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Distributed formation control for fractional-order systems: Dynamic interaction and absolute/relative damping

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ABSTRACT

This paper studies the distributed formation control problem for multiple fractional-order systems under dynamic interaction and with absolute/relative damping. In the context of this paper, formation control means that a group of systems reaches the desired state deviations via a local interaction. We first study a formation control algorithm in the case of a directed dynamic network topology. The convergence conditions on both the network topology and the fractional orders are presented. When the fractional-order α satisfies $\alpha \in (0, 1) \bigcup (1+\frac{2}{n})$, sufficient conditions on the network topology are given to ensure the formation control. We then propose fractional-order formation control algorithms with absolute/relative damping and study the conditions on the network topology and the control gains such that the formation control will be achieved under a directed fixed network topology. The final equilibria are also given explicitly. Finally, several simulation examples are presented as a proof of concept.

1. Introduction

In the past decade, distributed motion coordination has received significant research attention in the control community. Examples include rendezvous [1–4], consensus [5–9], formation control [10–13], and flocking [14–17]. One challenge in distributed motion coordination is that collective group behavior is achieved through local interaction.

While the existing results in distributed motion coordination primarily assume an integer-order dynamics, many phenomena cannot be explained in the framework of integer-order dynamics, for example, the synchronized motion of agents in fractional circumstances such as the macromolecule fluids and porous media [18], where each individual agent demonstrates a noninteger-order (i.e., fractional-order) dynamics rather than integer-order dynamics as shown in [19–21]. Also, many other phenomena can be explained naturally by a coordinated behavior of agents with fractional-order dynamics. Examples include chemotaxis behavior and food seeking of microbes and the collective motion of bacteria in lubrications perspired by themselves, where the behaviors of the microbes and bacteria are modeled by fractional-order dynamics [22,23]. Considering the fact that systems may demonstrate fractional-order dynamics under many complicated circumstances, the authors in [18] studied a formation control algorithm for the fractional-order systems. Conditions on both the network topology and the fractional orders were provided to ensure the formation control under a directed fixed network topology.

In this paper, we study the distributed formation control of the multiple fractional-order systems under dynamic interaction and with absolute/relative damping. In the context of the current paper, we use the term formation control to refer to the behavior that a group of systems reaches the desired state deviation via a local interaction. We extend the results in [18] in the following two aspects: (1) Analyze a formation control algorithm for the fractional-order systems under a directed dynamic network topology. The motivation for considering a directed dynamic network topology is from the observation that the interaction among different systems may be dynamic and directed due to unreliable communication, limited communication/sensing range, and/or sensing with a limited field of view. Therefore, it is necessary and meaningful to study the conditions on the network topology and the fractional orders such that the formation control will be achieved under a directed dynamic network topology. When the fractional-order α satisfies $\alpha \in (0, 1) \bigcup \left(1+\frac{2}{n}\right)$, we show the sufficient conditions on the network topology to ensure the formation control. (2) Study formation control algorithms for fractional-order systems with absolute/relative damping under a directed fixed network topology. The motivation for introducing fractional-order damping into the formation control algorithms is due to either the existence of a fractional-order damping when vehicles work in complicated environments or the fact that the fractional-order damping can be used to improve the stability margin. We derive the conditions on the network topology and the control gains such that formation control will be achieved when there exist, respectively, absolute and relative damping

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under a directed fixed network topology. In addition, the final equilibria are given explicitly. A preliminary version of the current paper was presented at the 48th IEEE Conference on Decision and Control [24].

The remainder of this paper is organized as follows: In Section 2, we introduce the graph theory notions, Caputo fractional operator and an existing formation control algorithm for fractional-order systems. This section provides the basic information that will be used in the following several sections. Section 3 focuses on the convergence analysis of the existing formation control algorithm for fractional-order systems under a directed dynamic network topology. Section 4 focuses on the convergence analysis of formation control algorithms for fractional-order systems with absolute/relative damping. In Section 5, several simulation examples are presented to show the effectiveness of the theoretical results. A short conclusion is given in Section 6.

2. Preliminaries

2.1. Graph theory notions

Given the n system, where $n \geq 2$, the interaction for them can be naturally modeled by a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{W})$, where $\mathcal{V} = \{v_1, v_2, \dots, v_n\}$ and $\mathcal{W} \subseteq \mathcal{V}^2$ represent, respectively, the system set and the edge set. Each edge denoted as (v_i, v_j) means that the system j can access the state information of the system i, but not necessarily vice versa. A directed path is a sequence of edges in a directed graph of the form $(v_1, v_2), (v_2, v_3), \dots$, where $v_i \in \mathcal{V}$. A directed graph has a directed spanning tree if there exists at least one system that has a directed path to all other systems. The union of a set of directed graphs $\mathcal{G}_{i_1}, \dots, \mathcal{G}_{i_m}$ is a directed graph with the edge set given by the union of the edge sets of the directed graphs $\mathcal{G}_{i_j}, j = 1, \dots, m$.

The interaction can be represented by two types of matrices: the adjacency matrix $A = [a_{ij}] \in \mathbb{R}^{n \times n}$ with $a_{ij} > 0$ if $(v_j, v_i) \in \mathcal{W}$ and $a_{ij} = 0$ otherwise, and the (nonsymmetric) Laplacian matrix $L = [\ell_{ij}] \in \mathbb{R}^{n \times n}$ with $\ell_{ii} = \sum_{j=1}^{n} a_{ij}$ and $\ell_{ij} = -a_{ij}$, $i \neq j$. It is straightforward to verify that L has at least one zero eigenvalue with a corresponding eigenvector $\mathbf{1}$, where $\mathbf{1}$ is an all-one column vector with a compatible size.

2.2. Caputo fractional operator

There are mainly two frequently used fractional operators: Caputo and Riemann–Liouville (R–L) fractional operators. In physical systems, the Caputo fractional operator is more practical than the R–L fractional operator because the R–L fractional operator has initial value problems [25]. Therefore, we will use the Caputo fractional operator in this paper to model the system's dynamics and analyze the stability of the proposed formation control algorithms. Generally, the Caputo fractional operator includes the Caputo integral and the Caputo derivative. The Caputo integral is defined from the Heaviside unit step function as

$$ {}_a^C D_t^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \frac{f(\tau)}{(t-\tau)^{1-\alpha}} \mathrm{d}\tau, $$

where $\Gamma(\cdot)$ is the Gamma function, $\alpha \in (0, 1]$, a is an arbitrary real number, and ${}_a^C D_t^{-\alpha}$ denotes the Caputo integral with an order α. For an arbitrary real number p, the Caputo derivative/integral is defined as

$$ {}_a^C D_t^{p} f(t) = {}_a^C D_t^{-\alpha} \left[\frac{\mathrm{d}^{[p]+1}}{\mathrm{d}t^{[p]+1}} f(t) \right], \tag{1} $$

where $\alpha = [p] + 1 - p \in (0, 1]$ and $[p]$ is the integer part of p. If p is an integer, then $\alpha = 1$ and (1) is equivalent to the integer-order derivative. For simplicity, in the following of this paper, a simple notation $f^{(\alpha)}$ will be used to replace ${}_a^C D_t^{\alpha} f$. When $\alpha > 0$, $f^{(\alpha)}$ is called a fractional derivative. When $\alpha < 0$, $f^{(\alpha)}$ is called a fractional integral.

Next we will introduce the Laplace transform of the Caputo derivative, and the Mittag–Leffler function which will be used in the following analysis. Let $\mathcal{L}\{\cdot\}$ denote the Laplace transform of a function. It follows from the formal definition of the Laplace transform $F(s) = \mathcal{L}\{f(t)\} = \int_{0^-}^{\infty} \mathrm{e}^{-st} f(t) \mathrm{d}t$ that the Laplace transform $\mathcal{L}\{f^{(\alpha)}(t)\}$ satisfies

$$ \mathcal{L}\{f^{(\alpha)}(t)\} = \begin{cases} s^{\alpha} F(s) + s^{\alpha-1} f(0^-), & \alpha \in (0, 1] \\ s^{\alpha} F(s) + s^{\alpha-1} f(0^-) + s^{\alpha-2} \dot{f}(0^-), & \alpha \in (1, 2], \end{cases} $$

where $f(0^-) = \lim_{\epsilon \to 0^-} f(\epsilon)$ and $\dot{f}(0^-) = \lim_{\epsilon \to 0^-} \dot{f}(\epsilon)$. Similar to the exponential function which is used frequently in integer-order systems, there is a frequently used function in fractional-order systems called the Mittag–Leffler function, which is defined as

$$ E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha + \beta)}, \tag{2} $$

where $\alpha, \beta \in \mathbb{C}$. When $\beta = 1$ and $\alpha > 0$, (2) can be written in a special case as $E_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha+1)}$. The Mittag–Leffler matrix function is defined as

$$ E_{\alpha,\beta}(B) = \sum_{k=0}^{\infty} \frac{B^k}{\Gamma(k\alpha + \beta)}, \tag{3} $$

where $B \in \mathbb{R}^{n \times n}$. Accordingly, when $\beta = 1$ and $\alpha > 0$, (2) becomes $E_{\alpha}(B) = \sum_{k=0}^{\infty} \frac{B^k}{\Gamma(k\alpha+1)}$.

2.3. Formation control algorithm for fractional-order systems

Define $\delta_{ij} \triangleq \delta_i - \delta_j$. Here δ_{ij} denotes the desired state deviation between the system i and the system j. For n systems with the fractional-order dynamics given by

$$ x_i^{(\alpha)}(t) = u_i(t), \tag{4} $$

where $x_i(t) \in \mathbb{R}$ and $u_i(t) \in \mathbb{R}$ represent, respectively, the state and the control input of system i, and $x_i^{(\alpha)}(t)$ is the αth Caputo derivative of $x_i(t)$ with $\alpha \in \mathbb{R}$,[1] a formation control algorithm for (4) was studied in [18] as

$$ u_i(t) = \sum_{j=1}^{n} a_{ij} [x_j(t) - x_i(t) - \delta_{ij}], \tag{5} $$

where a_{ij} is the (i, j)th entry of the adjacency matrix A. The objective of (5) is to achieve the formation control, i.e., $x_i(t) - x_j(t) \to \delta_{ij}$ as $t \to \infty$ for $i \neq j$. Using (5), (4) can be written in matrix form as

$$ \tilde{X}^{(\alpha)}(t) = -L\tilde{X}(t), \tag{6} $$

where $\tilde{X}(t) = [\tilde{x}_1(t), \dots, \tilde{x}_n(t)]^T \in \mathbb{R}^n$ with $\tilde{x}_i(t) = x_i(t) - \delta_i$ and L is the Laplacian matrix. For a given matrix L, the existence and uniqueness of the solution for (6) is always guaranteed [26].

3. Convergence analysis of the fractional-order systems under a directed dynamic interaction

In this section, we derive the conditions on the network topology and the fractional orders such that the formation control

[1] In contrast, α is an integer in integer-order systems.

will be achieved for the fractional-order system (6) under a directed dynamic network topology.

We assume that the interaction is constant over the time interval $[\sum_{j=1}^{k}\Delta_j, \sum_{j=1}^{k+1}\Delta_j)$ and switches at the time $t = \sum_{j=1}^{k}\Delta_j$ with $k = 0, 1, \ldots,$[2] where $\Delta_j > 0, j = 1, \ldots$. Let $\mathcal{G}_k$ and A_k denote, respectively, the directed graph and the adjacency matrix for $t \in [\sum_{j=1}^{k}\Delta_j, \sum_{j=1}^{k+1}\Delta_j)$. We also assume that each nonzero entry of A_k has a lower bound $\underline{a}$ and an upper bound $\bar{a}$, where $\underline{a}$ and $\bar{a}$ are positive constants with $\bar{a} \geq \underline{a}$. Then (6) becomes

$$\tilde{X}^{(\alpha)}[k+1] = -L_k\tilde{X}[k], \tag{7}$$

where $L_k \in \mathbb{R}^{n\times n}$ represents the Laplacian matrix associated with A_k.

3.1. Convergence analysis for $0 < \alpha < 1$

In this subsection, we focus on the case where $0 < \alpha < 1$. We have the following result.

Theorem 3.1. *Assume that $\alpha \in (0, 1)$. Using (5) for (4), a necessary condition to guarantee the formation control is that there exists a finite constant N such that the union of $\mathcal{G}_j, j = k, k+1, \ldots, k+N$, has a directed spanning tree for any finite k. Furthermore, if $\mathcal{G}_j, j = 0, 1, \ldots$, has a directed spanning tree at each time interval, there exists a positive $\overline{\Delta}_i$ such that the formation control will be achieved globally when $\Delta_i > \overline{\Delta}_i$.*[3]

Proof. For the first statement, when there does not exist a finite constant N such that the union of $\mathcal{G}_j, j = k, \ldots, k+N$, has a directed spanning tree for some k, it follows that at least one system, labeled as i, is separated from the other systems for $t \in [\sum_{j=1}^{k}\Delta_j, \infty)$. It follows that the state of the system i is independent of the states of the other systems for $t \geq \sum_{j=1}^{k}\Delta_j$, which implies that all systems cannot always achieve the formation control for arbitrary initial conditions.

For the second statement, it follows from Theorem 3.9 in [27] that

$$\tilde{X}(t) = E_\alpha(-Lt^\alpha)\tilde{X}(0).$$

Therefore, the solution to (7) is given by

$$\tilde{X}\left(\sum_{j=1}^{k}\Delta_j\right) = \prod_{i=1}^{k} E_\alpha\left(-L_k\left(\sum_{j=1}^{k+1}\Delta_j\right)^\alpha\right) \times \left[E_\alpha\left(-L_k\left(\sum_{j=1}^{k}\Delta_j\right)^\alpha\right)\right]^{-1} E_\alpha(-L_0\Delta_1)^\alpha\tilde{X}(0). \tag{8}$$

Define $\bar{x} \triangleq \max_i \tilde{x}_i$, $\underline{x} \triangleq \min_i \tilde{x}_i$, and $V \triangleq \max_i \tilde{x}_i - \min_i \tilde{x}_i$. It follows from Theorem 3.1 in [18] that $\tilde{x}_i$ converges to $\tilde{x}_j$ as $t \to \infty$ if the network topology has a directed spanning tree. That is, there exists a positive $\overline{\Delta}_1$ such that $V(t) < V(0)$ for any $t \geq \overline{\Delta}_1$. Similarly, by considering $[E_\alpha(-L_1(\Delta_1)^\alpha)]^{-1}E_\alpha(-L_0\Delta_1)^\alpha\tilde{X}(0)$ as the new initial state, it follows that there exists $\overline{\Delta}_2$ such that $V(t+\Delta_1) < V(\Delta_1)$ for any $t > \overline{\Delta}_2$. By following a similar analysis, there also exist $\overline{\Delta}_3, \ldots$. When $\Delta_i \geq \overline{\Delta}_i$, $V(\sum_{j=1}^{i+1}\Delta_k) < V(\sum_{j=1}^{i}\Delta_k)$. Therefore, $V(\sum_{j=1}^{i}\Delta_k) \to 0$ as $i \to \infty$. Therefore, $\tilde{x}_i[k] \to \tilde{x}_j[k]$, i.e., $x_i[k] - x_j[k] \to \delta_{ij}$ as $k \to \infty$ under the condition of the theorem. ■

Remark 3.2. For system $\dot{x}_i(t) = u_i(t)$, $x_i(t)$ will decrease if $u_i(t) < 0$ and $x_i(t)$ will increase if $u_i(t) > 0$. However, for system $x_i^{(\alpha)}(t) = u_i(t)$ with $\alpha \in (0, 1)$, due to the long memory process of the fractional calculus, the aforementioned properties do not necessarily hold. Therefore, even if the switching network topology has a directed spanning tree at each time interval, the formation control might not be achieved ultimately because the switching sequence also plays an important role.

3.2. Convergence analysis for $1 < \alpha < 1 + \frac{2}{n}$

In this subsection, we focus on the case where $1 < \alpha < 1 + \frac{2}{n}$, where $n \geq 2$. When the directed network topology is fixed, we have the following lemma regarding the solution of (6).

Lemma 3.1. *When $\alpha \in (1, 2)$, the solution of (6) is*

$$\tilde{X}(t) = E_\alpha(-Lt^\alpha)\tilde{X}(0) + tE_{\alpha,2}(-Lt^\alpha)\dot{\tilde{X}}(0). \tag{9}$$

Proof. Consider the fractional-order system given by (6). By applying the Laplace transform to both sides of (6), it follows that

$$s^{-(2-\alpha)}[\mathcal{L}\{\ddot{\tilde{X}}(t)\}] = -L\tilde{X}(s). \tag{10}$$

Eq. (10) can be written as

$$s^{-(2-\alpha)}[s^2\tilde{X}(s) - s\tilde{X}(0) - \dot{\tilde{X}}(0)] = -L\tilde{X}(s). \tag{11}$$

After some manipulation, (11) can be written as

$$\tilde{X}(s) = (s^\alpha I_n + L)^{-1}s^{\alpha-1}\tilde{X}(0) + (s^\alpha I_n + L)^{-1}s^{\alpha-2}\dot{\tilde{X}}(0). \tag{12}$$

By applying the inverse Laplace transform to (12), it follows from Theorem 3.2 in [27] that (9) is a solution of (6). Noting also that L is a constant matrix, it follows from the uniqueness and existence theorem of fractional equations in [26] that (9) is the unique solution of (6). ■

Taking the derivative of (9) with respect to t gives that

$$\dot{\tilde{X}}(t) = \frac{1}{t}E_{\alpha,0}(-Lt^\alpha)\tilde{X}(0) + E_\alpha(-Lt^\alpha)\dot{\tilde{X}}(0). \tag{13}$$

Combining (9) and (13) leads to the following matrix form

$$\begin{bmatrix}\tilde{X}(t)\\ \dot{\tilde{X}}(t)\end{bmatrix} = \begin{bmatrix}E_\alpha(-Lt^\alpha) & tE_{\alpha,2}(-Lt^\alpha)\\ \frac{1}{t}E_{\alpha,0}(-Lt^\alpha) & E_\alpha(-Lt^\alpha)\end{bmatrix}\begin{bmatrix}\tilde{X}(0)\\ \dot{\tilde{X}}(0)\end{bmatrix}. \tag{14}$$

Therefore, we can get that

$$\begin{bmatrix}\tilde{X}(\Delta_1)\\ \dot{\tilde{X}}(\Delta_1)\end{bmatrix} = \begin{bmatrix}E_\alpha(-L_0\Delta_1^\alpha) & \Delta_1 E_{\alpha,2}(-L_0\Delta_1^\alpha)\\ \frac{1}{\Delta_1}E_{\alpha,0}(-L_0\Delta_1^\alpha) & E_\alpha(-L_0\Delta_1^\alpha)\end{bmatrix}\begin{bmatrix}\tilde{X}(0)\\ \dot{\tilde{X}}(0)\end{bmatrix}.$$

Similarly, we can also get that

$$\begin{bmatrix}\tilde{X}(\Delta_k)\\ \dot{\tilde{X}}(\Delta_k)\end{bmatrix} = \prod_{i=1}^{k} C_{k-i}B_{0,0}\begin{bmatrix}\tilde{X}(0)\\ \dot{\tilde{X}}(0)\end{bmatrix}, \tag{15}$$

where $C_k = B_{k+1,k+1}B_{k+1,k}^{-1}$ with

$$B_{m,n} = \begin{bmatrix} E_\alpha\left(-L_m\left(\sum_{i=1}^{n+1}\Delta_i\right)^\alpha\right) & \sum_{i=1}^{n+1}\Delta_i E_{\alpha,2}\left(-L_m\left(\sum_{i=1}^{n+1}\Delta_i\right)^\alpha\right)\\ \dfrac{E_{\alpha,0}\left(-L_m\left(\sum_{i=1}^{n+1}\Delta_i\right)^\alpha\right)}{\sum_{i=1}^{n+1}\Delta_i} & E_\alpha\left(-L_m\left(\sum_{i=1}^{n+1}\Delta_i\right)^\alpha\right)\end{bmatrix}$$

where $C_0 \triangleq I_{2n}$ is the $2n$ by $2n$ identity matrix. Note that unlike the integer-order systems, there does not exist a transition matrix for

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[2] We define $\sum_{j=1}^{k}\Delta_j \triangleq 0$ when $k = 0$.

[3] Here the values of $\overline{\Delta}_i, i = 1, \ldots$, depend on the initial states, the fractional-order α, and $\mathcal{G}_k$.

fractional-order systems. Therefore, the analysis for the fractional-order systems is more challenging than that for the integer-order systems. Next we show the sufficient conditions on the directed dynamic network topology such that the formation control will be achieved.

Theorem 3.3. *Assume that $\alpha \in \left(1, 1+\frac{2}{n}\right)$ and $\mathcal{G}_k$ has a directed spanning tree. Define $V(t) \triangleq \max_j \tilde{x}_j(t) - \min_j \tilde{x}_j(t)$. For (15), there exists a positive $\bar{\Delta}_i$ such that $V(t) < V(\sum_{j=1}^{i-1} \Delta_j)$ for any $\Delta_i \geq \bar{\Delta}_i$ when $t \geq \sum_{j=1}^{i} \Delta_j$, $i = 1, \ldots$.[4] In addition, if $\Delta_i \geq \bar{\Delta}_i$, the formation control will be achieved globally.*

Proof. For the first statement, when the directed fixed network topology has a directed spanning tree, it follows from Theorem 3.3 in [18] that the formation control will be achieved for $\alpha \in \left(1, 1+\frac{2}{n}\right)$. It then follows that there exists a positive $\bar{\Delta}_1$ such that $V(t) < V(0)$ for any $t > \bar{\Delta}_1$. Similarly, by considering $B_{1,0}^{-1} B_{0,0} \begin{bmatrix} X(0) \\ \dot{X}(0) \end{bmatrix}$ the new initial state, it follows that there exists a positive $\bar{\Delta}_2$ such that $V(\Delta_1 + t) < V(\Delta_1)$ for any $t > \bar{\Delta}_2 + \Delta_1$. Similarly, we can also show the existence of Δ_i, $i = 3, \ldots$.

For the second statement, because $V(\sum_{j=1}^{i+1} \Delta_k) < V(\sum_{j=1}^{i} \Delta_j)$, it follows that $V(\sum_{j=1}^{i} \Delta_j) \to 0$ as $i \to \infty$. Therefore, we can get that $\tilde{x}_i[k] \to \tilde{x}_j[k]$, i.e., $x_i[k] - x_j[k] \to \delta_{ij}$ as $k \to \infty$ under the condition of the theorem. ■

Remark 3.4. Theorems 3.1 and 3.3 can be extended to the case when the fractional order $\alpha \in \left(1, 1+\frac{2}{n}\right)$ is constant for $t \in [\sum_{j=1}^{k} \Delta_j, \sum_{j=1}^{k+1} \Delta_j)$ and switches at $t = \sum_{j=1}^{k} \Delta_j$.

4. Convergence analysis of the fractional-order formation control algorithms with absolute/relative damping

In this section, we propose the fractional-order formation control algorithms with absolute/relative damping and then study the conditions on the network topology and the fractional orders such that the formation control will be achieved when using these algorithms for the fractional-order systems under a directed fixed network topology.

4.1. Absolute damping

For n ($n \geq 2$) systems with the dynamics given by (4), we propose the following fractional-order formation control algorithm with absolute damping as

$$u_i(t) = -\sum_{j=1}^{n} a_{ij}[x_i(t) - x_j(t) - (\delta_i - \delta_j)] - \beta x_i^{(\alpha/2)}(t), \tag{16}$$

where $\beta \in \mathbb{R}^+$ and $\delta_i \in \mathbb{R}$ is constant. Using (16), (4) can be written in matrix form as

$$\tilde{X}^{(\alpha)}(t) + \beta \tilde{X}^{(\alpha/2)}(t) + L\tilde{X}(t) = 0, \tag{17}$$

where $\tilde{X}(t)$ and L are defined in (6). It then follows that (17) can be written as

$$\begin{bmatrix} \tilde{X}(t) \\ \tilde{X}^{(\alpha/2)}(t) \end{bmatrix}^{(\alpha/2)} = \underbrace{\begin{bmatrix} \mathbf{0}_{n\times n} & I_n \\ -L & -\beta I_n \end{bmatrix}}_{F} \begin{bmatrix} \tilde{X}(t) \\ \tilde{X}^{(\alpha/2)}(t) \end{bmatrix}, \tag{18}$$

where $\mathbf{0}_{n\times n}$ is the n by n all-zero matrix. Note that each eigenvalue of L, λ_i, corresponds to two eigenvalues of F, denoted by $\mu_{2i-1} = \frac{-\beta+\sqrt{\beta^2-4\lambda_i}}{2}$ and $\mu_{2i} = \frac{-\beta-\sqrt{\beta^2-4\lambda_i}}{2}$ [28].

Note that F can be written in the Jordan canonical form as

$$F = P \underbrace{\begin{bmatrix} \Lambda_1 & 0 & \cdots & 0 \\ 0 & \Lambda_2 & \cdots & 0 \\ \cdots & \cdots & & \\ 0 & 0 & \cdots & \Lambda_k \end{bmatrix}}_{\Lambda} P^{-1},$$

where Λ_m, $m = 1, 2, \ldots, k$, are standard Jordan blocks. By defining $Z(t) = [z_1(t), \ldots, z_n(t)]^T \triangleq P^{-1} \begin{bmatrix} \tilde{X}(t) \\ \tilde{X}^{(\alpha/2)}(t) \end{bmatrix}$, (18) can be written as

$$Z^{(\alpha/2)}(t) = \Lambda Z(t). \tag{19}$$

Suppose that each diagonal entry of Λ_i is μ_i (i.e., an eigenvalue of F). Similar to the analysis in [18], by noting that the standard Jordan Block Λ_i has the form $\begin{bmatrix} \mu_i & 1 & 0 & \cdots & 0 \\ 0 & \mu_i & 1 & \cdots & 0 \\ \cdots & \cdots & & & \\ 0 & 0 & 0 & \cdots & \mu_i \end{bmatrix}$, (19) can be decoupled into n one-dimensional equations represented by either

$$z_i^{(\alpha/2)}(t) = \mu_i z_i(t) \tag{20}$$

which corresponds to the equation when the dimension of Λ_i is equal to 1, or the last equation when the dimension of Λ_i is larger than 1, or

$$z_i^{(\alpha/2)}(t) = \mu_i z_i(t) + z_{i+1}(t) \tag{21}$$

otherwise. Before moving on, we need the following lemmas.

Lemma 4.1 ([18]). *Let $\mathrm{Re}(\cdot)$ denote the real part of a complex number. The solution of (20) has the following properties:*

1. *When $\alpha \in \left(0, \frac{4\theta_i}{\pi}\right)$ and $\mathrm{Re}(\mu_i) < 0$, $\lim_{t\to\infty} z_i(t) \to 0$ as $t \to \infty$, where $\theta_i = \pi - |\arg\{-\mu_i\}|$ with $\arg\{-\mu_i\}$ denoting the phase of $-\mu_i$.*[5]
2. *When $\alpha \in (0, 2]$ and $\mu_i = 0$, $z_i(t) \equiv z_i(0)$, $\forall t$.*
3. *When $\alpha \in (2, 4)$ and $\mu_i = 0$, $z_i(t) = z_i(0) + \dot{z}_i(0)t$.*
4. *When $\alpha \in (4, \infty)$, the system is not stable.*

Lemma 4.2 ([18]). *Suppose that the continuous function $z_{i+1}(t)$ satisfies $\lim_{t\to\infty} z_{i+1}(t) = 0$. When $\alpha \in \left(0, \frac{4\theta_i}{\pi}\right)$, where $\theta_i = \pi - |\arg\{-\mu_i\}|$, the solution of (21) satisfies $\lim_{t\to\infty} z_i(t) = 0$.*

Lemma 4.3 ([29]). *Let λ_i be the ith eigenvalue of L, μ_{2i-1} and μ_{2i} are the two eigenvalues of F corresponding to λ_i, and $\mathrm{Im}(\cdot)$ denotes the imaginary part of a complex number. When $\mathrm{Re}(\lambda_i) > 0$, $\mathrm{Re}(\mu_{2i-1}) < 0$ and $\mathrm{Re}(\mu_{2i}) < 0$ if and only if $\beta > \sqrt{\frac{[\mathrm{Im}(\lambda_i)]^2}{\mathrm{Re}(\lambda_i)}}$.*

Theorem 4.1. *Let λ_i be the ith eigenvalue of L, and μ_{2i-1} and μ_{2i} be the two eigenvalues of F corresponding to λ_i. Define $\theta \triangleq \min_{\mu_i\neq 0, i=1,2,\ldots,2n} \theta_i$, where $\theta_i = \pi - |\arg\{-\mu_i\}|$. Using (16) for (4), the formation control will be achieved if the directed fixed network topology has a directed spanning tree and $\alpha \in \left(0, \frac{4\theta}{\pi}\right)$. In particular, the following properties hold.*

Case 1: $\beta > \max_{\lambda_i\neq 0} \sqrt{\frac{[\mathrm{Im}(\lambda_i)]^2}{\mathrm{Re}(\lambda_i)}}$. When $\alpha \in (0, 2]$, $\tilde{x}_i(t)$ and $\tilde{x}_j(t)$ converge to $\mathbf{p}^T \tilde{X}(0) + \frac{1}{\beta} \mathbf{p}^T \tilde{X}^{(\alpha/2)}(0)$ as $t \to \infty$, where $\mathbf{p}$ is the left eigenvector of L associated with the zero eigenvalue satisfying

[4] Here the values of $\bar{\Delta}_i$, $i = 1, \ldots$, depend on the initial states, the fractional-order α, and $\mathcal{G}_k$.

[5] We follow the convention that $\arg\{x\} \in (-\pi, \pi]$ for $x \in \mathbb{C}$.

$\mathbf{p}^T\mathbf{1} = 1$. *When* $\alpha \in \left(2, \frac{4\theta}{\pi}\right)$,[6] $\tilde{x}_i(t)$ *and* $\tilde{x}_j(t)$ *converge to* $\mathbf{p}^T\tilde{X}(0) + \frac{1}{\beta}\mathbf{p}^T\tilde{X}^{(\alpha/2)}(0) + \left[\mathbf{p}^T\dot{\tilde{X}}(0) + \frac{1}{\beta}\mathbf{p}^T\tilde{X}^{(1+\alpha/2)}(0)\right]t$, *and* $\dot{\tilde{x}}_i(t)$ *and* $\dot{\tilde{x}}_j(t)$ *converge to* $\mathbf{p}^T\dot{\tilde{X}}(0) + \frac{1}{\beta}\mathbf{p}^T\tilde{X}^{(1+\alpha/2)}(0)$ *as* $t \to \infty$.

Case 2: $0 < \beta \le \max_{\lambda_i \neq 0}\sqrt{\frac{[\mathrm{Im}(\lambda_i)]^2}{\mathrm{Re}(\lambda_i)}}$. *Then we have that* $\tilde{x}_i(t)$ *and* $\tilde{x}_j(t)$ *and* $\mathbf{p}^T\tilde{X}(0) + \frac{1}{\beta}\mathbf{p}^T\tilde{X}^{(\alpha/2)}(0)$ *as* $t \to \infty$.

Proof (*Proof of Case 1*). When the directed fixed network topology has a directed spanning tree, L has a simple zero eigenvalue and all the other eigenvalues have positive real parts [30,7]. Without loss of generality, let $\lambda_1 = 0$ and $\mathrm{Re}(\lambda_i) > 0, i \neq 1$. For $\lambda_1 = 0$, it follows that $\mu_1 = 0$ and $\mu_2 = -\beta$. Because $-\beta < 0$, it follows from Property 1 of Lemma 4.1 that $z_2(t) \to 0$ as $t \to \infty$. When $\alpha \in (0, 2]$, because $\mu_1 = 0$ is a simple zero eigenvalue, μ_1 satisfies (20). It follows from Property 2 in Lemma 4.1 that $z_1(t) \equiv z_1(0)$. When $\beta > \max_{\lambda_i \neq 0}\sqrt{\frac{[\mathrm{Im}(\lambda_i)]^2}{\mathrm{Re}(\lambda_i)}}$, it follows from Lemma 4.3 that $\mathrm{Re}(\mu_{2i-1}) < 0$ and $\mathrm{Re}(\mu_{2i}) < 0, i \neq 1$. When μ_{2i-1} and μ_{2i} satisfy (20), it then follows from Property 1 of Lemma 4.1 that $z_{2i-1}(t) \to 0$ and $z_{2i}(t) \to 0$ as $t \to \infty$. When μ_{2i-1} satisfies (20) and μ_{2i} satisfies (21), it then follows from Lemmas 4.1 and 4.2 that $z_{2i-1}(t) \to 0$ and $z_{2i}(t) \to 0$ as $t \to \infty$ as well. Recalling the structure of the standard Jordan block, by following the previous analysis, it can be shown that $z_{2i-1}(t) \to 0$ and $z_{2i}(t) \to 0$ as $t \to \infty$ when μ_{2i-1} and μ_{2i} satisfy (21). Combining the above arguments gives $\lim_{t\to\infty} Z(t) = [z_1(0), 0, \ldots, 0]^T$, which implies $\lim_{t\to\infty}\begin{bmatrix}\tilde{X}(t)\\ \tilde{X}^{(\alpha/2)}(t)\end{bmatrix} = \lim_{t\to\infty} PZ(t) = PSZ(0) = PSP^{-1}\begin{bmatrix}\tilde{X}(0)\\ \tilde{X}^{(\alpha/2)}(0)\end{bmatrix}$, where $S = [s_{ij}] \in \mathbb{R}^{n\times n}$ has only one nonzero entry $s_{11} = 1$. Note that the first column of P can be chosen as $[\mathbf{1}^T, \mathbf{0}^T]^T$ while the first row of P^{-1} can be chosen as $\left[\mathbf{p}^T, \frac{1}{\beta}\mathbf{p}^T\right]^T$ by noting that $[\mathbf{1}^T, \mathbf{0}^T]^T$ and $\left[\mathbf{p}^T, \frac{1}{\beta}\mathbf{p}^T\right]^T$ are, respectively, a right and left eigenvector of F associated with $\mu_1 = 0$ and $\left[\mathbf{p}^T, \frac{1}{\beta}\mathbf{p}^T\right][\mathbf{1}^T, \mathbf{0}^T]^T = 1$, where $\mathbf{0}$ is an all-zero column vector with a compatible size. Therefore, $\lim_{t\to\infty}\begin{bmatrix}\tilde{X}(t)\\ \tilde{X}^{(\alpha/2)}(t)\end{bmatrix} = PSP^{-1}\begin{bmatrix}\tilde{X}(0)\\ \tilde{X}^{(\alpha/2)}(0)\end{bmatrix} = [\mathbf{1}^T, \mathbf{0}^T]\left[\mathbf{p}^T, \frac{1}{\beta}\mathbf{p}^T\right]^T\begin{bmatrix}\tilde{X}(0)\\ \tilde{X}^{(\alpha/2)}(0)\end{bmatrix}$, that is, $\lim_{t\to\infty}\tilde{x}_i(t) = \mathbf{p}^T\tilde{X}(0) + \frac{1}{\beta}\mathbf{p}^T\tilde{X}^{(\alpha/2)}(0)$.

When $\alpha \in \left(2, \frac{4\theta}{\pi}\right)$, it follows from Property 3 of Lemma 4.1 that $z_1(t) = z_1(0) + \dot{z}_1(0)t$. A similar discussion to that for $\alpha \in (0, 2]$ shows that $z_i(t) \to 0$ as $t \to \infty$ for $i = 3, \ldots, 2n$. Therefore, it follows that $\lim_{t\to\infty} Z(t) = [z_1(0) + \dot{z}_1(0)t, 0, \ldots, 0]^T$, which implies that $\lim_{t\to\infty}\dot{Z}(t) = [\dot{z}_1(0), 0, \ldots, 0]^T$. Similar to the proof for $\alpha \in (0, 2]$, we can get that $\lim_{t\to\infty}\tilde{x}_i(t) = \mathbf{p}^T\tilde{X}(0) + \frac{1}{\beta}\mathbf{p}^T\tilde{X}^{(\alpha/2)}(0) + \left[\mathbf{p}^T\dot{\tilde{X}}(0) + \frac{1}{\beta}\mathbf{p}^T\tilde{X}^{(1+\alpha/2)}(0)\right]t$ and $\lim_{t\to\infty}\dot{\tilde{x}}_i(t) = \mathbf{p}^T\dot{\tilde{X}}(0) + \frac{1}{\beta}\mathbf{p}^T\tilde{X}^{(1+\alpha/2)}(0)$.

(Proof of Case 2) When $0 < \beta \le \max_{\lambda_i \neq 0}\sqrt{\frac{[\mathrm{Im}(\lambda_i)]^2}{\mathrm{Re}(\lambda_i)}}$, it follows from Lemma 4.3 that $\mathrm{Re}(\mu_{2i-1}) \ge 0$ for some i, which implies that $\frac{4\theta}{\pi} \le 2$. Therefore, we can get that $\alpha \in (0, 2)$. The proof then follows a similar analysis to that of Case 1 when $\alpha \in (0, 2]$. ■

Remark 4.2. From Theorem 4.1, it can be noted that the control gain β can be chosen as any positive number. In particular, the possible range of α to ensure formation control will be different depending on β. In addition, when there exists an absolute damping, the final velocity may not be zero as shown in Theorem 3.1, which is different from the results in [13,28]. The existing formation control algorithms for double-integrator dynamics with absolute damping studied in [13,28] can be viewed as a special case of Theorem 4.1 when $\alpha = 2$.

4.2. Relative damping

For n ($n \ge 2$) systems with the dynamics given by (4), we propose the following fractional-order formation control algorithm with the relative damping as

$$u_i(t) = -\sum_{j=1}^{n} a_{ij}\{x_i(t) - x_j(t) - (\delta_i - \delta_j) + \gamma[x_i^{(\alpha/2)}(t) - x_j^{(\alpha/2)}(t)]\}, \tag{22}$$

where $\gamma \in \mathbb{R}^+$ and $\delta_i \in \mathbb{R}$ is constant. Using (22), (4) can be written in matrix form as

$$\tilde{X}^{(\alpha)}(t) + \gamma L\tilde{X}^{(\alpha/2)}(t) + L\tilde{X}(t) = 0, \tag{23}$$

where $\tilde{X}(t)$ and L are defined in (6). It follows that (23) can be written as

$$\begin{bmatrix}\tilde{X}(t)\\ \tilde{X}^{(\alpha/2)}(t)\end{bmatrix}^{(\alpha/2)} = \underbrace{\begin{bmatrix}\mathbf{0}_{n\times n} & I_n\\ -L & -\gamma L\end{bmatrix}}_{G}\begin{bmatrix}\tilde{X}(t)\\ \tilde{X}^{(\alpha/2)}(t)\end{bmatrix}. \tag{24}$$

Note that each eigenvalue of L, λ_i, also corresponds to two eigenvalues of G, denoted by $\mu_{2i-1} = \frac{-\gamma\lambda_i + \sqrt{\gamma^2\lambda_i^2 - 4\lambda_i}}{2}$ and $\mu_{2i} = \frac{-\gamma\lambda_i - \sqrt{\gamma^2\lambda_i^2 - 4\lambda_i}}{2}$ [12].

Note that G can also be written in the Jordan canonical form as

$$G = Q\underbrace{\begin{bmatrix}\Sigma_1 & 0 & \cdots & 0\\ 0 & \Sigma_2 & \cdots & 0\\ \cdots & \cdots & & \\ 0 & 0 & \cdots & \Sigma_k\end{bmatrix}}_{\Sigma}Q^{-1},$$

where $\Sigma_m, m = 1, 2, \ldots, k$, are standard Jordan blocks. By defining $Z(t) = [z_1(t), \ldots, z_n(t)]^T \triangleq Q^{-1}\begin{bmatrix}\tilde{X}(t)\\ \tilde{X}^{(\alpha/2)}(t)\end{bmatrix}$, (24) can be written as

$$Z^{(\alpha/2)}(t) = \Sigma Z(t). \tag{25}$$

Suppose that each diagonal entry of Σ_i is μ_i (i.e., an eigenvalue of G). Similar to the analysis of (19), (25) can be decoupled into n one-dimensional equations represented by either (20) or (21). Before moving on, we need the following lemma.

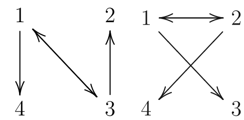
Lemma 4.4. *Let* λ_i *be the ith eigenvalue of* L, *and* μ_{2i-1} *and* μ_{2i} *be the two eigenvalues of* G *corresponding to* λ_i. *Suppose that* $\mathrm{Re}(\lambda_i) > 0$. *Then* $\mathrm{Re}(\mu_{2i-1}) < 0$ *and* $\mathrm{Re}(\mu_{2i}) < 0$ *if and only if* $\gamma > \bar{\gamma}_i$, *where* $\bar{\gamma}_i \triangleq \sqrt{\frac{\mathrm{Im}(\lambda_i)^2}{\mathrm{Re}(\lambda_i)|\lambda_i|^2}}$.

Proof. The characteristic polynomial of G is given by

$$s(s + \gamma\lambda_i) + \lambda_i = 0. \tag{26}$$

Letting s_1 and s_2 be the two roots of (26), it follows from (26) that $s_1 + s_2 = -\gamma\lambda_i$. Because $\mathrm{Re}(\lambda_i) > 0$, at least one of the two roots is in the open left half plane if $\gamma > 0$. Note that the bound of γ, $\bar{\gamma}_i$, can be obtained when one of the two roots is on the imaginary axis. Without loss of generality, we let $s_1 = z\mathbf{j}$, where z is a real constant and $\mathbf{j}$ is the imaginary unit. Substituting $s_1 = z\mathbf{j}$ into (26) gives that $-z^2 + \bar{\gamma}_i\lambda_i z\mathbf{j} + \lambda_i = 0$. After some manipulation, we can get that $\bar{\gamma}_i$ satisfies $-\mathrm{Im}(\lambda_i)^2 + \bar{\gamma}_i^2\mathrm{Im}(\lambda_i)^2\mathrm{Re}(\lambda_i) + \bar{\gamma}_i^2\mathrm{Re}(\lambda_i)^3 = 0$, which can be simplified as $\bar{\gamma}_i = \sqrt{\frac{\mathrm{Im}(\lambda_i)^2}{\mathrm{Re}(\lambda_i)|\lambda_i|^2}}$. ■

6 Note that $\frac{4\theta}{\pi} > 2$ because $\theta > \frac{\pi}{2}$ according to Lemma 4.1.



(a) $\mathcal{G}_1$. (b) $\mathcal{G}_2$.

Fig. 1. Directed network topology for four systems. An arrow from *j* to *i* denotes that system *i* can receive information from system *j*.

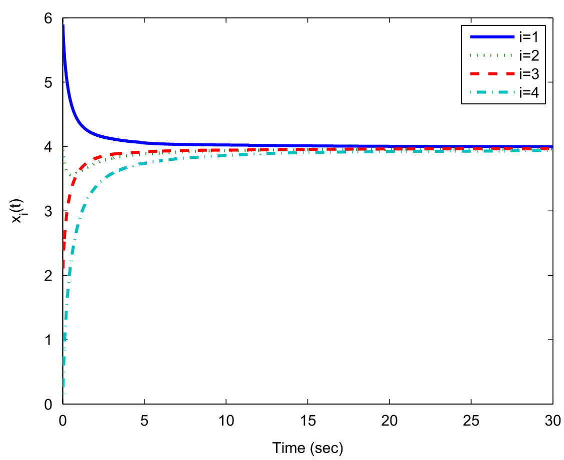


Fig. 2. States of the four systems using (5) when the network topology switches between Fig. 1(a) and (b) every 3 s with $\alpha = 0.8$.

Theorem 4.3. *Let λ_i be the ith eigenvalue of L, and μ_{2i-1} and μ_{2i} be the two eigenvalues of G corresponding to λ_i. Define $\bar{\gamma} \triangleq \max_{\lambda_i \neq 0} \bar{\gamma}_i$ with $\bar{\gamma}_i$ being defined in Lemma 4.4, and $\theta = \min_{\mu_i \neq 0, i=1,2,\ldots,2n} \theta_i$, where $\theta_i = \pi - |\arg\{-\mu_i\}|$. Using (23) for (4), the formation control will be achieved if the directed fixed network topology has a directed spanning tree and $\alpha \in \left(0, \frac{4\theta}{\pi}\right)$. In addition, the following properties hold.*

Case 1: $\gamma > \bar{\gamma}$. When $\alpha \in (0, 2]$, $\tilde{x}_i(t)$ and $\tilde{x}_j(t)$ converge to $\mathbf{p}^T \tilde{X}(0) + \frac{t^{\alpha/2}}{\Gamma(1+\alpha/2)} \mathbf{p}^T \tilde{X}^{(\alpha/2)}(0)$ as $t \to \infty$, where $\mathbf{p}$ is defined in Theorem 4.1. When $\alpha \in \left(2, \frac{4\theta}{\pi}\right)$,[7] $\tilde{x}_i(t)$ and $\tilde{x}_j(t)$ converge to $\mathbf{p}^T \tilde{X}(0) + \frac{t^{\alpha/2}}{\Gamma(1+\alpha/2)} \mathbf{p}^T \tilde{X}^{(\alpha/2)}(0) + \frac{t^{1+\alpha/2}}{\Gamma(\alpha/2+2)} \tilde{X}^{(\alpha/2+1)}(0)$ as $t \to \infty$.

Case 2: $\gamma \leq \bar{\gamma}$. Then we have that $\tilde{x}_i(t)$ and $\tilde{x}_j(t)$ converge to $\mathbf{p}^T \tilde{X}(0) + \frac{t^{\alpha/2}}{\Gamma(1+\alpha/2)} \mathbf{p}^T \tilde{X}^{(\alpha/2)}(0)$ as $t \to \infty$.

Proof (*Proof of Case 1*). When the directed fixed network topology has a directed spanning tree, L has a simple zero eigenvalue and all other eigenvalues have positive real parts [30,7]. Without loss of generality, let $\lambda_1 = 0$ and $\text{Re}(\lambda_i) > 0, i \neq 1$. For $\lambda_1 = 0$, it follows from (26) that $\mu_1 = 0$ and $\mu_2 = 0$. Because G has two zero eigenvalues whose geometric multiplicity is 1, it follows that $\mu_2 = 0$ satisfies (20) and $\mu_1 = 0$ satisfies (21). When $\alpha \in (0, 2]$, it follows from Property 2 in Lemma 4.1 that $z_2(t) \equiv z_2(0)$. By substituting $z_2(t) = z_2(0)$ into (21), it follows that

$$z_1(t) = z_2(0) \frac{t^{\alpha/2}}{\Gamma(1+\alpha/2)} + z_1(0). \tag{27}$$

We next study the case of $\lambda_i, i \neq 1$. Because $\text{Re}(\lambda_i) > 0, i \neq 1$, it follows from Lemma 4.3 that $\text{Re}(\mu_{2i-1}) < 0$ and $\text{Re}(\mu_{2i}) < 0$ when $\gamma > \bar{\gamma}$. By following a similar analysis to that in the proof of Theorem 4.1, it can be shown that $z_{2i-1}(t) \to 0$ and $z_{2i}(t) \to 0$ as $t \to \infty$ as well. Similar to the analysis in the proof of Theorem 4.1, it can also be computed that $w_1 = [\mathbf{1}^T, \mathbf{0}^T]^T$ and $v_1 = [\mathbf{0}^T, \mathbf{p}^T]^T$ are the right and left eigenvectors corresponding to $\mu_1 = 0$. Meanwhile, $w_2 = [\mathbf{0}^T, \mathbf{1}^T]^T$ and $v_2 = [\mathbf{p}^T, \mathbf{0}^T]^T$ are the generalized right and left eigenvectors corresponding to $\mu_2 = 0$, where $v_1^T w_2 = 1$ and $v_2^T w_1 = 1$. Therefore, the first and second columns of Q can be chosen as $[\mathbf{1}^T, \mathbf{0}^T]^T$ and $[\mathbf{0}^T, \mathbf{1}^T]^T$ while the first and second rows of Q^{-1} can be chosen as $[\mathbf{p}^T, \mathbf{0}^T]^T$ and $[\mathbf{0}^T, \mathbf{p}^T]^T$. Therefore, $\lim_{t\to\infty} \begin{bmatrix} \tilde{X}(t) \\ \tilde{X}^{(\alpha/2)}(t) \end{bmatrix} = \lim_{t\to\infty} QZ(t) = \lim_{t\to\infty} QSZ(0) = \lim_{t\to\infty} QSQ^{-1} \begin{bmatrix} \tilde{X}(0) \\ \tilde{X}^{(\alpha/2)}(0) \end{bmatrix}$, where $S = [s_{ij}] \in \mathbb{R}^{n \times n}$ has three entries which are not equal to zero, $s_{11} = 1$, $s_{12} = \frac{t^{\alpha/2}}{\Gamma(1+\alpha/2)}$ and $s_{22} = 1$, where s_{12} is derived from (27). After some manipulation, we can get that $\lim_{t\to\infty} \begin{bmatrix} \tilde{X}(t) \\ \tilde{X}^{(\alpha/2)}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{1}\mathbf{p}^T \tilde{X}(0) + \frac{t^{\alpha/2}}{\Gamma(1+\alpha/2)} \mathbf{1}\mathbf{p}^T \tilde{X}^{(\alpha/2)}(0) \\ \mathbf{1}\mathbf{p}^T \tilde{X}^{(\alpha/2)}(0) \end{bmatrix}$, that is, $\lim_{t\to\infty} \tilde{x}_i(t) = \mathbf{p}^T \tilde{X}(0) + \frac{t^{\alpha/2}}{\Gamma(1+\alpha/2)} \mathbf{p}^T \tilde{X}^{(\alpha/2)}(0)$.

When $\alpha \in \left(2, \frac{4\theta}{\pi}\right)$, it follows from Property 3 of Lemma 4.1 that $z_2(t) = z_2(0) + \dot{z}_2(0)t$. Because $z_1(t)$ satisfies (21), we can get that $z_1(t) = z_1(0) + z_2(0) \frac{t^{\alpha/2}}{\Gamma(\alpha/2+1)} + \dot{z}_2(0) \frac{t^{1+\alpha/2}}{\Gamma(\alpha/2+2)}$. A similar discussion to that for $\alpha \in (0, 2]$ shows that $z_i(t) \to 0$ as $t \to \infty$ for $i = 3, \ldots, 2n$. Therefore, it follows that $\lim_{t\to\infty} Z(t) = [z_1(0) + z_2(0) \frac{t^{\alpha/2}}{\Gamma(\alpha/2+1)} + \dot{z}_2(0) \frac{t^{1+\alpha/2}}{\Gamma(\alpha/2+2)}, z_2(0) + \dot{z}_2(0)t, 0, \ldots, 0]^T$. Similar to the proof for $\alpha \in (0, 2]$, we can get that $\lim_{t\to\infty} \tilde{x}_i(t) = \mathbf{p}^T \tilde{X}(0) + \frac{t^{\alpha/2}}{\Gamma(1+\alpha/2)} \mathbf{p}^T \tilde{X}^{(\alpha/2)}(0) + \frac{t^{1+\alpha/2}}{\Gamma(\alpha/2+2)} \tilde{X}^{(\alpha/2+1)}(0)$.

(Proof of Case 2) When $\gamma \leq \bar{\gamma}$, it follows from Lemma 4.4 that $\text{Re}(\mu_{2i-1}) \geq 0$ for some i, which implies that $\frac{4\theta}{\pi} \leq 2$. Therefore, we

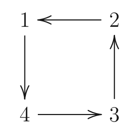


Fig. 3. Directed network topology for four systems. An arrow from *j* to *i* denotes that system *i* can receive information from system *j*.

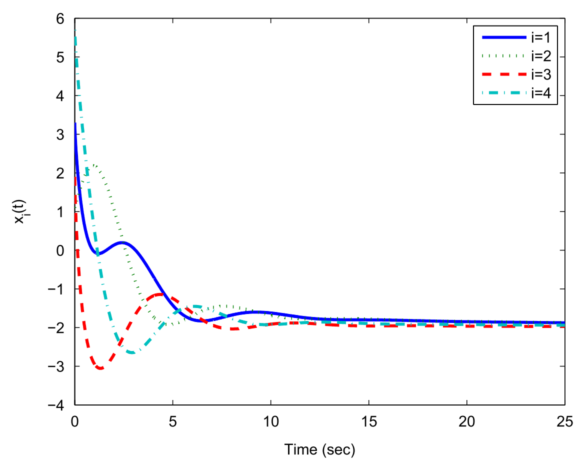
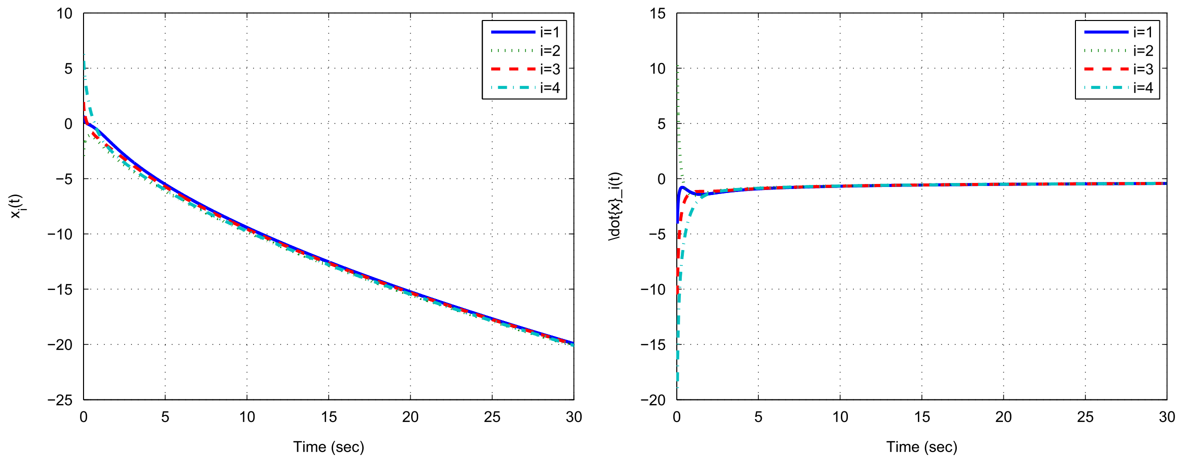


Fig. 4. States of the four systems using (16) with $\alpha = 1.6$ and $\beta = 1$ with the directed fixed network topology given by Fig. 3.

[7] Note that $\frac{4\theta}{\pi} > 2$ because $\theta > \frac{\pi}{2}$ according to Lemma 4.4.



(a) $x_i(t)$. (b) $\dot{x}_i(t)$.

Fig. 5. States of the four systems using (22) with $\alpha = 1.2$ and $\gamma = 1$ with the directed fixed network topology given by Fig. 3.

can get that $\alpha \in (0, 2)$. The proof then follows a similar analysis to that of Case 1 when $\alpha \in (0, 2]$. ■

Remark 4.4. From Theorem 4.3, it can be noted that the control gain γ can also be chosen to be any positive number. In particular, the range of α will be different depending on γ. In addition, when there exists a relative damping, the final velocity may not be constant as shown in Theorem 4.3, which is different from the results in [12]. The existing formation control algorithms for double-integrator dynamics with relative damping studied in [12] can be viewed as a special case of Theorem 4.3 when $\alpha = 2$.

5. Simulation

In this section, we present several simulation results to illustrate the theoretical results in Sections 3 and 4. We consider a network of four systems.

To illustrate the results in Section 3, we consider the case of a directed dynamic network topology with the interaction pattern chosen from Fig. 1. Note that both Fig. 1(a) and (b) have a directed spanning tree. Fig. 2 shows the states of the four systems using (5) when the network topology switches between Fig. 1(a) and (b) every 3 s with $\alpha = 0.8$. Here for simplicity we have chosen $\delta_i = 0$. It can be noted that the formation control is achieved with a directed dynamic network topology and time-varying fractional orders.

To illustrate the results in Section 4, we consider the case of a directed fixed network topology shown by Fig. 3 which has a directed spanning tree. The simulation result using (16) is shown in Fig. 4 when $\alpha = 1.6$ and $\beta = 1$. The simulation result using (22) is shown in Fig. 5 when $\alpha = 1.2$ and $\gamma = 1$. Here for simplicity we have again chosen $\delta_i = 0$. It can be noted from Figs. 4 and 5 that the formation control is achieved. In particular, it can be seen from the bottom subplot of Fig. 5 that using (22) the final velocity $\dot{x}_i(t)$ is no longer constant when $\alpha = 1.2$ and $\gamma = 1$.

6. Conclusion

In this paper, we studied the distributed formation control problem for multiple fractional-order systems under a dynamic interaction and with absolute/relative damping. We first studied a closed-loop fractional-order system in the case of a directed dynamic network topology. We derived the conditions on both the network topology and the fractional orders to ensure a formation control. When the order $\alpha \in (0, 1) \bigcup \left(1, 1 + \frac{2}{n}\right)$, sufficient conditions on the network topology were given to ensure the formation control. Then we proposed the fractional-order formation control algorithms with an absolute/relative damping and studied the conditions on the network topology and the control gains to ensure the formation control under a directed fixed network topology. In addition, the final equilibria were given explicitly.

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