}\{\sum_{i=1}^n a_{ji}\epsilon|\widetilde V(t)|\} + \max_{k\in\arg\min r_k}\{\sum_{i=1}^n a_{ki}\epsilon|\widetilde V(t)|\}$. Noting also that $r_i(t)$ is continuous in t and $\widetilde V(t)$ is bounded, it then follows that $\dot{\widetilde V}(t)$ is continuous in t and locally Lipschitz in $\max_i r_i(t) - \min_i r_i(t)$ for $t \in [t_0, t_1)$. From (17) and (19), we can get that $D^+V[t, \max_i r_i(t) - \min_i r_i(t)] \le \dot{\widetilde V}[t, \max_i r_i(t) - \min_i r_i(t)]$ for $t \in [t_0, t_1)$. When $r_i(0) = x_i(0)$, $i = 0, 1, \ldots, n$, it follows that $V(0) = \widetilde V(0)$. It then follows from Lemma 4.1 that $V(t) \le \widetilde V(t)$ for $t \in [t_0, t_1)$. Because both $V(t)$ and $\widetilde V(t)$ are continuous in t, by repeating the previous analysis for $t \in [t_i, t_{i+1})$, $i = 1, \ldots$, it follows that $V(t) \le \widetilde V(t)$ for $t \in [t_i, t_{i+1})$, $i = 1, \ldots$. Therefore, $V(t) \le \widetilde V(t)$ for all $t \ge t_0 \equiv 0$. Note that consensus can be achieved as $t \to \infty$ for the closed-loop system (18) under the condition of the lemma (Moreau, 2004b; Olfati-Saber & Murray, 2004). That is, $\widetilde V(t) \to 0$ as $t \to \infty$ under the condition of the lemma. Combining with the facts $V(t) \le \widetilde V(t)$ for all $t \ge t_0 \equiv 0$ and $V(t) \ge 0$ shows that $V(t) \to 0$ as $t \to \infty$ under the condition of the lemma. Therefore, $r_i(t) \to 0$ as $t \to \infty$ using (14) for (1) under (15) and (16) under the condition of the lemma. $\square$

Theorem 4.2. *Suppose that the directed network topology is fixed, $\alpha > 0$, and $\beta \geq \gamma_l$, where $\gamma_l \triangleq \sup_{i\in\mathcal{R}} \|v_i(t)\|$. Using (13) for (1), all followers will always converge to the dynamic convex hull $\mathbf{Co}\{x_j(t), j \in \mathcal{R}\}$ as $t \to \infty$ for arbitrary initial conditions $x_i(0)$, $i \in \mathcal{F}$, if and only if Condition 2.2 is satisfied in the directed graph $\mathcal{G}$. In particular, the final positions of the followers are given by $-(\mathcal{L}_1^{-1}\mathcal{L}_2 \otimes I_p)x_L(t)$, where x_L is the column stack vector of the leaders' positions, and $\mathcal{L}_1$ and $\mathcal{L}_2$ are given in (5).*

Proof (*Necessity*). The necessity proof is similar to that in Theorem 3.1 and hence omitted here.

(Sufficiency) Without loss of generality, suppose that agents 1 to $n-m$ are followers and agents $n-m+1$ to n are leaders. Denote $X(t) \triangleq [x_1^T(t), \ldots, x_n^T(t)]^T$ and let $\mathcal{L} \in \mathbb{R}^{n\times n}$ be the (nonsymmetric) Laplacian matrix for the n agents. It can be noted that the last m rows of $\mathcal{L}$ are all equal to zero. Using (13), (1) can be written in a matrix form as

$$\dot{X}(t) = -\alpha(\mathcal{L} \otimes I_p)X(t) - \beta\mathrm{sgn}[(\mathcal{L} \otimes I_p)X(t)] + V(t), \tag{20}$$

where $V(t) = [\mathbf{0}_p^T, \ldots, v_{(n-m+1)}^T(t), \ldots, v_n^T(t)]^T$. Let $Z(t) = [z_1^T(t), \ldots, z_n^T(t)]^T = (\mathcal{L} \otimes I_p)X(t)$. It follows

$$\begin{aligned}\dot{Z}(t) &= (\mathcal{L} \otimes I_p)\dot{X}(t)\\ &= -\alpha(\mathcal{L} \otimes I_p)Z(t) - \beta(\mathcal{L} \otimes I_p)\mathrm{sgn}[Z(t)]\\ &\quad + (\mathcal{L} \otimes I_p)V(t).\end{aligned} \tag{21}$$

Because the last m rows of $\mathcal{L}$ are equal to zero, we get that $z_i(t) \equiv \mathbf{0}_p$, $i = n-m+1, \ldots, n$. We can thus view agents $n-m+1$ to n as a single agent, labeled as 0, instead of m agents. It thus follows that $z_0(t) \equiv \mathbf{0}_p$. When Condition 2.2 is satisfied in $\mathcal{G}$, it follows that agent 0 has a directed path to the $n-m$ followers.

Considering the group consisting of agents 0 to $n-m$, we know that $z_0(t) \equiv \mathbf{0}_p$ and

$$\begin{aligned}\dot{z}_i(t) &= -\alpha\sum_{j=1}^{n-m} a_{ij}\big([z_i(t) - z_j(t)]\\ &\quad + \beta\{\mathrm{sgn}[z_i(t)] - \mathrm{sgn}[z_j(t)]\}\big)\\ &\quad - \sum_{j=n-m+1}^{n} a_{ij}\{z_i(t) + \beta\mathrm{sgn}[z_i(t)]\\ &\quad - v_j(t)\},\quad i = 1, \ldots, n-m,\end{aligned}$$

where we have used (21) by noting that $z_i(t) \equiv \mathbf{0}_p$, $i = n-m+1, \ldots, n$. In order to use Lemma 4.1, we next show that (21) satisfies the condition (15) or (16) in each dimension. Denote $z_{i(k)}$, $k = 1, \ldots, p$, as the kth component of z_i (i.e., the projection of the position of agent i to the kth axis of $\mathcal{C}_0$), and $v_{i(k)}$, $k = 1, \ldots, p$, as the kth component of v_i (i.e., the projection of the velocity of agent i to the kth axis of $\mathcal{C}_0$). When $\beta \geq \gamma_l$, $\alpha[z_{i(k)}(t) - z_{j(k)}(t)] + \beta[\mathrm{sgn}(z_{i(k)}(t)) - \mathrm{sgn}(z_{j(k)}(t))]$ satisfies the condition (15). When $\beta \geq \gamma_l$, it follows that $\alpha z_{i(k)}(t) + \beta\mathrm{sgn}(z_{i(k)}(t)) - v_{j(k)}(t)$ satisfies the condition (16). Because agent 0 has directed paths to agents 1 to $n-m$ (i.e., the network topology for agents 0 to $n-m$ has a directed spanning tree), it follows from Lemma 4.2 that $z_{i(k)}(t) \to 0$, $i = 1, \ldots, n-m$, as $t \to \infty$. Therefore, $z_i(t) \to \mathbf{0}_p$. It follows from the definition that $Z(t) = (\mathcal{L} \otimes I_p)X(t)$ that $(\mathcal{L} \otimes I_p)X(t) \to \mathbf{0}_{np}$ as $t \to \infty$. When partitioning $\mathcal{L}$ as in (5), it follows from the proof of Theorem 3.1 that the final positions of the followers are given by $-(\mathcal{L}_1^{-1}\mathcal{L}_2 \otimes I_p)x_L(t)$. Noting that each row of $-\mathcal{L}_1^{-1}\mathcal{L}_2$ has a sum equal to 1 from the proof of Theorem 3.1, it follows that all followers will always converge to the dynamic convex hull $\mathbf{Co}\{x_j(t), j \in \mathcal{R}\}$ as $t \to \infty$ under the condition of the theorem. □

Note that the results in Theorem 3.1 can be obtained by letting $\beta = 0$ in Theorem 4.2.

4.2. Switching directed interaction

In this subsection, we assume that the adjacency matrix $\mathcal{A}(t)$ (and hence the interaction) is switching over time but remains constant for $t \in [\sum_{j=1}^{k}\Delta_j, \sum_{j=1}^{k+1}\Delta_j)$ as in Section 3.2.

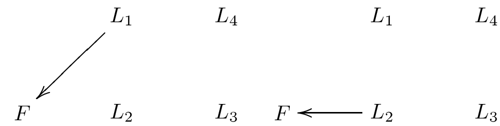
Theorem 4.3. *Suppose that $\beta > \gamma_l$, where γ_l is defined in Theorem 4.2. Using (13) for (1), all followers will always converge to the dynamic minimal hyperrectangle that contains the dynamic leaders and each of whose hyperplanes is normal to one axis of $\mathcal{C}_0$ as $t \to \infty$ for arbitrary initial conditions $x_i(0)$, $i \in \mathcal{F}$, if Condition 2.2 is satisfied in the directed graph $\mathcal{G}(t)$ at each time interval $[\sum_{j=1}^{k}\Delta_j, \sum_{j=1}^{k+1}\Delta_j)$.*

Proof. Let $x_{i(k)}, v_{i(k)}, x^+_{L(k)}, x^-_{L(k)}, x^+_{F(k)}$, and $x^-_{F(k)}$ be defined the same as those in the proof of Lemma 3.2 without the explicit introduction of the superscript 0. Different from the proof of Lemma 3.2 where $x_{j(k)}, j \in \mathcal{R}, x^+_{L(k)}$ and $x^-_{L(k)}$ are constant, $x_{j(k)}, j \in \mathcal{R}, x^+_{L(k)}$, and $x^-_{L(k)}$ here are dynamic. To prove the theorem, it is sufficient to show that $x_{i(k)}(t)$, $i \in \mathcal{F}$, will always converge to the dynamic set $\mathcal{S}_k \triangleq [\min_{j\in\mathcal{R}} x_{j(k)}, \max_{j\in\mathcal{R}} x_{j(k)}]$ for each $k = 1, \ldots, p$, as $t \to \infty$. We study the following four cases[7]:

Case 1: $x_{i(k)}(0) \in \mathcal{S}_k(0)$, $i \in \mathcal{F}$. We next show that $x_{i(k)}(t) \in \mathcal{S}_k(t)$, $i \in \mathcal{F}$ for any $t > 0$. We prove this by contradiction, which is motivated by the proof of Case 1 in Lemma 3.2. Assume that there exists some follower, labeled as s, satisfying that $x_{s(k)}(\tilde{t}) > x^+_{L(k)}(\tilde{t})$ (respectively, $x_{s(k)}(\tilde{t}) < x^-_{L(k)}(\tilde{t})$) for some $\tilde{t} > 0$. Because $x_{s(k)}(0) \in \mathcal{S}_k(0)$, we have that $x_{s(k)}(0) \leq x^+_{L(k)}(0)$ (respectively, $x_{s(k)}(0) \geq x^-_{L(k)}(0)$). Noting that Condition 2.2 is satisfied at each time interval, each follower has at least one neighbor at each time interval. In order to guarantee that $x_{s(k)}(\tilde{t}) > x^+_{L(k)}(\tilde{t})$, there must exist at least one neighbor of s, labeled as r, satisfying that for some $0 \leq t_1 \leq t_2 \leq \tilde{t}$, $x_{r(k)}(t) > x^+_{L(k)}(t)$ (respectively, $x_{r(k)}(t) < x^-_{L(k)}(t)$) when $t \in [t_1, t_2]$ because otherwise $x_{s(k)}(t)$ cannot increase to be larger than $x^+_{L(k)}(t)$ (respectively, $x_{s(k)}(t)$ cannot decrease to be smaller than $x^-_{L(k)}(t)$) by noting that $\dot{x}_{s(k)}(t) = -\alpha\sum_{i\in\mathcal{F}\bigcup\mathcal{R}} a_{si}(t)[x_{s(k)}(t) - x_{i(k)}(t)] - \beta\mathrm{sgn}\left\{\sum_{i\in\mathcal{F}\bigcup\mathcal{R}} a_{si}(t)[x_{s(k)}(t) - x_{i(k)}(t)]\right\} \leq -\beta < -\gamma_\ell \leq v_{i(k)}(t)$ once $x_{s(k)}(t) > x^+_{L(k)}(t)$ (respectively, $\dot{x}_{s(k)}(t) \geq \beta > \gamma_\ell \geq v_{i(k)}(t)$ once $x_{s(k)}(t) < x^-_{L(k)}(t)$). By repeating the previous analysis and following a similar argument in the proof of Case 1 in Lemma 3.2, it can be obtained that $x_{i(k)}(t) \in \mathcal{S}_k(t)$, $i \in \mathcal{F}$ for any $t > 0$.

Case 2: $x_{i(k)}(0) > x^+_{L(k)}(0)$, $i \in \mathcal{F}_1 \subseteq \mathcal{F}$, while $x_{i(k)}(0) \in \mathcal{S}_k(0)$, $i \in \mathcal{F} \setminus \mathcal{F}_1$, where $\mathcal{F}_1$ is a nonempty set. We next show that $x^+_{F(k)} \leq x^+_{L(k)}$ as $t \to \infty$ and $x^-_{F(k)} \geq x^-_{L(k)}$ for any $t > 0$. We study how $x^+_{F(k)} - x^+_{L(k)}$ evolves when $x^+_{F(k)} > x^+_{L(k)}$. Because the derivative of $x^+_{F(k)} - x^+_{L(k)}$ might be nonsmooth, we use the differential inclusion (Clarke, 1990; Cortes, 2008; Filippov, 1988; Paden & Sastry, 1987) in the following analysis. When there exists a unique follower whose kth component of the position is equal to $x^+_{F(k)}$, the generalized derivative of $x^+_{F(k)} - x^+_{L(k)}$ satisfies that $(x^+_{F(k)} - x^+_{L(k)})^o \in K[\dot{x}^+_{F(k)} - \dot{x}^+_{L(k)}]$, where $K[\cdot]$ denotes the differential inclusion and $(\cdot)^o$ denotes the generalized derivative. Noting that $\dot{x}^+_{F(k)} < -\beta$ and $K[\dot{x}^+_{L(k)}] \subseteq [-\gamma_\ell, \gamma_\ell]$ in this case, it follows that $\max(x^+_{F(k)} - x^+_{L(k)})^o < -\beta + \gamma_\ell < 0$. When there exist multiple followers whose kth components of the positions are equal to $x^+_{F(k)}$, the generalized derivative of $x^+_{F(k)} - x^+_{L(k)}$ satisfies

[7] Although the four cases are similar to those in the proof of Lemma 3.2, the corresponding technical analysis is fairly different from that in the proof of Lemma 3.2.



(a) Graph 1. (b) Graph 2.

Fig. 3. Switching directed network topologies for a group of agents with four leaders and one follower. Here L_i, $i = 1, \ldots, 4$, denote the leaders while F denote the follower.

that $(x^+_{F(k)} - x^+_{L(k)})^o \in K[\dot{x}^+_{F(k)} - \dot{x}^+_{L(k)}]$. For the simplicity of representation, we denote $\Upsilon \triangleq \sum_{i\in\mathcal{F}\bigcup\mathcal{R}} a_{ji}(t)[x_{j(k)}(t) - x_{i(k)}(t)]$. It follows from the definition of $x^+_{F(k)}$ that $K[\dot{x}^+_{F(k)}] = [\vartheta_1, \vartheta_2]$, where $\vartheta_1 = -\beta\mathrm{sgn}\left(\max_{j\in\arg\max x_{j(k)}} \Upsilon\right) - \alpha \max_{j\in\arg\max x_{j(k)}} \Upsilon$ and $\vartheta_2 = -\beta\mathrm{sgn}\left(\min_{j\in\arg\max x_{j(k)}} \Upsilon\right) - \alpha \min_{j\in\arg\max x_{j(k)}} \Upsilon$. Note that if $\vartheta_1 \neq 0$ (respectively, $\vartheta_2 \neq 0$), then $\vartheta_1 < -\beta$ (respectively, $\vartheta_2 < -\beta$). We next show that $\vartheta_2 = 0$ only happens at some time instants but not in a time interval when $x^+_{F(k)} > x^+_{L(k)}$. We prove this by contradiction. Assume that $\vartheta_2 = 0$ for $t \in [t_1, t_2]$, where $0 \le t_1 < t_2$. This implies that $x^+_{F(k)}$ is constant for $t \in [t_1, t_2]$. That is, there exists some follower(s), labeled as r, satisfying that $x_{r(k)} = x^+_{F(k)}$ for $t \in [t_3, t_4]$ where $t_1 \le t_3 < t_4 \le t_2$. It then follows from the closed-loop system of (1) using (13) that $x_{j(k)}(t) = x_{r(k)}(t) = x^+_{F(k)}, j \in \mathcal{N}_r$, for $t \in [t_3, t_4]$. Because Condition 2.2 is satisfied in the directed graph $\mathcal{G}(t)$ at each time interval, repeating the previous analysis shows that there exists at least one leader, labeled as o, satisfying that $x_{o(k)}(t) = x^+_{F(k)}$ for $t \in [t_3, t_4]$. This contradicts the fact that $x^+_{F(k)} > x^+_{L(k)}$. Combining with $K[\dot{x}^+_{F(k)}] = [\vartheta_1, \vartheta_2]$, $K[\dot{x}^+_{L(k)}] \subseteq [-\gamma_\ell, \gamma_\ell]$, and $(x^+_{F(k)} - x^+_{L(k)})^o \in K[\dot{x}^+_{F(k)} - \dot{x}^+_{L(k)}]$ shows that $\max(x^+_{F(k)} - x^+_{L(k)})^o < -\beta + \gamma_\ell < 0$ almost everywhere when $x^+_{F(k)} > x^+_{L(k)}$ under the condition of the theorem. It follows that $x^+_{F(k)} \le x^+_{L(k)}$ as $t \to \infty$. By following the analysis in Case 1, when $x_{i(k)}(0) \in \mathcal{S}_k(0), i \in \mathcal{F} \setminus \mathcal{F}_1$, it can be shown that $x^-_{F(k)} \ge x^-_{L(k)}$ for any $t > 0$. This completes the proof of Case 2.

Case 3. $x_{i(k)}(0) < x^-_{L(k)}(0), i \in \mathcal{F}_2 \subseteq \mathcal{F}$, while $x_{i(k)}(0) \in \mathcal{S}_k(0), i \in \mathcal{F} \setminus \mathcal{F}_2$, where $\mathcal{F}_2$ is a nonempty set. The proof follows the same analysis to that in the proof of Case 2 by showing that $x^-_{F(k)} \ge x^-_{L(k)}$ as $t \to \infty$ and $x^+_{F(k)} \le x^+_{L(k)}$ for all t.

Case 4. $x_{i(k)}(0) > x^+_{L(k)}(0), i \in \mathcal{F}_3 \subseteq \mathcal{F}, x_{i(k)}(0) < x^-_{L(k)}(0), i \in \mathcal{F}_4 \subseteq \mathcal{F}$, and $x_{i(k)}(0) \in \mathcal{S}_k(0), i \in \mathcal{F} \setminus (\mathcal{F}_3 \bigcup \mathcal{F}_4)$, where $\mathcal{F}_3$ and $\mathcal{F}_4$ are two nonempty sets satisfying $\mathcal{F}_3 \bigcap \mathcal{F}_4 = \emptyset$. The proof follows the same four steps as in the proof of Case 4 in Lemma 3.2 and the analysis in Cases 1 and 2 by showing that $x^+_{F(k)} \le x^+_{L(k)}$ and $x^-_{F(k)} \ge x^-_{L(k)}$ as $t \to \infty$. □

Remark 4.4. By comparing Lemma 3.2 and Theorem 4.3, it can be observed that the introduction of the nonsmooth term in (13) is helpful if at least one leader has a nonzero velocity (i.e., $\gamma_\ell > 0$), but could be harmful if all leaders are stationary (i.e., $\gamma_\ell = 0$). The main reason is that while the existence of the nonsmooth term in (13) helps compensate for the unknown bounded time-varying velocities of the dynamic leaders and hence has a beneficial effect on the convergence to the minimal hyperrectangle, it has an adverse effect on the convergence to the convex hull due to the fact that the coordinate transformation technique used in the proof of Theorem 3.2 is not applicable to the closed-loop system of (1) using (13).

Remark 4.5. Considering a special case of Theorem 4.3 when there exists only one dynamic leader, the dynamic minimal hyperrectangle that contains the dynamic leader becomes the state of the dynamic leader. According to Theorem 4.3, all followers' states will converge to that of the dynamic leader, which is consistent with that in Cao, Ren, and Meng (2010).

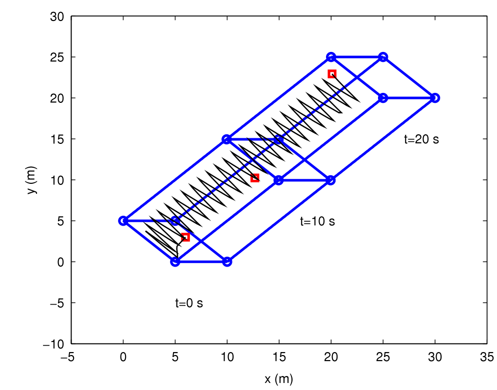


Fig. 4. A counterexample to illustrate that the follower cannot converge to the dynamic convex hull in the 2D space. The red square represents the position of the follower and the blue circles represent the positions of the four leaders. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

Remark 4.6. Under a switching directed network topology, the case of the stationary leaders (c.f. Section 3.2) is not a special case of the dynamic leaders (c.f. Section 4.2). The reasons are: (i) the convex hull in Section 3.2 is a tighter set than the minimum hyperrectangle in Section 4.2, (ii) weaker condition on the network topology is required in Section 3.2 than in Section 4.2, and (iii) the coordinate transformation technique used in Section 3.2 cannot be used in Section 4.2.

Remark 4.7. To illustrate that all followers might not converge to the dynamic convex hull formed by the dynamic leaders except for the 1-D case in the case of dynamic leaders under a switching directed network topology, we present the following counterexample. Consider a group of five agents with four leaders and one follower where the leaders have the same velocity. The network topology switches from Fig. 3(a) to (b) every 0.4 s and the process repeats. Simulation results using (13) in the 2D space are given in Fig. 4 where the red square represents the position of the follower and the blue circles represent the positions of the four leaders. From the simulation results, it can be seen that even if the follower is originally within the convex hull, it cannot always stay within the convex hull although Condition 2.2 is satisfied in the directed graph at each time interval. Instead, the follower will converge to the minimal rectangle that contains the dynamic leaders and each of whose edges is normal to one axis of $\mathcal{C}_0$.

Remark 4.8. For a high-dimensional space, the $\mathrm{sgn}(\nu)$ function in (13) can also be defined as[8]

$$\mathrm{sgn}(\nu) = \begin{cases} \mathbf{0}_p, & \nu = \mathbf{0}_p, \\ \dfrac{\nu}{\|\nu\|}, & \text{otherwise}. \end{cases} \tag{22}$$

Under this definition, the closed-loop system is independent of the coordinate frame. However, all followers might still not converge

8 In a one-dimensional space, $\mathrm{sgn}(\nu)$ becomes the standard signum function.

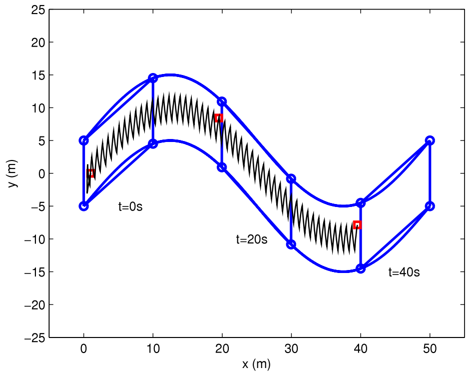


Fig. 5. A counterexample to illustrate that the follower cannot converge to the dynamic convex hull in the 2D space when sgn(·) is defined in (22). The red square represents the position of the follower and the blue circles represent the positions of the four leaders. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

to the dynamic convex hull formed by the dynamic leaders. Similar to the example in Remark 4.7, we consider four leaders and one follower where the leaders have the same velocity and let the network topology switch according to the same pattern as in Remark 4.7. Simulation results are given in Fig. 5. It can be noted that the follower cannot converge to the dynamic convex hull formed by the dynamic leaders even if the follower is initially within the convex hull.[9]

Remark 4.9. For distributed containment control without velocity measurements in the presence of multiple dynamic leaders under a switching network topology, it is, in general, impossible to find distributed tracking control algorithms without velocity measurements to guarantee that all followers will converge to the dynamic convex hull formed by the dynamic leaders in a high-dimensional space. In a one-dimensional space, the degree of freedom of the dynamic leaders is 1 and only the minimum and maximum states of the dynamic leaders are required to determine the dynamic convex hull formed by the dynamic leaders. Therefore, the signum function can be used to drive all followers to the dynamic convex hull formed by the dynamic leaders under a switching network topology given that the network topology and the control gain satisfy the conditions in Theorem 4.3. However, in a high-dimensional space, the degree of freedom of the dynamic leaders is larger than 1. The dynamic convex hull formed by the dynamic leaders might depend on a number of leaders' states (instead of only the minimum and maximum states of the dynamic leaders in a one-dimensional space). Therefore, the signum function, in general, does not have the capability to drive all followers to the dynamic convex hull formed by the dynamic leaders in a high-dimensional space under a switching network topology. Similarly, without velocity measurements, the basic linear distributed control algorithms do not have such capability either. Therefore, more information (i.e., velocity measurements, topology switching sequence, topologies, etc.) is needed in order to guarantee distributed containment control with multiple dynamic leaders under a switching network topology in a high-dimensional space.

5. Simulation

In this section, we present several simulation results to validate the previous theoretical results. We consider a group of agents with 4 leaders and 6 followers.

[9] The followers might not even converge to the minimal hyperrectangle that contains the dynamic leaders and each of whose hyperplanes is normal to one axis of $\mathcal{C}_0$ in this case.

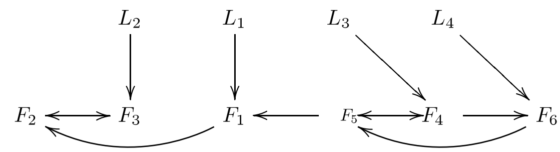


Fig. 6. Fixed directed network topology for a group of agents with four leaders and six followers. Here L_i, $i = 1, \ldots, 4$, denote the leaders while F_i, $i = 1, \ldots, 6$, denote the followers.

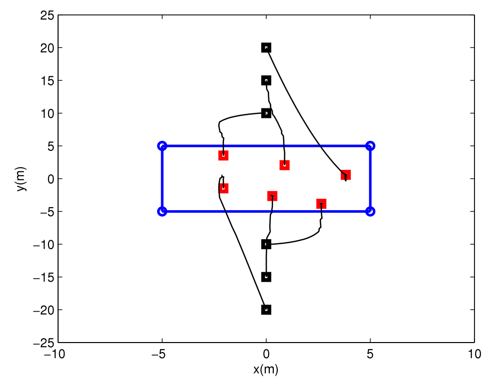


Fig. 7. Trajectories of the agents using (3) under a fixed directed network topology in the 2D space.

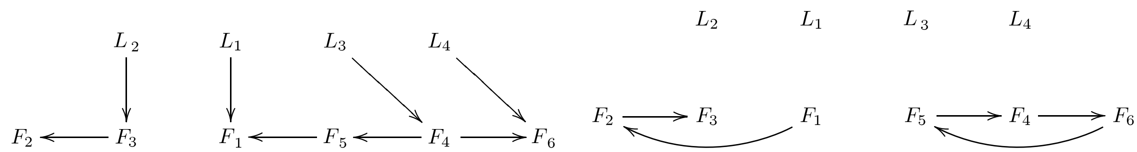
When the directed graph $\mathcal{G}$ is fixed, the interaction pattern is chosen as in Fig. 6. It can be noted that Condition 2.2 is satisfied in $\mathcal{G}$. The simulation result using (3) for (1) is shown in Fig. 7. We can see that all followers ultimately converge to the stationary convex hull formed by the stationary leaders and the final positions of the followers are constant.

When the directed graph $\mathcal{G}$ is switching, the interaction pattern is chosen as in Fig. 8. Note that Condition 2.2 is not satisfied in either Fig. 8(a) or (b). However, the union of Fig. 8(a) and (b) is Fig. 3(a), in which Condition 2.2 is satisfied. The simulation result using (3) for (1) is shown in Fig. 9 when the directed network topology switches between Fig. 8(a) and (b) every 1 s. Fig. 9(a) and (b) show, respectively, the trajectories of the agents from $t = 0$ to 20 s and the trajectories of the agents from $t = 5$ to 20 s. From these two figures, it can be seen that the follows ultimately converge to the stationary convex hull formed by the stationary leaders despite the fact that the directed network topology is switching. In particular, the final positions of the followers are not constant because the directed network topology is switching.

Fig. 10 shows the trajectories of the agents using (13) for (1) when the fixed directed graph $\mathcal{G}$ is given by Fig. 3(a). It can be noted that all followers ultimately converge to the dynamic convex hull formed by the dynamic leaders.

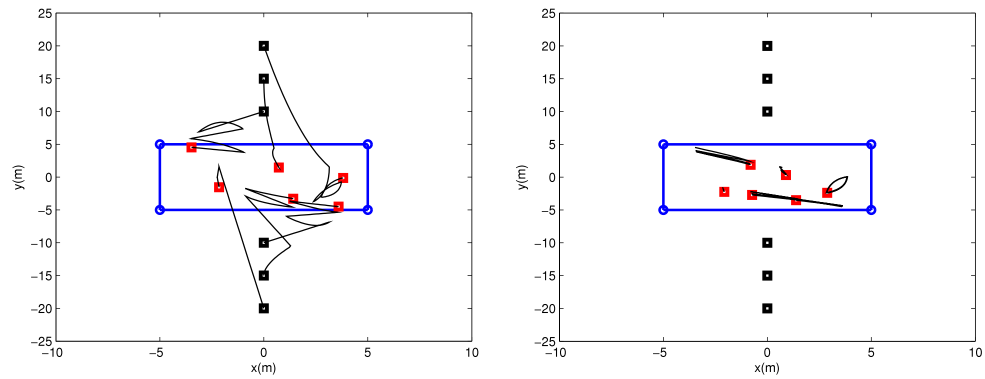
6. Conclusion and future works

This paper studied the distributed containment control problem of mobile autonomous agents with multiple stationary or dynamic leaders under both fixed and switching directed network topologies. In the case of stationary leaders, we showed necessary and sufficient conditions on the directed network topology to guarantee distributed containment control in a space of any finite dimension. In the case of dynamic leaders, we proposed a distributed tracking control algorithm without velocity measurements and studied the condition on the directed network topology and the control gains to guarantee distributed containment control. When the directed network topology is fixed, it was shown that the proposed algorithm can guarantee distributed containment control in a space of any finite dimension. When the directed network topology is switching, we showed that the proposed algorithm can guarantee distributed containment control



(a) Graph $\mathcal{G}_1$. (b) Graph $\mathcal{G}_2$.

Fig. 8. Switching directed network topologies for a group of agents with four leaders and six followers. Here L_i, $i = 1, \ldots, 4$, denote the leaders while F_i, $i = 1, \ldots, 6$, denote the followers.



(a) $t \in (0, 20)$ s. (b) $t \in (5, 20)$ s.

Fig. 9. Trajectories of the agents using (3) under a switching directed network topology in the 2D space. Circles denote the starting positions of the stationary leaders while the black and red squares denote, respectively, the starting and ending positions of the followers. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

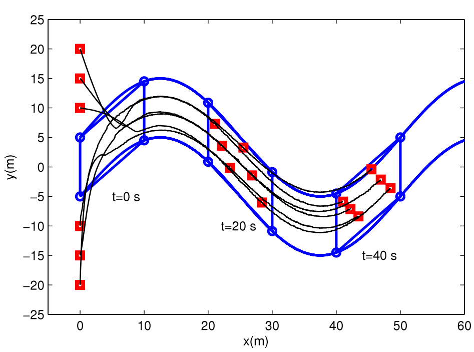


Fig. 10. Trajectories of the agents using (13) under a fixed directed network topology in the 2D space. Circles denote the positions of the dynamic leaders while the squares denote the positions of the followers. Two snapshots at $t = 20$ and $t = 40$ s show that all followers converge to the dynamic convex hull formed by the dynamic leaders.

only in a one-dimensional space. We also showed via some counterexamples that it is, in general, impossible to find distributed containment control algorithms without velocity measurements to guarantee distributed containment control in a high-dimensional space when the network topology is switching. Future work includes the study of containment control with dynamic leaders in a high-dimensional space with switching interaction.

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