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Multi-vehicle coordination for double-integrator dynamics under fixed undirected/directed interaction in a sampled-data setting

Yongcan Cao and Wei Ren*,†

Department of Electrical and Computer Engineering, Utah State University, Logan, UT 84322-4120, U.S.A.

SUMMARY

This paper studies the convergence of two coordination algorithms for double-integrator dynamics under fixed undirected/directed interaction in a sampled-data setting. The first algorithm guarantees that a team of vehicles achieves coordination on their positions with a zero final velocity while the second algorithm guarantees that a team of vehicles achieves coordination on their positions with a constant final velocity. We show necessary and sufficient conditions on the sampling period, the control gain, and the communication graph such that coordination is achieved using these two algorithms under, respectively, an undirected interaction topology and a directed interaction topology. Tools like matrix theory, bilinear transformation, and Cauchy theorem are used for convergence analysis. Coordination equilibria for both algorithms are also given. Simulation results are presented as a proof of concept. Copyright © 2009 John Wiley & Sons, Ltd.

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KEY WORDS: consensus; coordination; sampled-data; double-integrator systems; decentralized control

1. INTRODUCTION

Distributed multi-vehicle coordination has received significant attention in the control community in recent years. Examples include formation control [1], coverage control [2], flocking [3, 4], distributed estimation [5], and consensus [6]. Consensus plays an important role in achieving distributed multi-vehicle coordination. The basic idea of consensus is that a team of vehicles achieves an agreement on a common value by negotiating with their neighbors. By specifying desired separations among different vehicles, consensus algorithms can be applied to achieve distributed multi-vehicle formation control. Consensus algorithms for single-integrator kinematics have been studied extensively in the literature (see [6] and references therein).

Taking into account the fact that equations of motion of a broad class of vehicles require a double-integrator dynamic model, coordination algorithms for double-integrator dynamics are studied in [7–17]. In particular, [8–11, 17] derive conditions on the interaction topology and the control gains under which consensus is guaranteed. References [12, 13] study formation keeping problems while [14–16] study flocking of multiple vehicle systems. All these algorithms are studied in a continuous-time setting.

In multi-vehicle coordination, vehicles may only be able to exchange information periodically but not continuously, which results in discrete-time or

*Correspondence to: Wei Ren, Department of Electrical and Computer Engineering, Utah State University, Logan, UT 84322-4120, U.S.A.
†E-mail: wren@engineering.usu.edu

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sampled-data formulation. Current discrete-time coordination algorithms are primarily studied for first-order kinematic models [18–21]. The algorithms are essentially distributed weighted averaging algorithms [22–24]. Discrete-time coordination for double-integrator dynamics is studied in [25] where a simple forward difference is used to approximate both the velocities and accelerations in the continuous-time setting. However, existing discrete-time coordination algorithms do not often explicitly consider the effect of sampled-data control on stability of vehicles with mass despite the fact the sampling period plays an important role in stability when there are physical vehicle dynamics involved. Few works study coordination algorithms for double-integrator dynamics in a sampled-data setting except [26, 27]. In [26], an algorithm is studied for double-integrator dynamics through average-energy-like Lyapunov functions. The analysis in [26] is limited to an undirected interaction topology. However, in coordination applications, information flow may often be directed, either due to heterogeneity, nonuniform communication powers, or sensing with a limited field of view. The case of directed interaction is much more challenging than that of undirected interaction. Reference [27] studies the case of directed switching interaction and derive sufficient conditions by using the property of an infinity product of stochastic matrices. However, only sufficient conditions are derived due to the switching interaction topologies. In contrast to [26, 27], we focus on studying necessary and sufficient conditions for convergence in the case of fixed undirected/directed interaction in this paper. In contrast to [26] (respectively, [27]), our analysis in this paper is based on tools like matrix theory, bilinear transformation, and Cauchy theorem rather than a Lyapunov approach as in [26] (respectively, the property of an infinite product of stochastic matrices as in [27]). Our results generalize the convergence conditions derived in [26] and complement those derived in [27].

In this paper, we study the convergence of two sampled-data coordination algorithms for double-integrator dynamics under fixed undirected/directed interaction by expanding on our preliminary work reported in [28]. The first algorithm guarantees that a team of vehicles achieves coordination on their positions with a zero final velocity while the second algorithm guarantees that a team of vehicles achieves coordination on their positions with a constant final velocity. We show necessary and sufficient conditions on the sampling period, the control gain, and the communication graph such that coordination is achieved using these two algorithms under, respectively, an undirected interaction topology and a directed interaction topology. Tools like matrix theory, bilinear transformation, and Cauchy theorem are used for convergence analysis. Coordination equilibria for both algorithms are also given.

2. BACKGROUND AND PRELIMINARIES

2.1. Graph theory notions

It is natural to model interaction among vehicles by directed or undirected graphs. Suppose that a team consists of n vehicles. A weighted graph $\mathscr{G}$ consists of a node set $\mathscr{V}=\{1,\dots,n\}$, an edge set $\mathscr{E}\subseteq\mathscr{V}\times\mathscr{V}$, and a weighted adjacency matrix $\mathscr{A}=[a_{ij}]\in\mathbb{R}^{n\times n}$. An edge (i,j) in a weighted directed graph denotes that vehicle j can obtain information from vehicle i, but not necessarily vice versa. In contrast, the pairs of nodes in a weighted undirected graph are unordered, where an edge (i,j) denotes that vehicles i and j can obtain information from one another. Weighted adjacency matrix $\mathscr{A}$ of a weighted directed graph is defined such that a_{ij} is a positive weight if $(j,i)\in\mathscr{E}$, while $a_{ij}=0$ if $(j,i)\in\mathscr{E}$. Weighted adjacency matrix $\mathscr{A}$ of a weighted undirected graph is defined analogously except that $a_{ij}=a_{ji}$, $\forall i\neq j$, since $(j,i)\in\mathscr{E}$ implies $(i,j)\in\mathscr{E}$.

A directed path is a sequence of edges in a directed graph of the form $(i_1,i_2),(i_2,i_3),\dots$, where $i_j\in\mathscr{V}$. An undirected path in an undirected graph is defined analogously. A directed graph has a directed spanning tree if there exists at least one node having a directed path to all other nodes. An undirected graph is connected if there is an undirected path between every pair of distinct nodes.

Let the (nonsymmetric) Laplacian matrix $\mathscr{L}=[\ell_{ij}]\in\mathbb{R}^{n\times n}$ associated with $\mathscr{A}$ be defined as [29] $\ell_{ii}=\sum_{j=1,j\neq i}^{n}a_{ij}$ and $\ell_{ij}=-a_{ij},\ i\neq j$. For an undirected graph, $\mathscr{L}$ is symmetric positive semi-definite.

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However, $\mathscr{L}$ for a directed graph does not have this property. In both the undirected and directed cases, 0 is an eigenvalue of $\mathscr{L}$ with associated eigenvector $\mathbf{1}_n$, where $\mathbf{1}_n$ is the $n\times 1$ column vector of all ones.

2.2. Continuous-time coordination algorithms for double-integrator dynamics

Consider vehicles with double-integrator dynamics given by

$$\dot{r}_i = v_i, \quad \dot{v}_i = u_i, \quad i=1,\ldots,n \tag{1}$$

where $r_i \in \mathbb{R}^m$ and $v_i \in \mathbb{R}^m$ are, respectively, the position and velocity of the ith vehicle, and $u_i \in \mathbb{R}^m$ is the control input.

A coordination algorithm for (1) is studied in [8, 17] as

$$u_i = -\sum_{j=1}^{n} a_{ij}[(r_i - r_j) - (\delta_i - \delta_j)] - \alpha v_i, \quad i=1,\ldots,n \tag{2}$$

where $\delta_i, i=1,\ldots,n$, are real constants, a_{ij} is the (i,j)th entry of the weighted adjacency matrix $\mathscr{A}$ associated with graph $\mathscr{G}$, and α is a positive gain introducing absolute damping.[‡] Define $\Delta_{ij} = \delta_i - \delta_j$. Coordination is achieved for (2) if for all $r_i(0)$ and $v_i(0)$, $r_i(t) - r_j(t) \to \Delta_{ij}$ and $v_i(t) \to 0$ as $t \to \infty$.

A coordination algorithm for (1) is studied in [10] as

$$u_i = -\sum_{j=1}^{n} a_{ij}[(r_i - r_j) - (\delta_i - \delta_j) + \alpha(v_i - v_j)], \quad i=1,\ldots,n \tag{3}$$

where δ_i and a_{ij} are defined as in (2) and α is a positive gain introducing relative damping. Coordination is achieved for (3) if for all $r_i(0)$ and $v_i(0)$, $r_i(t) - r_j(t) \to \Delta_{ij}$ and $v_i(t) \to v_j(t)$ as $t \to \infty$.

[‡] In [8, 17] (respectively, [10] in the next paragraph), $\delta_i, i=1,\ldots,n$, are set to zero. When $\delta_i, i=1,\ldots,n$, are real constants, the results in [8, 17] (respectively, [10] in the next paragraph) still hold.

2.3. Sampled-data coordination algorithms for double-integrator dynamics

We consider a sampled-data setting where the vehicles have continuous-time dynamics while the measurements are made at discrete sampling times and the control inputs are based on zero-order hold as

$$u_i(t) = u_i[k], \quad kT \leqslant t < (k+1)T \tag{4}$$

where k denotes the discrete-time index, T denotes the sampling period, and $u_i[k]$ is the control input at $t=kT$. By using direct discretization in [30], the continuous-time system (1) can be discretized as

$$\begin{aligned} r_i[k+1] &= r_i[k] + T v_i[k] + \frac{T^2}{2} u_i[k] \\ v_i[k+1] &= v_i[k] + T u_i[k] \end{aligned} \tag{5}$$

where $r_i[k]$ and $v_i[k]$ denote, respectively, the position and velocity of the ith vehicle at $t=kT$. Note that (5) is the exact discrete-time dynamics for (1) based on zero-order hold in a sampled-data setting.

We study the following two algorithms:

$$u_i[k] = -\sum_{j=1}^{n} a_{ij}[(r_i[k] - r_j[k]) - (\delta_i - \delta_j)] - \alpha v_i[k] \tag{6}$$

which corresponds to continuous-time algorithm (2) and

$$u_i[k] = -\sum_{j=1}^{n} a_{ij}[(r_i[k] - r_j[k]) - (\delta_i - \delta_j) + \alpha(v_i[k] - v_j[k])] \tag{7}$$

which corresponds to continuous-time algorithm (3). Note that [26] shows conditions for (7) under an undirected interaction topology through average-energy-like Lyapunov functions. Relying on algebraic graph theory and matrix theory, we will show necessary and sufficient conditions for convergence of both (6) and (7) under fixed undirected/directed interaction.

In the remainder of the paper, we assume that $\Delta_{ij} = 0$. However, all the results hereafter are valid for $\Delta_{ij} \neq 0$ by using $r_i - \delta_i$ to replace r_i. For simplicity, we suppose that $r_i \in \mathbb{R}$, $v_i \in \mathbb{R}$, and $u_i \in \mathbb{R}$. However, all results still hold for $r_i \in \mathbb{R}^m$, $v_i \in \mathbb{R}^m$, and $u_i \in \mathbb{R}^m$ by use of the properties of the Kronecker product.

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3. CONVERGENCE ANALYSIS OF THE SAMPLED-DATA ALGORITHM WITH ABSOLUTE DAMPING

In this section, we analyze algorithm (6) under, respectively, an undirected and a directed interaction topology. Before moving on, we need the following lemmas:

Lemma 3.1 (*Schur's formula* [*31*])
Let $A, B, C, D \in \mathbb{R}^{n\times n}$. Let $M = [\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}]$. Then $\det(M) = \det(AD - BC)$, where $\det(\cdot)$ denotes the determinant of a matrix, if A, B, C, and D commute pairwise.

Lemma 3.2
Let $\mathscr{L}$ be the nonsymmetric Laplacian matrix (respectively, Laplacian matrix) associated with directed graph $\mathscr{G}$ (respectively, undirected graph $\mathscr{G}$). Then $\mathscr{L}$ has a simple zero eigenvalue and all other eigenvalues have positive real parts (respectively, are positive) if and only if $\mathscr{G}$ has a directed spanning tree (respectively, is connected). In addition, there exist $\mathbf{1}_n$ satisfying $\mathscr{L}\mathbf{1}_n = 0$ and $\mathbf{p} \in \mathbb{R}^n$ satisfying $\mathbf{p} \geqslant 0$, $\mathbf{p}^{\mathrm{T}}\mathscr{L} = 0$, and $\mathbf{p}^{\mathrm{T}}\mathbf{1} = 1$.[§]

Proof
See [32] for the case of undirected graphs and [20] for the case of directed graphs. □

Lemma 3.3 ([*33*] *Lemma 8.2.7 part*(*i*), *p. 498*)
Let $A \in \mathbb{R}^{n\times n}$ be given, let $\lambda \in \mathbb{C}$ be given, and suppose x and y are vectors such that (i) $Ax = \lambda x$, (ii) $A^{\mathrm{T}}y = \lambda y$, and (iii) $x^{\mathrm{T}}y = 1$. If $|\lambda| = \rho(A) > 0$, where $\rho(A)$ denotes the spectral radius of A, and λ is the only eigenvalue of A with modulus $\rho(A)$, then $\lim_{m\to\infty}(\lambda^{-1}A)^m \to xy^{\mathrm{T}}$.

Using (6), (5) can be written in matrix form as

$$\begin{bmatrix} r[k+1] \\ v[k+1] \end{bmatrix} = \underbrace{\begin{bmatrix} I_n - \frac{T^2}{2}\mathscr{L} & \left(T - \frac{\alpha T^2}{2}\right) I_n \\ -T\mathscr{L} & (1-\alpha T) I_n \end{bmatrix}}_{F} \begin{bmatrix} r[k] \\ v[k] \end{bmatrix} \tag{8}$$

where $r = [r_1, \ldots, r_n]^{\mathrm{T}}$, $v = [v_1, \ldots, v_n]^{\mathrm{T}}$ and I_n denote the $n \times n$ identity matrix. To analyze (8), we first study the property of F. Note that the characteristic polynomial of F is given by

$$\begin{aligned} &\det(sI_{2n} - F) \\ &= \det\left(\begin{bmatrix} sI_n - (I_n - \frac{T^2}{2}\mathscr{L}) & -(T - \frac{\alpha T^2}{2}) I_n \\ T\mathscr{L} & sI_n - (1-\alpha T) I_n \end{bmatrix}\right) \\ &= \det\left(\left[sI_n - \left(I_n - \frac{T^2}{2}\mathscr{L}\right)\right][sI_n - (1-\alpha T) I_n] - \left(T\mathscr{L}\left[-\left(T - \frac{\alpha T^2}{2}\right) I_n\right]\right)\right) \\ &= \det\left((s^2 - 2s + \alpha T s + 1 - \alpha T) I_n + \frac{T^2}{2}(1+s)\mathscr{L}\right) \end{aligned}$$

where we have used Lemma 3.1 to obtain the second to the last equality.

Letting μ_i be the ith eigenvalue of $-\mathscr{L}$, we get $\det(sI_n + \mathscr{L}) = \prod_{i=1}^n (s - \mu_i)$. It thus follows that

$$\det(sI_{2n} - F) = \prod_{i=1}^{n} \left(s^2 - 2s + \alpha T s + 1 - \alpha T - \frac{T^2}{2}(1+s)\mu_i\right)$$

Therefore, the roots of $\det(sI_{2n} - F) = 0$ (i.e. the eigenvalues of F) satisfy

$$s^2 + \left(\alpha T - 2 - \frac{T^2}{2}\mu_i\right)s + 1 - \alpha T - \frac{T^2}{2}\mu_i = 0 \tag{9}$$

Note that each eigenvalue of $-\mathscr{L}$, μ_i, corresponds to two eigenvalues of F, denoted by λ_{2i-1} and λ_{2i}.

Without loss of generality, let $\mu_1 = 0$. It follows from (9) that $\lambda_1 = 1$ and $\lambda_2 = 1 - \alpha T$. Therefore, F has at least one eigenvalue equal to one. Let $[p^{\mathrm{T}}, q^{\mathrm{T}}]^{\mathrm{T}}$, where $p, q \in \mathbb{R}^n$, be the right eigenvector of F associated with eigenvalue $\lambda_1 = 1$. It follows that

$$\begin{bmatrix} I_n - \frac{T^2}{2}\mathscr{L} & \left(T - \frac{\alpha T^2}{2}\right) I_n \\ -T\mathscr{L} & (1-\alpha T) I_n \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} p \\ q \end{bmatrix}$$

After some manipulation, it follows from Lemma 3.2 that we can choose $p = \mathbf{1}_n$ and $q = \mathbf{0}_n$, where $\mathbf{0}_n$ is the $n \times 1$ column vector of all zeros. Similarly, it can be

[§]That is, $\mathbf{1}_n$ and $\mathbf{p}$ are, respectively, the right and left eigenvectors of $\mathscr{L}$ associated with the zero eigenvalue.

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shown that $[\mathbf{p}^{\mathrm{T}}, (1/\alpha - T/2)\mathbf{p}^{\mathrm{T}}]^{\mathrm{T}}$ is a left eigenvector of F associated with eigenvalue $\lambda_1 = 1$.

Lemma 3.4
Using (6) for (5), $r_i[k] \to \mathbf{p}^{\mathrm{T}} r[0] + (1/\alpha - T/2)\mathbf{p}^{\mathrm{T}} v[0]$ and $v_i[k] \to 0$ as $k \to \infty$ if and only if one is the unique eigenvalue of F with maximum modulus, where $\mathbf{p}$ is defined in Lemma 3.2.

Proof (*Sufficiency*)
Note that $x = [\mathbf{1}_n^{\mathrm{T}}, \mathbf{0}_n^{\mathrm{T}}]^{\mathrm{T}}$ and $y = [\mathbf{p}^{\mathrm{T}}, (1/\alpha - T/2)\mathbf{p}^{\mathrm{T}}]^{\mathrm{T}}$ are, respectively, a right and left eigenvector of F associated with eigenvalue one. Also note that $x^{\mathrm{T}} y = 1$. If one is the unique eigenvalue with maximum modulus, then it follows from Lemma 3.3 that

$$\lim_{k\to\infty} F^k \to \begin{bmatrix} \mathbf{1}_n \\ \mathbf{0}_n \end{bmatrix} \left[\mathbf{p}^{\mathrm{T}}, \left(\frac{1}{\alpha} - \frac{T}{2}\right)\mathbf{p}^{\mathrm{T}} \right]$$

Therefore, it follows that

$$\lim_{k\to\infty} \begin{bmatrix} r[k] \\ v[k] \end{bmatrix} = \lim_{k\to\infty} F^k \begin{bmatrix} r[0] \\ v[0] \end{bmatrix} = \begin{bmatrix} r[0] + \left(\frac{1}{\alpha} - \frac{T}{2}\right)\mathbf{p}^{\mathrm{T}} v[0] \\ \mathbf{0}_n \end{bmatrix}$$

(Necessity) Note that F can be written in Jordan canonical form as $F = PJP^{-1}$, where J is the Jordan block matrix. If $r_i[k] \to \mathbf{p}^{\mathrm{T}} r[0] + (1/\alpha - T/2)\mathbf{p}^{\mathrm{T}} v[0]$ and $v_i[k] \to 0$ as $k \to \infty$, it follows that

$$\lim_{k\to\infty} F^k \to \begin{bmatrix} \mathbf{1}_n \\ \mathbf{0}_n \end{bmatrix} \left[\mathbf{p}^{\mathrm{T}}, \left(\frac{1}{\alpha} - \frac{T}{2}\right)\mathbf{p}^{\mathrm{T}} \right]$$

which has rank one. It thus follows that $\lim_{k\to\infty} J^k$ has rank one, which implies that all but one eigenvalue are within the unit circle. Noting that F has at least one eigenvalue equal to one, it follows that one is the unique eigenvalue of F with maximum modulus. □

3.1. Undirected interaction

In this subsection, we show necessary and sufficient conditions on α and T such that coordination is achieved using (6) under an undirected interaction topology. Note that all eigenvalues of $\mathscr{L}$ are real for undirected graphs.

Lemma 3.5
The polynomial

$$s^2 + as + b = 0 \tag{10}$$

where $a, b \in \mathbb{C}$, has all roots within the unit circle if and only if all roots of

$$(1 + a + b)t^2 + 2(1 - b)t + b - a + 1 = 0 \tag{11}$$

are in the open left half plane (LHP).

Proof
By applying bilinear transformation $s = (t+1)/(t-1)$ [34], polynomial (10) can be rewritten as

$$(t+1)^2 + a(t+1)(t-1) + b(t-1)^2 = 0$$

which implies (11). Note that the bilinear transformation maps the open LHP one-to-one onto the interior of the unit circle. The lemma follows directly. □

Lemma 3.6
Suppose that the undirected graph $\mathscr{G}$ is connected. All eigenvalues of F, where F is defined in (8), are within the unit circle except one eigenvalue equal to one if and only if α and T are chosen from the set[¶]

$$S_r = \left\{ (\alpha, T) \mid -\frac{T^2}{2} \min_i \mu_i < \alpha T < 2 \right\} \tag{12}$$

where $\cap$ denotes the intersection of sets.

Proof
When undirected graph $\mathscr{G}$ is connected, it follows from Lemma 3.2 that $\mu_1 = 0$ and $\mu_i < 0$, $i = 2, \ldots, n$. Because $\mu_1 = 0$, it follows that $\lambda_1 = 1$ and $\lambda_2 = 1 - \alpha T$. To ensure $|\lambda_2| < 1$, it is required that $0 < \alpha T < 2$.

Let $a = \alpha T - 2 - (T^2/2)\mu_i$ and $b = 1 - \alpha T - (T^2/2)\mu_i$. It follows from Lemma 3.5 that for $\mu_i < 0, i = 2, \ldots, n$, the roots of (9) are within the unit circle if and only if all roots of

$$-T^2 \mu_i t^2 + (T^2 \mu_i + 2\alpha T)t + 4 - 2\alpha T = 0 \tag{13}$$

are in the open LHP. Because $-T^2\mu_i > 0$, the roots of (13) are always in the open LHP if and only if $T^2\mu_i + 2\alpha T > 0$ and $4 - 2\alpha T > 0$, which implies that $-(T^2/2)\mu_i < \alpha T < 2$, $i = 2, \ldots, n$. Combining the above arguments proves the lemma. □

¶Note that S_r is nonempty.

Theorem 3.1

Suppose that undirected graph $\mathscr{G}$ is connected. Let $\mathbf{p}$ be defined in Lemma 3.2. Using (6) for (5), $r_i[k] \to \mathbf{p}^{\mathrm{T}} r[0] + (1/\alpha - T/2)\mathbf{p}^{\mathrm{T}} v[0]$ and $v_i[k] \to 0$ as $k \to \infty$ if and only if α and T are chosen from S_r, where S_r is defined by (12).

Proof

The statement follows directly from Lemmas 3.4 and 3.6. □

Remark 3.2

From Lemma 3.6, we can get $T < 2/(\sqrt{-\mu_i})$. From the Gershgorin circle theorem, we know that $|\mu_i| \leqslant 2\max_i \ell_{ii}$. Therefore, if $T < \sqrt{2/\max_i \ell_{ii}}$, then we have $T < 2/(\sqrt{-\mu_i})$. Note that $\max_i \ell_{ii}$ represents the maximal in-degree of a graph. Therefore, the sufficient bound of the sampling period is related to the maximal in-degree of a graph.

3.2. Directed interaction

In this subsection, we first show necessary and sufficient conditions on α and T such that coordination is achieved using (6) under a directed interaction topology. Because it is not easy to find the explicit bounds for α and T such that the necessary and sufficient conditions are satisfied, we present sufficient conditions that can be used to compute the explicit bounds for α and T. Note that the eigenvalues of $\mathscr{L}$ may be complex for directed graphs, which makes the analysis more challenging.

Lemma 3.7

Suppose that the directed graph $\mathscr{G}$ has a directed spanning tree. Let $\mathrm{Re}(\cdot)$ and $\mathrm{Im}(\cdot)$ denote, respectively, the real and imaginary part of a number. There exist α and T such that the following three conditions are satisfied:

1. $0 < \alpha T < 2$;
2. When $\mathrm{Re}(\mu_i) < 0$ and $\mathrm{Im}(\mu_i) = 0$, $(\alpha, T) \in S_r$, where S_r is defined in (12);
3. When $\mathrm{Re}(\mu_i) < 0$ and $\mathrm{Im}(\mu_i) \neq 0$, α and T satisfy $\alpha/T > -|\mu_i|^2/2\mathrm{Re}(\mu_i)$ and $T < \overline{T}_i$, where

$$\overline{T}_i = \frac{-2\alpha \mathrm{Re}(\mu_i)[\mathrm{Re}(\mu_i)+\alpha] - 2\sqrt{\alpha^2[\mathrm{Re}(\mu_i)]^2[\mathrm{Re}(\mu_i)+\alpha]^2 - \mathrm{Re}(\mu_i)|\mu_i|^2[\mathrm{Im}(\mu_i)]^2}}{\mathrm{Re}(\mu_i)|\mu_i|^2} \quad (14)$$

In addition, all eigenvalues of F, where F is defined in (8), are within the unit circle except one eigenvalue equal to one if and only if the previous three conditions are satisfied.

Proof

For the first statement, when T is sufficiently small, there always exists α such that conditions (1), (2), and (3) are satisfied.

For the second statement, when $\mu_1 = 0$, it follows that $\lambda_1 = 1$ and $\lambda_2 = 1 - \alpha T$. Therefore, condition (1) guarantees that λ_2 is within the unit circle. When $\mathrm{Re}(\mu_i) < 0$ and $\mathrm{Im}(\mu_i) = 0$, it follows from Lemma 3.6 that all roots of F corresponding to μ_i are within the unit circle if and only if condition (2) is satisfied.

We next consider the case when $\mathrm{Re}(\mu_i) < 0$ and $\mathrm{Im}(\mu_i) \neq 0$. Letting t_1 and t_2 be the two roots of (13), it follows that

$$\mathrm{Re}(t_1) + \mathrm{Re}(t_2) = 1 + 2\frac{\alpha}{T}\frac{\mathrm{Re}(\mu_i)}{|\mu_i|^2}$$

Therefore, both t_1 and t_2 are in the open LHP only if

$$1 + 2\frac{\alpha}{T}\frac{\mathrm{Re}(\mu_i)}{|\mu_i|^2} < 0$$

i.e. $\alpha/T > -|\mu_i|^2/2\mathrm{Re}(\mu_i)$. To find the bound on T, we assume that one root of (13) is on the imaginary axis. Without loss of generality, let $t_1 = \chi\mathbf{j}$, where χ is a real constant and $\mathbf{j}$ is the imaginary unit. Substituting $t_1 = \chi\mathbf{j}$ into (13) and separating the corresponding real and imaginary parts give that

$$T^2 \mathrm{Re}(\mu_i)\chi^2 - T^2 \mathrm{Im}(\mu_i)\chi + 4 - 2\alpha T = 0 \quad (15)$$

$$T^2 \mathrm{Im}(\mu_i)\chi^2 + [T^2 \mathrm{Re}(\mu_i) + 2\alpha T]\chi = 0 \quad (16)$$

It follows from (16) that

$$\chi = -\frac{T\,\mathrm{Re}(\mu_i) + 2\alpha}{T\,\mathrm{Im}(\mu_i)} \quad (17)$$

By substituting (17) into (15) gives that

$$\frac{\mathrm{Re}(\mu_i)[T\,\mathrm{Re}(\mu_i)+2\alpha]^2}{[\mathrm{Im}(\mu_i)]^2} + T[T\,\mathrm{Re}(\mu_i)+2\alpha]+4-2\alpha T=0$$

After some simplifications, we get that

$$\mathrm{Re}(\mu_i)|\mu_i|^2T^2+4\alpha\mathrm{Re}(\mu_i)[\mathrm{Re}(\mu_i)+\alpha]T + 4[\mathrm{Im}(\mu_i)]^2=0 \tag{18}$$

Note that when $T=0$, the left side of (18) is larger than zero. Because $\mathrm{Re}(\mu_i)|\mu_i|^2<0$, there exists a unique positive $\overline{T}_i$ such that (14) holds when $T=\overline{T}_i$, where $\overline{T}_i$ is given by (14).

Combining the previous arguments completes the proof. □

Theorem 3.3
Suppose that directed graph $\mathscr{G}$ has a directed spanning tree. Let $\mathbf{p}$ be defined in Lemma 3.2. Using (6) for (5), $r_i[k]\to\mathbf{p}^{\mathrm{T}}r[0]+(1/\alpha-T/2)\mathbf{p}^{\mathrm{T}}v[0]$ and $v_i[k]\to 0$ as $k\to\infty$ if and only if α and T are chosen satisfying the conditions in Lemma 3.7.

Proof
The statement follows directly from Lemmas 3.4 and 3.7. □

From Lemma 3.7, it is not easy to find α and T explicitly such that the conditions in Lemma 3.7 are satisfied. We next present a sufficient condition in which α and T can be easily determined. Before moving on, we need the following lemmas.

Lemma 3.8 ([*35, 36*])
All the zeros of the complex polynomial

$$P(z)=z^n+\alpha_1 z^{n-1}+\cdots+\alpha_{n-1}z+\alpha_n$$

satisfy $|z|\leqslant r_0$, where r_0 is the unique nonnegative solution of the equation

$$r^n-|\alpha_1|r^{n-1}-\cdots-|\alpha_{n-1}|r-|\alpha_n|=0$$

The bound r_0 is attained if $\alpha_i=-|\alpha_i|$.

Corollary 3.4
All roots of polynomial (10) are within the unit circle if $|a|+|b|<1$. Moreover, if $|a+b|+|a-b|<1$, all roots of (10) are still within the unit circle.

Proof
According to Lemma 3.8, the roots of (10) are within the unit circle if the unique nonnegative solution s_0 of $s^2-|a|s-|b|=0$ satisfies $s_0<1$. It is straightforward to show that $s_0=(|a|+\sqrt{|a|^2+4|b|})/2$. Therefore, the roots of (10) are within the unit circle if

$$|a|+\sqrt{|a|^2+4|b|}<2 \tag{19}$$

We next discuss the condition under which (19) holds. If $b=0$, then the statements of the corollary hold trivially. If $|b|\neq 0$, we have

$$\frac{(|a|+\sqrt{|a|^2+4|b|})(-|a|+\sqrt{|a|^2+4|b|})}{-|a|+\sqrt{|a|^2+4|b|}}<2$$

After some computation, it follows that condition (19) is equivalent to $|a|+|b|<1$. Therefore, the first statement of the corollary holds. For the second statement, because $|a|+|b|\leqslant|a+b|+|a-b|$, if $|a+b|+|a-b|<1$, then $|a|+|b|<1$, which implies that the second statement of the corollary also holds. □

Lemma 3.9
Suppose that directed graph $\mathscr{G}$ has a directed spanning tree. There exist positive α and T such that $S_c\cap S_r$ is nonempty, where

$$S_c=\bigcap_{\forall\mathrm{Re}(\mu_i)<0 \text{ and } \mathrm{Im}(\mu_i)\neq 0}\{(\alpha,T)||1+T^2\mu_i| + |3-2\alpha T|<1\} \tag{20}$$

and S_r is defined by (12). If α and T are chosen from $S_c\cap S_r$, then all eigenvalues of F are within the unit circle except one eigenvalue equal to one.

Proof
For the first statement, we let $\alpha T=\frac{3}{2}$. When $\mathrm{Re}(\mu_i)<0$ and $\mathrm{Im}(\mu_i)\neq 0$, $|1+T^2\mu_i|+|3-2\alpha T|<1$ implies

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$|1+T^2\mu_i|<1$ because $\alpha T=\frac{3}{2}$. It thus follows that

$$0<T<\frac{\sqrt{-2\mathrm{Re}(\mu_i)}}{|\mu_i|}$$
$$\forall \mathrm{Re}(\mu_i)<0 \quad \text{and} \quad \mathrm{Im}(\mu_i)\neq 0$$

When $\mu_i\leqslant 0$, $-(T^2/2)\mu_i<\alpha T<2$ can be simplified as $-T^2\mu_i<\frac{3}{2}$ because $\alpha T=\frac{3}{2}$. It thus follows that $0<T<(\sqrt{3/-\mu_i})$, $\forall \mu_i\leqslant 0$. Let[‖]

$$T_c=\bigcap_{\forall \mathrm{Re}(\mu_i)<0 \text{ and } \mathrm{Im}(\mu_i)\neq 0}\left\{T\,|\,0<T<\frac{\sqrt{-2\mathrm{Re}(\mu_i)}}{|\mu_i|}\right\}$$

$$\text{and} \quad T_r=\bigcap_{\forall \mu_i\leqslant 0}\left\{T\,|\,0<T<\sqrt{\frac{3}{-\mu_i}}\right\}$$

It is straightforward to see that $T_c\cap T_r$ is nonempty. Recalling that $\alpha T=\frac{3}{2}$, it follows that $S_c\cap S_r$ is nonempty as well.

For the second statement, note that if directed graph $\mathscr{G}$ has a directed spanning tree, then it follows from Lemma 3.2 that $\mu_1=0$ and $\mathrm{Re}(\mu_i)<0$, $i=2,\ldots,n$. Note that $\mu_1=0$ implies that $\lambda_1=1$ and $\lambda_2=1-\alpha T$. To ensure that $|\lambda_2|<1$, it is required that $0<\alpha T<2$. When $\mathrm{Re}(\mu_i)<0$ and $\mathrm{Im}(\mu_i)\neq 0$, it follows from Corollary 3.4 that the roots of (9) are within the unit circle if $|1+T^2\mu_i|+|3-2\alpha T|<1$, where we have used the second statement of Corollary 3.4 by letting $a=\alpha T-2-(T^2/2)\mu_i$ and $b=1-(T^2/2)\mu_i-\alpha T$. When $\mu_i<0$, it follows from the proof of Lemma 3.6 that the roots of (9) are within the unit circle if $-(T^2/2)\mu_i<\alpha T<2$. Combining the above arguments proves the second statement. □

Remark 3.5

According to Lemmas 3.4 and 3.9, if α and T are chosen from $S_c\cap S_r$ and directed graph $\mathscr{G}$ has a directed spanning tree, coordination can be achieved ultimately. An easy way to choose α and T is to let $\alpha T=\frac{3}{2}$. It then follows that T can be chosen satisfying

$$T<\min_{\forall \mathrm{Re}(\mu_i)<0 \text{ and } \mathrm{Im}(\mu_i)\neq 0}\frac{|\mu_i|}{\sqrt{-\mathrm{Re}(\mu_i)}}$$

$$\text{and} \quad T<\min_{\forall \mathrm{Re}(\mu_i)<0 \text{ and } \mathrm{Im}(\mu_i)=0}\sqrt{\frac{3}{-\mu_i}}$$

‖When $\mu_i=0$, $T>0$ can be chosen arbitrarily.

4. CONVERGENCE ANALYSIS OF THE SAMPLED-DATA ALGORITHM WITH RELATIVE DAMPING

In this section, we analyze algorithm (7) under, respectively, an undirected and an directed interaction topology.

Using (7), (5) can be written in matrix form as

$$\begin{bmatrix} r[k+1] \\ v[k+1] \end{bmatrix}=\underbrace{\begin{bmatrix} I_n-\frac{T^2}{2}\mathscr{L} & TI_n-\frac{T^2}{2}\mathscr{L} \\ -T\mathscr{L} & I_n-\alpha T\mathscr{L} \end{bmatrix}}_{G}\begin{bmatrix} r[k] \\ v[k] \end{bmatrix} \quad (21)$$

A similar analysis to that for (8) shows that the roots of $\det(sI_{2n}-G)=0$ (i.e. the eigenvalues of G) satisfy

$$s^2-\left(2+\alpha T\mu_i+\frac{1}{2}T^2\mu_i\right)s+1+\alpha T\mu_i-\frac{1}{2}T^2\mu_i=0 \quad (22)$$

Similarly, each eigenvalue of $-\mathscr{L}$, μ_i, corresponds to two eigenvalues of G, denoted by ρ_{2i-1} and ρ_{2i}. Without loss of generality, let $\mu_1=0$, which implies that $\rho_1=\rho_2=1$. Therefore, G has at least two eigenvalues equal to one.

Lemma 4.1

Using (7) for (5), $r_i[k]\to \mathbf{p}^{\mathrm{T}}r[0]+kT\mathbf{p}^{\mathrm{T}}v[0]$ and $v_i[k]\to \mathbf{p}^{\mathrm{T}}v[0]$ for large k if and only if G has exactly two eigenvalues equal to one and all other eigenvalues have modulus smaller than one.

Proof (*Sufficiency*)

Note from (22) that if G has exactly two eigenvalues equal to one (i.e. $\rho_1=\rho_2=1$), then $-\mathscr{L}$ has exactly one eigenvalue equal to zero. Let $[p^{\mathrm{T}},q^{\mathrm{T}}]^{\mathrm{T}}$, where

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$p, q \in \mathbb{R}^n$, be the right eigenvector of G associated with eigenvalue one. It follows that

$$\begin{bmatrix} I_n - \frac{T^2}{2}\mathscr{L} & T I_n - \frac{T^2}{2}\mathscr{L} \\ -T\mathscr{L} & I_n - \alpha T\mathscr{L} \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} p \\ q \end{bmatrix}$$

After some computation, it follows that eigenvalue one has geometric multiplicity equal to one even if it has algebraic multiplicity equal to two. It also follows from Lemma 3.2 that we can choose $p = \mathbf{1}_n$ and $q = \mathbf{0}_n$. In addition, a generalized right eigenvector associated with eigenvalue one can be chosen as $[\mathbf{0}_n^{\mathrm{T}}, (1/T)\mathbf{1}_n^{\mathrm{T}}]^{\mathrm{T}}$. Similarly, it can be shown that $[\mathbf{0}_n^{\mathrm{T}}, T\mathbf{p}_n^{\mathrm{T}}]^{\mathrm{T}}$ and $[\mathbf{p}^{\mathrm{T}}, \mathbf{0}_n^{\mathrm{T}}]^{\mathrm{T}}$ are, respectively, a left eigenvector and generalized left eigenvector associated with eigenvalue one. Note that G can be written in Jordan canonical form as $G = PJP^{-1}$, where the columns of P, denoted by p_k, $k=1, \ldots, 2n$, can be chosen to be the right eigenvectors or generalized right eigenvectors of G, the rows of P^{-1}, denoted by q_k^{T}, $k=1, \ldots, 2n$, can be chosen to be the left eigenvectors or generalized left eigenvectors of G such that $p_k^{\mathrm{T}} q_k = 1$ and $p_k^{\mathrm{T}} q_\ell = 0$, $k \neq \ell$, and J is the Jordan block diagonal matrix with the eigenvalues of G being the diagonal entries. Note that $\rho_1 = \rho_2 = 1$ and $\mathrm{Re}(\rho_k) < 0$, $k = 3, \ldots, 2n$. Also note that we can choose $p_1 = [\mathbf{1}_n^{\mathrm{T}}, \mathbf{0}_n^{\mathrm{T}}]^{\mathrm{T}}$, $p_2 = [\mathbf{0}_n^{\mathrm{T}}, (1/T)\mathbf{1}_n^{\mathrm{T}}]^{\mathrm{T}}$, $q_1 = [\mathbf{p}^{\mathrm{T}}, \mathbf{0}_n^{\mathrm{T}}]^{\mathrm{T}}$, and $q_2 = [\mathbf{0}_n^{\mathrm{T}}, T\mathbf{p}_n^{\mathrm{T}}]^{\mathrm{T}}$. It follows that

$$G^k \to PJ^kP^{-1} \to \begin{bmatrix} \mathbf{1}_n & \mathbf{0}_n \\ \mathbf{0}_n & \frac{1}{T}\mathbf{1}_n \end{bmatrix} \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{p}^{\mathrm{T}} & \mathbf{0}_n^{\mathrm{T}} \\ \mathbf{0}_n^{\mathrm{T}} & T\mathbf{p}^{\mathrm{T}} \end{bmatrix} = \begin{bmatrix} \mathbf{1}_n\mathbf{p}^{\mathrm{T}} & kT\mathbf{1}_n\mathbf{p}^{\mathrm{T}} \\ \mathbf{0}_n & \mathbf{1}_n\mathbf{p}^{\mathrm{T}} \end{bmatrix}$$

Therefore, it follows that $r_i[k] \to \mathbf{p}^{\mathrm{T}} r[0] + kT\mathbf{p}^{\mathrm{T}} v[0]$ and $v_i[k] \to \mathbf{p}^{\mathrm{T}} v[0]$ for large k.

(Necessity) Note that G has at least two eigenvalues equal to one. If $r_i[k] \to \mathbf{p}^{\mathrm{T}} r[0] + kT\mathbf{p}^{\mathrm{T}} v[0]$ and $v_i[k] \to \mathbf{p}^{\mathrm{T}} v[0]$ for large k, it follows that F^k has rank two for large k, which in turn implies that J^k has rank two for large k. It follows that G has exactly two eigenvalues equal to one and all other eigenvalues have modulus smaller than one. □

4.1. Undirected interaction

In this subsection, we show necessary and sufficient conditions on α and T such that coordination is achieved using (7) under an undirected interaction topology.

Lemma 4.2
Suppose that undirected graph $\mathscr{G}$ is connected. All eigenvalues of G are within the unit circle except two eigenvalues equal to one if and only if α and T are chosen from the set**

$$Q_r = \left\{ (\alpha, T) \,\middle|\, \frac{T^2}{2} < \alpha T < -\frac{2}{\min_i \mu_i} \right\} \tag{23}$$

Proof
Because undirected graph $\mathscr{G}$ is connected, it follows that $\mu_1 = 0$ and $\mu_i < 0$, $i = 2, \ldots, n$. Note that $\rho_1 = \rho_2 = 1$ because $\mu_1 = 0$. Let $a = -(2 + \alpha T\mu_i + \frac{1}{2}T^2\mu_i)$ and $b = 1 + \alpha T\mu_i - \frac{1}{2}T^2\mu_i$. It follows from Lemma 3.5 that for $\mu_i < 0$, $i = 2, \ldots, n$, the roots of (22) are within the unit circle if and only if all roots of

$$-T^2\mu_i t^2 + (T^2\mu_i - 2\alpha T\mu_i)t + 4 + 2\alpha T\mu_i = 0 \tag{24}$$

are in the open LHP. Because $-T^2\mu_i > 0$, the roots of (24) are always in the open LHP if and only if $4 + 2\alpha T\mu_i > 0$ *and* $T^2\mu_i - 2\alpha T\mu_i > 0$, which implies that $T^2/2 < \alpha T < -(2/\mu_i)$, $i = 2, \ldots, n$. Combining the above arguments proves the lemma. □

Theorem 4.1
Suppose that undirected graph $\mathscr{G}$ is connected. Let $\mathbf{p}$ be defined in Lemma 3.2. Using (7), $r_i[k] \to \mathbf{p}^{\mathrm{T}} r[0] + kT\mathbf{p}^{\mathrm{T}} v[0]$ and $v_i[k] \to \mathbf{p}^{\mathrm{T}} v[0]$ for large k if and only if α and T are chosen from Q_r, where Q_r is defined by (23).

Proof
The statement follows directly from Lemmas 4.1 and 4.2. □

4.2. Directed interaction

In this subsection, we show necessary and sufficient conditions on α and T such that coordination is achieved

**Note that Q_r is nonempty.

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using (7) under a directed interaction topology. Note again that the eigenvalues of $\mathscr{L}$ may be complex for directed graphs, which makes the analysis more challenging.

Lemma 4.3
Suppose that $\text{Re}(\mu_i)<0$ and $\text{Im}(\mu_i)\neq 0$. All roots of (22) are within the unit circle if and only if $\alpha/T>\frac{1}{2}$ and $B_i<0$, where

$$B_i \triangleq \left(\frac{4\,\text{Re}(\mu_i)}{|\mu_i|^2T^2}+2\frac{\alpha}{T}\right)\left(1-2\frac{\alpha}{T}\right)^2+\frac{16\,\text{Im}(\mu_i)^2}{|\mu_i|^4T^4} \quad (25)$$

Proof
As in the proof of Lemma 4.2, all roots of (22) are within the unit circle if and only if all roots of (24) are in the open LHP. Letting s_1 and s_2 denote the roots of (24), it follows that

$$s_1+s_2=1-2\frac{\alpha}{T} \quad (26)$$

and

$$s_1s_2=-\frac{4}{\mu_iT^2}-2\frac{\alpha}{T} \quad (27)$$

Noting that (26) implies that $\text{Im}(s_1)+\text{Im}(s_2)=0$, we define $s_1=a_1+\mathbf{j}b$ and $s_2=a_2-\mathbf{j}b$, where $\mathbf{j}$ is the imaginary unit. Note that s_1 and s_2 have negative real parts if and only if $a_1+a_2<0$ and $a_1a_2>0$. Note from (26) that $a_1+a_2<0$ is equivalent to $(\alpha/T)>\frac{1}{2}$. We next show conditions on α and T such that $a_1a_2>0$ holds. Substituting the definitions of s_1 and s_2 into (27), gives

$$a_1a_2+b^2+\mathbf{j}(a_2-a_1)b=\frac{-4}{\mu_iT^2}-2\frac{\alpha}{T}$$

which implies that

$$(a_2-a_1)b=\frac{4\,\text{Im}(\mu_i)}{|\mu_i|^2T^2} \quad (28)$$

$$a_1a_2+b^2=\frac{-4\,\text{Re}(\mu_i)}{|\mu_i|^2T^2}-2\frac{\alpha}{T} \quad (29)$$

It follows from (28) that

$$b=\frac{4\,\text{Im}(\mu_i)}{|\mu_i|^2T^2(a_2-a_1)}$$

Consider also the fact that $(a_2-a_1)^2=(a_2+a_1)^2-4a_1a_2=(1-2\alpha/T)^2-4a_1a_2$. After some manipulation, (29) can be written as

$$4(a_1a_2)^2+A_ia_1a_2-B_i=0 \quad (30)$$

where

$$A_i\triangleq 4\left(\frac{4\,\text{Re}(\mu_i)}{|\mu_i|^2T^2}+2\frac{\alpha}{T}\right)-\left(1-2\frac{\alpha}{T}\right)^2$$

and B_i is defined in (25). It follows that

$$A_i^2+16B_i=\left[4\left(\frac{4\,\text{Re}(\mu_i)}{|\mu_i|^2T^2}+2\frac{\alpha}{T}\right)+\left(1-2\frac{\alpha}{T}\right)^2\right]^2+\frac{16\,\text{Im}(\mu_i)^2}{|\mu_i|^4T^4}\geqslant 0$$

which implies that (30) has two real roots. Therefore, necessary and sufficient conditions for $a_1a_2>0$ are $B_i<0$ and $A_i<0$. Because

$$\frac{16\,\text{Im}(\mu_i)^2}{|\mu_i|^4T^4}>0$$

if $B_i<0$, then

$$4\left(\frac{4\,\text{Re}(\mu_i)}{|\mu_i|^2T^2}+2\alpha/T\right)<0$$

which implies $A_i<0$ as well. Combining the previous arguments proves the lemma. □

Lemma 4.4
Suppose that directed graph $\mathscr{G}$ has a directed spanning tree. There exist positive α and T such that $Q_c\cap Q_r$ is nonempty, where

$$Q_c=\bigcap_{\forall \text{Re}(\mu_i)<0 \text{ and } \text{Im}(\mu_i)\neq 0}\left\{(\alpha,T)\,\middle|\,\frac{1}{2}<\frac{\alpha}{T},\ B_i<0\right\} \quad (31)$$

where B_i is defined by (25) and Q_r is defined by (23). All eigenvalues of G are within the unit circle except two eigenvalues equal to one if and only if α and T are chosen from $Q_r\cap Q_c$.

Proof
For the first statement, we let $\alpha>T>0$. When $\text{Re}(\mu_i)<0$ and $\text{Im}(\mu_i)\neq 0$, it follows that $\alpha/T>\frac{1}{2}$ holds apparently. Note that $\alpha>T$ implies $(T-2\alpha)^2>\alpha^2$. Therefore,

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a sufficient condition for $B_i < 0$ is

$$\alpha T < -\frac{8\,\mathrm{Im}(\mu_i)^2}{|\mu_i|^4\alpha^2} - \frac{2\,\mathrm{Re}(\mu_i)}{|\mu_i|^2} \tag{32}$$

To ensure that there are feasible $\alpha > 0$ and $T > 0$ satisfying (32), we first need to ensure that the right side of (32) is positive, which requires $\alpha > 2|\mathrm{Im}(\mu_i)|/(|\mu_i|\sqrt{-\mathrm{Re}(\mu_i)})$. It also follows from (32) that

$$T < -\frac{8\,\mathrm{Im}(\mu_i)^2}{|\mu_i|^4\alpha^3} - \frac{2\,\mathrm{Re}(\mu_i)}{|\mu_i|^2\alpha}, \ \forall \mathrm{Re}(\mu_i) < 0 \quad \text{and} \quad \mathrm{Im}(\mu_i) \neq 0$$

Therefore, (31) is ensured to be nonempty if α and T are chosen from, respectively,

$$\alpha_c = \bigcap_{\forall \mathrm{Re}(\mu_i)<0 \text{ and } \mathrm{Im}(\mu_i)\neq 0} \left\{ \alpha \,\middle|\, \alpha > \frac{2|\mathrm{Im}(\mu_i)|}{|\mu_i|\sqrt{-\mathrm{Re}(\mu_i)}} \right\}$$

and

$$T_c = \bigcap_{\forall \mathrm{Re}(\mu_i)<0 \text{ and } \mathrm{Im}(\mu_i)\neq 0} \left\{ T \,\middle|\, T < -\frac{8\,\mathrm{Im}(\mu_i)^2}{|\mu_i|^4\alpha^3} - \frac{2\,\mathrm{Re}(\mu_i)}{|\mu_i|^2\alpha} \text{ and } 0 < T < \alpha \right\}$$

Note that (23) is ensured to be nonempty if α and T are chosen from, respectively, $\alpha_r = \{\alpha | \alpha > 0\}$ and

$$T_r = \bigcap_{\forall \mu_i < 0} \left\{ T \,\middle|\, 0 < T < 2\alpha \text{ and } T < -\frac{2}{\mu_i \alpha} \right\}$$

It is straightforward to see that both $\alpha_c \cap \alpha_r$ and $T_c \cap T_r$ are nonempty. Combining the above arguments shows that $Q_c \cap Q_r$ is nonempty.

For the second statement, note that if directed graph $\mathcal{G}$ has a directed spanning tree, it follows from Lemma 3.2 that $\mu_1 = 0$ and $\mathrm{Re}(\mu_i) < 0$, $i = 2, \ldots, n$. Note that $\mu_1 = 0$ implies that $\rho_1 = 1$ and $\rho_2 = 1$. When $\mathrm{Re}(\mu_i) < 0$ and $\mathrm{Im}(\mu_i) \neq 0$, it follows from Lemma 4.3 that the roots of (22) are within unit circle if and only if $\alpha/T > \frac{1}{2}$ and $B_i < 0$. When $\mu_i < 0$, it follows from Lemma 4.2 that the roots of (22) are within unit circle if and only if $(T^2/2) < \alpha T < (-2/\mu_i)$. Combining the above arguments shows that all eigenvalues of G are within the unit circle except two eigenvalues equal to one if and only if α and T are chosen from $Q_c \cap Q_r$. □

Remark 4.2

From the proof of the first statement of Lemma 4.4, an easy way to choose α and T is to let $\alpha > T$. Then α is chosen from α_c and T is chosen from $T_c \cap T_r$, where α_c, T_c, and T_r are defined in the proof of Lemma 4.4.

Theorem 4.3

Suppose that directed graph $\mathcal{G}$ has a directed spanning tree. Using (7), $r_i[k] \to \mathbf{p}^{\mathrm{T}} r[0] + kT\mathbf{p}^{\mathrm{T}} v[0]$ and $v_i[k] \to \mathbf{p}^{\mathrm{T}} v[0]$ for large k if and only if α and T are chosen from $Q_c \cap Q_r$, where Q_c and Q_r are defined in (31) and (23), respectively.

Proof

The proof follows directly from Lemma 4.2 and Theorem 4.4. □

Remark 4.4

Note that it is required in Theorems 3.3 and 4.3 that the communication graph has a directed spanning tree in order to guarantee coordination. The connectivity requirement in Theorems 3.3 and 4.3 can be interpreted as follows. For a group of vehicles, if the communication graph does not have a directed spanning tree, then the group of vehicles can be divided into at least two disconnected subgroups. Because there is no communication among these subgroups, the final states of the subgroups in general cannot achieve coordination.

5. SIMULATION

In this section, we present simulation results to validate the theoretical results derived in Sections 3 and 4. We consider a team of four vehicles with directed graph $\mathcal{G}$ shown by Figure 1. Note that $\mathcal{G}$ has a directed spanning

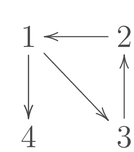

Figure 1. Directed graph $\mathcal{G}$ for four vehicles. An arrow from j to i denotes that vehicle i can receive information from vehicle j.

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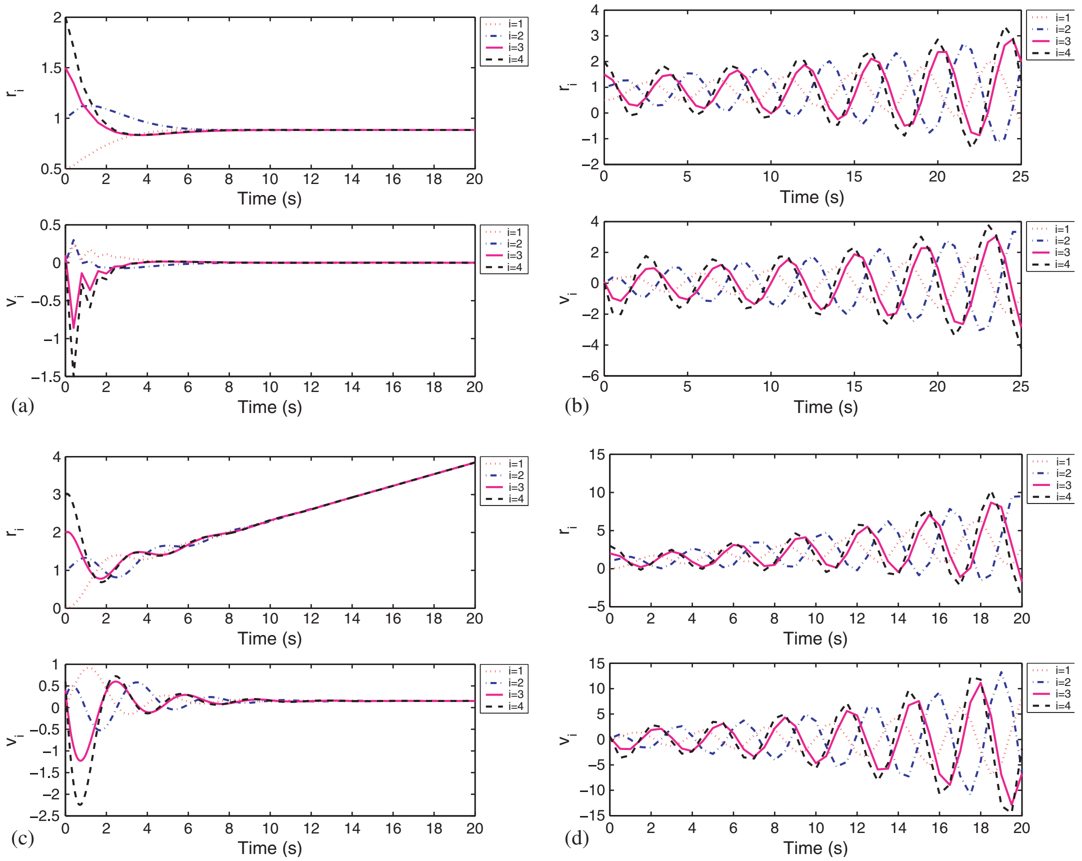

Figure 2. Convergence results using (6) and (7) with different α and T values. Note that coordination is achieved in (a) and (c) but not in (b) and (d) depending on different choices of α and T. (a) Convergence resulting using (6) ($\alpha=4$ and $T=0.4$ s); (b) Convergence resulting using (6) ($\alpha=1.2$ and $T=0.5$ s); (c) Convergence resulting using (7) ($\alpha=0.6$ and $T=0.02$ s); and (d) Convergence resulting using (7) ($\alpha=0.6$ and $T=0.5$ s).

tree. The nonsymmetric Laplacian matrix associated with $\mathscr{G}$ is chosen as

$$\mathscr{L}=\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1.5 & -1.5 & 0 \\ -2 & 0 & 2 & 0 \\ -2.5 & 0 & 0 & 2.5 \end{bmatrix}$$

It can be computed that for $\mathscr{L}$, $\mathbf{p}=[0.4615, 0.3077, 0.2308, 0]^{\mathrm{T}}$. Here for simplicity, we have chosen $\delta_i=0, i=1,\ldots,4$.

For coordination algorithm (6), let $r[0]=[0.5, 1, 1.5, 2]^{\mathrm{T}}$ and $v[0]=[-0.1, 0, 0.1, 0]^{\mathrm{T}}$. Figure 2 shows the convergence result using (6) with $\alpha=4$ and $T=0.4$ s. Note that the conditions in Theorem 3.3 are satisfied. It can be seen that coordination is achieved with the final equilibrium for $r_i[k]$ being 0.8835, which is equal to $\mathbf{p}^{\mathrm{T}}r[0]+(1/\alpha-T/2)\mathbf{p}^{\mathrm{T}}v[0]$ as argued in Theorem 3.3. Figure 2 shows the convergence result using (6) with $\alpha=1.2$ and $T=0.5$ s. Note that coordination is not achieved in this case.

For coordination algorithm (7), let $r[0]=[0, 1, 2, 3]^{\mathrm{T}}$ and $v[0]=[0, 0.2, 0.4, 0.6]^{\mathrm{T}}$. Figure 2 shows the convergence result using (7) with $\alpha=0.6$ and $T=0.02$ s.

Note that the conditions in Theorem 4.3 are satisfied. It can be seen that coordination is achieved with the final equilibrium for $v_i[k]$ being 0.1538, which is equal to $\mathbf{p}^{\mathrm{T}}v[0]$ as argued in Theorem 4.3. Figure 2(d) the convergence result using (7) with $\alpha = 0.6$ and $T = 0.5$ s. Note that coordination is not achieved in this case.

6. CONCLUSION

We have studied the sampled-data coordination algorithms for double-integrator dynamics under fixed undirected/directed interaction. Two sampled-date coordination algorithms with, respectively, absolute damping and relative damping have been studied under both undirected and directed interaction topologies. Necessary and sufficient conditions for convergence are given in both undirected and directed cases. The final coordination equilibria for both algorithms have also been given. Simulation results have illustrated the effectiveness of the results.

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