Analysis of Positioning Uncertainty in Reconfigurable Networks of Heterogeneous Mobile Robots

Anastasios I. Mourikis and Stergios I. Roumeliotis
{mourikis|stergios}@cs.umn.edu

Dept. of Computer Science & Engineering
University of Minnesota
Minneapolis, MN 55455

Center for
Distributed Robotics

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Dept. of Computer Science & Engineering
University of Minnesota
4-192 EE/CS Building
200 Union St. S.E.
Minneapolis, MN 55455
Tel: (612) 625-2217
Fax: (612) 625-0572
URL: http://www.cs.umn.edu/˜mourikis

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Anastasios I. Mourikis and Stergios I. Roumeliotis
{mourikis|stergios}@cs.umn.edu

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Minneapolis, MN 55455

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Abstract

This Technical Report studies the accuracy of position estimation for groups of mobile robots performing Cooperative Localization (CL). We consider the case of teams comprising of possibly heterogeneous robots and provide analytical expressions for upper bounds on the worst-case as well as the expected positioning uncertainty. These bounds are determined as a function of the sensors' noise covariance and the eigenvalues of the Relative Position Measurement Graph (RPMG), i.e., the weighted directed graph which represents the network of robot-to-robot exteroceptive measurements. The RPMG is employed as a key element in this analysis and its properties are related to the localization performance of the team. It is shown that for a robot group of certain size, the maximum expected rate of uncertainty increase is *independent* of the accuracy and number of relative position measurements and depends only on the accuracy of the proprioceptive and orientation sensors on the robots. Additionally, the effect of changes in the topology of the RPMG are studied and it is shown that at steady state, these reconfigurations do *not* inflict any loss in localization precision. The theoretical results are validated by simulations.

1 Introduction

This Technical Report presents a theoretical analysis of the positioning uncertainty of a team of mobile robots performing Cooperative Localization (CL). We consider a group of N robots that employs an Extended Kalman Filter (EKF) estimator to perform CL. *Proprioceptive* measurements (i.e., velocity) are integrated to propagate the state estimates, while *exteroceptive* measurements (i.e., robot-to-robot relative position measurements and potentially absolute position measurements) are processed to update these estimates. In our formulation, we assume that an upper bound on the variance of the robots' orientation estimates can be a priori determined. This is the case, for example, when each robot is equipped with a heading sensor of limited accuracy (e.g., a compass [1, 2] or a sun sensor [3, 4]) that directly measures its orientation, or if the robots infer their orientation from measurements of the structure of the environment in their surroundings [5, 6]. The ensuing analysis holds even if only a conservative upper bound on the orientation uncertainty can be determined, e.g., by estimating the maximum orientation error, accumulated over a certain period of time, due to the integration of the odometric measurements [7].

We should note here that the condition for bounded orientation uncertainty is satisfied in most cases in practice. If instead, special care is not taken and the errors in the orientation estimates of the robots are allowed to grow unbounded, any EKF-based estimator of their position will eventually diverge [8]. Thus, the requirement for bounded orientation errors is *not* an artificially imposed assumption; it is essentially a *prerequisite* for performing EKF-based localization. In fact, if we can determine the maximum tolerable value of the orientation variance, so that the linearization errors are acceptably small, we can use this variance value in the derivations that follow.

The availability of an upper bound on each robot's orientation uncertainty enables us to decouple the task of position estimation from that of orientation estimation, for the purpose of determining upper bounds on the performance of CL. Specifically, we formulate a state vector comprising of only the positions of the N robots, and the orientation estimates are used as inputs to the system, of which noise-corrupted observations are available. Clearly, the resulting EKF-based estimator is a suboptimal one, since the correlations that exist between the position and orientation estimates of the robots are discarded. Thus, by deriving an upper bound on the covariance of the estimates produced with this suboptimal, "position-only" estimator, we simultaneously determine an upper bound on the covariance of the position estimates that would result from using a "full-state" EKF estimator.

Throughout this paper, we consider that all robots move constantly in a random fashion (i.e., no specific formation is assumed [9]). At every time step, some (or all) robots record relative position measurements, and use this information to improve the position estimates for all members of the group. During each EKF update cycle, all exteroceptive measurements, as well as the current position estimates of the robots, must be available to the estimator [10]. Therefore, it is assumed that a communication network exists enabling all robots to transmit such information. These can then be fused either in a distributed scheme, or at a central fusion center.

A key element in this analysis is the Relative Position Measurement Graph (RPMG), which is defined as a graph whose vertices represent robots in the group and its directed edges correspond to relative position measurements (Fig. 8). That is, if robot i measures the relative position of robot j, the RPMG contains a directed edge from vertex i to vertex j. In this work, we primarily consider the most challenging scenario where the absolute positions of the robots cannot be measured or inferred. The case where global positioning information is available to at least one of the robots in the group, is subsumed in our formulation and is treated as a special one.

2 Discrete-Time Analysis

In this section we present a discrete-time analysis of CL, and derive performance bounds that are applicable for the covariance estimates output by the discrete-time EKF. For this analysis, we assume that both odometric and exteroceptive measurements are processed at the same rate. However, this not always the case, since odometric data are commonly available at a higher rate. To address this problem, a *continuous-time* analysis of the time evolution of the covariance has also been conducted, and is presented in Section 3.

2.1 Propagation Model

We consider a team of N non-holonomic robots, $r_1, r_2, \ldots, r_N$ moving in a planar environment. The discrete-time kinematic equations for the i-th robot are

$$x_i(k+1) = x_i(k) + V_i(k)\delta t \cos(\phi_i(k)) \quad (1)$$
$$y_i(k+1) = y_i(k) + V_i(k)\delta t \sin(\phi_i(k)) \quad (2)$$

where $V_i(k)$ denotes the robot's translational velocity at time k and δt is the sampling period. In the Kalman filter framework, the estimates of the robot's position are propagated using the measurements of the robot's velocity, $V_{m_i}(k)$, and the estimates of the robot's orientation, $\hat{\phi}_i(k)$:

$$\hat{x}_{i_{k+1|k}} = \hat{x}_{i_{k|k}} + V_{m_i}(k)\delta t \cos(\hat{\phi}_i(k))$$
$$\hat{y}_{i_{k+1|k}} = \hat{y}_{i_{k|k}} + V_{m_i}(k)\delta t \sin(\hat{\phi}_i(k))$$

Clearly, these equations are time varying and nonlinear due to the dependence on the robot's orientation. By linearizing Eqs. (1) and (2), the error propagation equation for the robot's position is readily derived:

$$\begin{bmatrix} \widetilde{x}_{i_{k+1|k}} \\ \widetilde{y}_{i_{k+1|k}} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \widetilde{x}_{i_{k|k}} \\ \widetilde{y}_{i_{k|k}} \end{bmatrix} + \begin{bmatrix} \delta t \cos(\hat{\phi}_i(k)) & -V_{m_i}(k)\delta t \sin(\hat{\phi}_i(k)) \\ \delta t \sin(\hat{\phi}_i(k)) & V_{m_i}(k)\delta t \cos(\hat{\phi}_i(k)) \end{bmatrix} \begin{bmatrix} w_{V_i}(k) \\ \widetilde{\phi}_i(k) \end{bmatrix}$$
$$\Leftrightarrow \widetilde{X}_{i_{k+1|k}} = I_{2\times 2} \; \widetilde{X}_{i_{k|k}} + G_i(k) \; W_i(k) \quad (3)$$

where[1] $w_{V_i}(k)$ is a zero-mean white Gaussian noise sequence of variance $\sigma^2_{V_i}$, affecting the velocity measurements and $\widetilde{\phi}_i(k)$ is the error in the robot's orientation estimate at time k. This is modeled as a zero-mean white Gaussian noise sequence of variance $\sigma^2_{\phi_i}$.

From Eq. (3), we deduce that the covariance matrix of the system noise affecting the i-th robot is:

$$\begin{aligned}
Q_i(k) &= E\{G_i(k)W_i(k)W_i^T(k)G_i^T(k)\} \\
&= G_i(k)E\{W_i(k)W_i^T(k)\}G_i^T(k) \\
&= \begin{bmatrix} \delta t\cos(\hat{\phi}_i(k)) & -V_{m_i}(k)\delta t\sin(\hat{\phi}_i(k)) \\ \delta t\sin(\hat{\phi}_i(k)) & V_{m_i}(k)\delta t\cos(\hat{\phi}_i(k)) \end{bmatrix} \begin{bmatrix} \sigma^2_{V_i} & 0 \\ 0 & \sigma^2_{\phi_i} \end{bmatrix} \begin{bmatrix} \delta t\cos(\hat{\phi}_i(k)) & -V_{m_i}(k)\delta t\sin(\hat{\phi}_i(k)) \\ \delta t\sin(\hat{\phi}_i(k)) & V_{m_i}(k)\delta t\cos(\hat{\phi}_i(k)) \end{bmatrix}^T \\
&= \begin{bmatrix} \cos(\hat{\phi}_i(k)) & -\sin(\hat{\phi}_i(k)) \\ \sin(\hat{\phi}_i(k)) & \cos(\hat{\phi}_i(k)) \end{bmatrix} \begin{bmatrix} \delta t^2\sigma^2_{V_i} & 0 \\ 0 & \delta t^2 V^2_{m_i}(k)\sigma^2_{\phi_i} \end{bmatrix} \begin{bmatrix} \cos(\hat{\phi}_i(k)) & -\delta t\sin(\hat{\phi}_i(k)) \\ \sin(\hat{\phi}_i(k)) & \delta t\cos(\hat{\phi}_i(k)) \end{bmatrix}^T \\
&= C(\hat{\phi}_i(k)) \begin{bmatrix} \delta t^2\sigma^2_{V_i} & 0 \\ 0 & \delta t^2 V^2_{m}(k)\sigma^2_{\phi_i} \end{bmatrix} C^T(\hat{\phi}_i(k))
\end{aligned} \tag{4}$$

where $C(\hat{\phi}_i)$ denotes the rotation matrix associated with $\hat{\phi}_i$.

Using these results we can now write the error propagation equations for the entire system, comprising of N robots:

$$\begin{aligned}
\widetilde{X}_{k+1|k} &= I_{2N\times 2N}\widetilde{X}_{k|k} + \begin{bmatrix} G_1(k) & \mathbf{0}_{2\times 2} & \cdots & \mathbf{0}_{2\times 2} \\ \mathbf{0}_{2\times 2} & G_2(k) & \cdots & \mathbf{0}_{2\times 2} \\ & & \ddots & \\ \mathbf{0}_{2\times 2} & & & G_N(k) \end{bmatrix} \begin{bmatrix} w_{V_1}(k) \\ \widetilde{\phi}_1(k) \\ w_{V_2}(k) \\ \widetilde{\phi}_2(k) \\ \vdots \\ w_{V_N}(k) \\ \widetilde{\phi}_N(k) \end{bmatrix} \\
\Leftrightarrow \widetilde{X}_{k+1|k} &= \mathbf{\Phi}(k)\widetilde{X}_{k|k} + \qquad \mathbf{G}_t(k) \qquad \mathbf{W}(k)
\end{aligned} \tag{5}$$

where we have defined the state vector of the entire system as the stacked vector comprising of the positions of all the robots:

$$X = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_N \end{bmatrix}$$

The covariance matrix of the system noise is given by

$$\begin{aligned}
\mathbf{Q}(k) &= E\{\mathbf{G}_t(k)\mathbf{W}(k)\mathbf{W}^T(k)\mathbf{G}_t^T(k)\} \\
&= \begin{bmatrix} E\{G_1(k)W_1(k)W_1^T(k)G_1^T(k)\} & \cdots & \mathbf{0}_{2\times 2} \\ \vdots & \ddots & \vdots \\ \mathbf{0}_{2\times 2} & \cdots & E\{G_N(k)W_Nkfd W_N^T(k)G_N^T(k)\} \end{bmatrix} \\
&= \begin{bmatrix} Q_1(k) & \cdots & \mathbf{0}_{2\times 2} \\ \vdots & \ddots & \vdots \\ \mathbf{0}_{2\times 2} & \cdots & Q_N(k) \end{bmatrix} \\
&= \mathbf{Diag}(Q_i(k))
\end{aligned} \tag{6}$$

Thus the equation for propagating the covariance matrix of the state error is written as

$$\mathbf{P}_{k+1|k} = \mathbf{P}_{k|k} + \mathbf{Q}(k) \tag{7}$$

where $\mathbf{P}_{k+1|k} = E\{\widetilde{X}_{k+1|k}\widetilde{X}^T_{k+1|k}\}$ and $\mathbf{P}_{k|k} = E\{\widetilde{X}_{k|k}\widetilde{X}^T_{k|k}\}$ are the covariance of the error in the estimate of $X(k+1)$ and $X(k)$ respectively, after measurements up to time k have been processed.

[1]Throughout this document, $\mathbf{0}_{m\times n}$ denotes the $m \times n$ matrix of zeros, $\mathbf{1}_{m\times n}$ denotes the $m \times n$ matrix of ones, $I_{n\times n}$ denotes the $n \times n$ identity matrix, and $\mathbf{Diag}(\cdot)$ denotes a block diagonal matrix.

2.2 Exteroceptive Measurement Model

2.2.1 Relative Position Measurements

At every time step, the robots perform robot-to-robot relative position measurements. Assuming that robot r_i performs M_i relative position measurements at each time step, we denote by $\mathcal{N}_i \subset \{r_1, r_2, \cdots, r_N\} \setminus \{r_i\}$ the subset of robots and observed by robot i. We denote by T_{ij} the target of the j-th measurement performed by robot r_i, i.e.,

$$T_{ij} \in \mathcal{N}_i$$

where the index j in assumes integer values in the range $[1, M_i]$ to describe the M_i relative position measurements of robot r_i.

With this notation, the relative position measurement between robots r_i and T_{ij} is given by:

$$z_{ij}(k+1) = C^T(\phi_i(k+1))\big(X_{T_{ij}}(k+1) - X_i(k+1)\big) + n_{z_{ij}}(k+1) \tag{8}$$

By linearizing the last expression, the measurement error equation is obtained:

$$\begin{aligned}
\widetilde{z}_{ij}(k+1) &= z_{ij}(k+1) - \hat{z}_{ij}(k+1) \\
&= C^T(\hat{\phi}_i(k+1))\left(\widetilde{X}_{T_{ij}\ k+1|k} - \widetilde{X}_{i\ k+1|k}\right) - C^T(\hat{\phi}_i(k+1))J\left(\hat{X}_{T_{ij}\ k+1|k} - \hat{X}_{i\ k+1|k}\right)\widetilde{\phi}_i(k+1) + n_{z_{ij}}(k+1) \\
&= C^T(\hat{\phi}_i(k+1))\left[\begin{array}{ccccccc} 0_{2\times 2} & \cdots & \underbrace{-I_{2\times 2}}_{r_i} & \cdots & \underbrace{I_{2\times 2}}_{T_{ij}} & \cdots & 0_{2\times 2}\end{array}\right]\left[\begin{array}{c}\vdots \\ \widetilde{X}_i \\ \vdots \\ \widetilde{X}_{T_{ij}} \\ \vdots\end{array}\right]_{k+1|k} \\
&+ \left[\begin{array}{cc} I_{2\times 2} & -C^T(\hat{\phi}_i(k+1))J\widehat{\Delta p}_{ij\ k+1|k}\end{array}\right]\left[\begin{array}{c} n_{z_{ij}}(k+1) \\ \widetilde{\phi}_i(k+1)\end{array}\right] \\
&= H_{ij}(k+1)\widetilde{X}_{k+1|k} + \Gamma_{ij}(k+1)n_{ij}(k+1)
\end{aligned} \tag{9}$$

where

$$J = \left[\begin{array}{cc} 0 & -1 \\ 1 & 0\end{array}\right], \quad \widehat{\Delta p}_{ij_{k+1|k}} = \hat{X}_{T_{ij}\ k+1|k} - \hat{X}_{r_i\ k+1|k}$$

and we note that the measurement matrix for this relative position measurement can be written as

$$H_{ij}(k+1) = C^T(\hat{\phi}_i(k+1))\left[\begin{array}{ccccccc} 0_{2\times 2} & \cdots & \underbrace{-I_{2\times 2}}_{r_i} & \cdots & \underbrace{I_{2\times 2}}_{T_{ij}} & \cdots & 0_{2\times 2}\end{array}\right] = C^T(\hat{\phi}_i(k+1))H_{o_{ij}} \tag{10}$$

At each time instant robot i records M_i relative position measurements, described by the measurement matrix $\boldsymbol{H}_i(k+1)$, i.e., a matrix whose block rows are $H_{ij}(k+1)$, $j = 1 \ldots M_i$:

$$\boldsymbol{H}_i(k+1) = \left[\begin{array}{c} C^T(\hat{\phi}_i(k+1))H_{o_{i1}} \\ C^T(\hat{\phi}_i(k+1))H_{o_{i2}} \\ \vdots \\ C^T(\hat{\phi}_i(k+1))H_{o_{iM_i}}\end{array}\right] = \boldsymbol{\Xi}^T_{\hat{\phi}_i}(k+1)\boldsymbol{H}_{o_i} \tag{11}$$

in the last expression $\boldsymbol{H}_{o_i}$ is a constant matrix whose block rows are $H_{o_{ij}}$, $j = 1 \ldots M_i$, and

$$\boldsymbol{\Xi}_{\hat{\phi}_i}(k+1) = I_{M_i\times M_i} \otimes C(\hat{\phi}_i(k+1)) \tag{12}$$

with $\otimes$ denoting the Kronecker matrix product. The covariance for the error of the j-th measurement of robot i is given by

$$\begin{aligned}
{}^iR_{jj}(k+1) &= \Gamma_{ij}(k+1)E\{n_{ij}(k+1)n^T_{ij}(k+1)\}\Gamma^T_{ij}(k+1) \\
&= R_{z_{ij}}(k+1) + R_{\tilde{\phi}_{ij}}(k+1)
\end{aligned} \tag{13}$$

This expression encapsulates all sources of noise and uncertainty that contribute to the measurement error $\widetilde{z}_{ij}(k+1)$. More specifically, $R_{z_{ij}}(k+1)$ is the covariance of the noise $n_{ij}(k+1)$ in the recorded relative position measurement $z_{ij}(k+1)$ and $R_{\tilde{\phi}_{ij}}(k+1)$ is the additional covariance term due to the error $\tilde{\phi}_i(k+1)$ in the orientation estimate of the measuring robot. This is given by:

$$\begin{aligned} R_{\tilde{\phi}_{ij}}(k+1) &= C^T(\hat{\phi}_i(k+1)) J \widehat{\Delta p}_{ij_{k+1|k}} E\{\tilde{\phi}_i^{\,2}\} \widehat{\Delta p}^T_{ij_{k+1|k}} J^T C(\hat{\phi}_i(k+1)) \\ &= \sigma^2_{\phi_i} C^T(\hat{\phi}_i(k+1)) J \widehat{\Delta p}_{ij_{k+1|k}} \widehat{\Delta p}^T_{ij_{k+1|k}} J^T C(\hat{\phi}_i(k+1)) \end{aligned} \quad (14)$$

From this expression we conclude that the uncertainty $\sigma^2_{\phi_i}$ in the orientation estimate $\hat{\phi}_i(k+1)$ of the robot is amplified by the distance between the two robots.

Each relative position measurement is comprised of the distance ρ_{ij} and bearing θ_{ij} to the target, expressed in the measuring robot's local coordinate frame, i.e.,

$$z_{ij}(k+1) = \begin{bmatrix} \rho_{ij}(k+1)\cos\theta_{ij}(k+1) \\ \rho_{ij}(k+1)\sin\theta_{ij}(k+1) \end{bmatrix} + n_{z_{ij}}(k+1)$$

By linearizing, the noise in this measurement can be expressed as:

$$n_{z_{ij}}(k+1) \simeq \begin{bmatrix} \cos\hat{\theta}_{ij} & -\hat{\rho}_{ij}\sin\hat{\theta}_{ij} \\ \sin\hat{\theta}_{ij} & \hat{\rho}_{ij}\cos\hat{\theta}_{ij} \end{bmatrix} \begin{bmatrix} n_{\rho_{ij}}(k+1) \\ n_{\theta_{ij}}(k+1) \end{bmatrix}$$

where $n_{\rho_{ij}}$ is the error in the range measurement, $n_{\theta_{ij}}$ is the error in the bearing measurement, assumed to be independent white zero-mean Gaussian sequences, and

$$\begin{aligned} \hat{\rho}^2_{ij} &= \widehat{\Delta p}^T_{ij_{k+1|k}} \widehat{\Delta p}_{ij_{k+1|k}} \\ \hat{\theta}_{ij} &= \mathrm{Atan2}(\widehat{\Delta y}_{ij_{k+1|k}}, \widehat{\Delta x}_{ij_{k+1|k}}) - \hat{\phi}_i(k+1) \end{aligned}$$

are the estimates of the range and bearing to robot r_j, expressed with respect to the robot's coordinate frame. At this point we note that

$$\begin{aligned} C(\hat{\phi}_i(k+1)) n_{z_{ij}}(k+1) &= \begin{bmatrix} \cos\hat{\phi}_i(k+1) & -\sin\hat{\phi}_i(k+1) \\ \sin\hat{\phi}_i(k+1) & \cos\hat{\phi}_i(k+1) \end{bmatrix} \begin{bmatrix} \cos\hat{\theta}_{ij} & -\hat{\rho}_{ij}\sin\hat{\theta}_{ij} \\ \sin\hat{\theta}_{ij} & \hat{\rho}_{ij}\cos\hat{\theta}_{ij} \end{bmatrix} \begin{bmatrix} n_{\rho_{ij}}(k+1) \\ n_{\theta_{ij}}(k+1) \end{bmatrix} \\ &= \begin{bmatrix} \cos(\hat{\phi}_i(k+1)+\hat{\theta}_{ij}) & -\hat{\rho}_{ij}\sin(\hat{\phi}_i(k+1)+\hat{\theta}_{ij}) \\ \sin(\hat{\phi}_i(k+1)+\hat{\theta}_{ij}) & \hat{\rho}_{ij}\cos(\hat{\phi}_i(k+1)+\hat{\theta}_{ij}) \end{bmatrix} \begin{bmatrix} n_{\rho_{ij}}(k+1) \\ n_{\theta_{ij}}(k+1) \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{\hat{\rho}_{ij}}\widehat{\Delta p}_{ij} & J\widehat{\Delta p}_{ij} \end{bmatrix} \begin{bmatrix} n_{\rho_{ij}}(k+1) \\ n_{\theta_{ij}}(k+1) \end{bmatrix} \end{aligned}$$

and therefore the quantity $R_{z_{ij}}(k+1)$ can be written as:

$$\begin{aligned} R_{z_{ij}}(k+1) &= E\{n_{z_{ij}}(k+1) n^T_{z_{ij}}(k+1)\} \\ &= C^T(\hat{\phi}_i(k+1)) \begin{bmatrix} \frac{1}{\hat{\rho}_{ij}}\widehat{\Delta p}_{ij} & J\widehat{\Delta p}_{ij} \end{bmatrix} E\left\{ \begin{bmatrix} n_{\rho_{ij}} \\ n_{\theta_{ij}} \end{bmatrix} \begin{bmatrix} n_{\rho_{ij}} \\ n_{\theta_{ij}} \end{bmatrix}^T \right\} \begin{bmatrix} \frac{1}{\hat{\rho}_{ij}}\widehat{\Delta p}_{ij} & J\widehat{\Delta p}_{ij} \end{bmatrix}^T C(\hat{\phi}_i(k+1)) \\ &= C^T(\hat{\phi}_i(k+1)) \begin{bmatrix} \frac{1}{\hat{\rho}_{ij}}\widehat{\Delta p}_{ij} & J\widehat{\Delta p}_{ij} \end{bmatrix} \begin{bmatrix} \sigma^2_{\rho_i} & 0 \\ 0 & \sigma^2_{\theta_i} \end{bmatrix} \begin{bmatrix} \frac{1}{\hat{\rho}_{ij}}\widehat{\Delta p}_{ij} & J\widehat{\Delta p}_{ij} \end{bmatrix}^T C(\hat{\phi}_i(k+1)) \\ &= C^T(\hat{\phi}_i(k+1)) \left(\frac{\sigma^2_{\rho_i}}{\hat{\rho}^2_{ij}} \widehat{\Delta p}_{ij} \widehat{\Delta p}^T_{ij} + \sigma^2_{\theta_i} J \widehat{\Delta p}_{ij} \widehat{\Delta p}^T_{ij} J^T \right) C(\hat{\phi}_i(k+1)) \\ &= C^T(\hat{\phi}_i(k+1)) \left(\frac{\sigma^2_{\rho_i}}{\hat{\rho}^2_{ij}} \left(\hat{\rho}^2_{ij} I_{2\times 2} - J \widehat{\Delta p}_{ij} \widehat{\Delta p}^T_{ij} J^T \right) + \sigma^2_{\theta_i} J \widehat{\Delta p}_{ij} \widehat{\Delta p}^T_{ij} J^T \right) C(\hat{\phi}_i(k+1)) \\ &= C^T(\hat{\phi}_i(k+1)) \left(\sigma^2_{\rho_i} I_{2\times 2} + \left(\sigma^2_{\theta_i} - \frac{\sigma^2_{\rho_i}}{\hat{\rho}^2_{ij}} \right) J \widehat{\Delta p}_{ij} \widehat{\Delta p}^T_{ij} J^T \right) C(\hat{\phi}_i(k+1)) \end{aligned} \quad (15)$$

where the variance of the noise in the distance and bearing measurements is given by

$$\sigma_{\rho_i}^2 = E\{n_{\rho_i}^2\} \ , \ \ \sigma_{\theta_i}^2 = E\{n_{\theta_i}^2\}$$

respectively. Due to the existence of the error component attributed to $\tilde{\phi}_i(k+1)$, the exteroceptive measurements that each robot performs at a given time instant are correlated. The matrix of correlation between the errors in the measurements $z_{ij}(k+1)$ and $z_{i\ell}(k+1)$ is

$$\begin{aligned} {}^iR_{j\ell}(k+1) &= \Gamma_{ij}(t)E\{n_{ij}(k+1)n_{i\ell}^T(k+1)\}\Gamma_{i\ell}^T(t) \\ &= \sigma_{\phi_i}^2 C^T(\hat{\phi}_i(k+1))J\widehat{\Delta p}_{ij_{k+1|k}}\widehat{\Delta p}_{i\ell_{k+1|k}}^T J^T C(\hat{\phi}_i(k+1)) \end{aligned} \quad (16)$$

The covariance matrix of all the measurements performed by robot i at the time instant $k+1$ can now be computed. This is a block matrix whose mn-th 2×2 submatrix element is ${}^iR_{mn}$, for $m, n = 1 \dots M_i$. Using the results of Eqs. (14), (15), and (16), this matrix can be written as

$$\boldsymbol{R}_i(k+1) = \boldsymbol{\Xi}_{\hat{\phi}_i}^T(k+1)\boldsymbol{R}_{o_i}(k+1)\boldsymbol{\Xi}_{\hat{\phi}_i}(k+1) \quad (17)$$

where

$$\boldsymbol{R}_{o_i}(k+1) = \begin{bmatrix} \sigma_{\rho_i}^2 I_{2\times 2} + \left(\sigma_{\phi_i}^2 + \sigma_{\theta_i}^2 - \frac{\sigma_{\rho_i}^2}{\hat{\rho}_{i1}^2}\right) J\widehat{\Delta p}_{i1}\widehat{\Delta p}_{i1}^T J^T & \dots & \sigma_{\phi_i}^2 J\widehat{\Delta p}_{i1}\widehat{\Delta p}_{iM_i}^T J^T \\ \vdots & \ddots & \vdots \\ \sigma_{\phi_i}^2 J\widehat{\Delta p}_{iM_i}\widehat{\Delta p}_{i1}^T J^T & \dots & \sigma_{\rho_i}^2 I_{2\times 2} + \left(\sigma_{\phi_i}^2 + \sigma_{\theta_i}^2 - \frac{\sigma_{\rho_i}^2}{\hat{\rho}_{iM_i}^2}\right) J\widehat{\Delta p}_{iM_i}\widehat{\Delta p}_{iM_i}^T J^T \end{bmatrix}$$

$$= \sigma_{\rho_i}^2 I_{2M_i\times 2M_i} + D_i(k+1)\left(\sigma_{\theta_i}^2 I_{M_i\times M_i} + \sigma_{\phi_i}^2 \mathbf{1}_{M_i\times M_i} - \text{diag}\left(\frac{\sigma_{\rho_{ij}}^2}{\hat{\rho}_i^2}\right)\right) D_i^T(k+1)$$

$$= \underbrace{\sigma_{\rho_i}^2 I_{2M_i\times 2M_i} - D_i(k+1)\,\text{diag}\left(\frac{\sigma_{\rho_{ij}}^2}{\hat{\rho}_i^2}\right) D_i^T(k+1)}_{R_1(k+1)} + \underbrace{\sigma_{\theta_i}^2 D_i(k+1) D_i^T(k+1)}_{R_2(k+1)} + \underbrace{\sigma_{\phi_i}^2 D_i(k+1)\mathbf{1}_{M_i\times M_i} D_i^T(k+1)}_{R_3(k+1)} \quad (18)$$

where

$$D_i(k+1) = \begin{bmatrix} J\widehat{\Delta p}_{i1_{k+1|k}} & \dots & 0_{2\times 1} \\ \vdots & \ddots & \vdots \\ 0_{2\times 1} & \dots & J\widehat{\Delta p}_{iM_{i_{k+1|k}}} \end{bmatrix} = \textbf{Diag}\left(J\widehat{\Delta p}_{ij_{k+1|k}}\right)$$

is a $2M_1 \times M_i$ block diagonal matrix, depending on the estimated positions of the robots. In Eq. (18) the covariance term $R_1(k+1)$ is the covariance of the error due to the noise in the range measurements, $R_2(k+1)$ is the covariance term due to the error in the bearing measurements, and $R_3(k+1)$ is the covariance term due to the error in the orientation estimates of the robot. The measurement matrix $\mathbf{H}(k+1)$ describing the measurements that are performed by all the robots of the team at time step $k+1$ is a matrix with block rows $\boldsymbol{H}_i(k+1)$, $i = 1 \dots M$, i.e.,

$$\mathbf{H}(k+1) = \begin{bmatrix} \boldsymbol{\Xi}_{\hat{\phi}_1}^T(k+1)\boldsymbol{H}_{o_1} \\ \boldsymbol{\Xi}_{\hat{\phi}_2}^T(k+1)\boldsymbol{H}_{o_2} \\ \vdots \\ \boldsymbol{\Xi}_{\hat{\phi}_M}^T(k+1)\boldsymbol{H}_{o_N} \end{bmatrix} = \textbf{Diag}\left(\boldsymbol{\Xi}_{\hat{\phi}_i}^T(k+1)\right)\begin{bmatrix} \boldsymbol{H}_{o_1} \\ \boldsymbol{H}_{o_2} \\ \vdots \\ \boldsymbol{H}_{o_M} \end{bmatrix} = \boldsymbol{\Xi}^T(k+1)\mathbf{H}_o \quad (19)$$

where

$$\boldsymbol{\Xi}(k+1) = \textbf{Diag}\left(\boldsymbol{\Xi}_{\hat{\phi}_i}(k+1)\right) \quad (20)$$

is a block diagonal matrix with block elements $\boldsymbol{\Xi}_{\hat{\phi}_i}(k+1)$, for $i = 1 \dots M$, and $\mathbf{H}_o$ is a matrix with block rows $\boldsymbol{H}_{o_i}$, $i = 1 \dots M$. Since the measurements performed by different robots are independent, the measurement covariance matrix for the entire system is given by

$$\mathbf{R}(k+1) = \textbf{Diag}\left(\boldsymbol{R}_i(k+1)\right) = \textbf{Diag}\left(\boldsymbol{\Xi}_{\hat{\phi}_i}^T\boldsymbol{R}_{o_i}(k+1)\boldsymbol{\Xi}_{\hat{\phi}_i}\right) = \boldsymbol{\Xi}^T(k+1)\mathbf{R}_o(k+1)\boldsymbol{\Xi}(k+1) \quad (21)$$

where $\mathbf{R}_o$ is a block diagonal matrix with block elements $\boldsymbol{R}_{o_i}$, $i = 1 \dots N$.

2.2.2 Absolute Position Measurements

If, in addition to relative position measurements, any of the robots, e.g., robot ℓ, has access to absolute positioning information, such as GPS measurements or from a map of the area, the corresponding submatrix element of $\mathbf{H}(k+1)$ is:

$$\boldsymbol{H}_{a_\ell} = \begin{bmatrix} 0_{2\times 2} & \cdots & \underbrace{I_{2\times 2}}_{\ell} & \cdots & 0_{2\times 2} \end{bmatrix} \tag{22}$$

while $\mathbf{R}_{a_\ell}$, the covariance of the absolute position measurement, is a constant provided by the specifications of the absolute positioning sensor.

To account for the absolute position measurements, the matrix $\mathbf{H}_o$ in Eq. (19) is augmented by simply appending the appropriate block rows $\boldsymbol{H}_{a_\ell}$, while $\mathbf{R}_o$ is augmented by appending the matrices R_{a_ℓ} on the diagonal, yielding

$$\mathbf{R}_o(k+1) = \begin{bmatrix} \mathbf{Diag}\left(\boldsymbol{R}_{o_i}(k+1)\right) & \mathbf{0} \\ \mathbf{0} & \mathbf{Diag}\left(\mathbf{R}_{a_\ell}\right) \end{bmatrix} \tag{23}$$

Additionally, in this case, the matrix $\boldsymbol{\Xi}^T(k+1)$ is also augmented as follows:

$$\boldsymbol{\Xi}(k+1) = \begin{bmatrix} \mathbf{Diag}\left(\Xi_{\hat{\phi}_i}(k+1)\right) & \mathbf{0} \\ \mathbf{0} & I_{2M_a\times 2M_a} \end{bmatrix} \tag{24}$$

where we have assumed that M_a absolute position measurements are available to the robots of the team.

2.2.3 Covariance update equation

We now write the covariance update equation, which is

$$\begin{aligned} \mathbf{P}_{k+1|k+1} &= \mathbf{P}_{k+1|k} - \mathbf{P}_{k+1|k}\mathbf{H}^T(k+1)\left(\mathbf{H}(k+1)\mathbf{P}_{k+1|k}\mathbf{H}^T(k+1) + \mathbf{R}(k+1)\right)^{-1}\mathbf{H}(k+1)\mathbf{P}_{k+1|k} \\ &= \mathbf{P}_{k+1|k} \\ &\quad - \mathbf{P}_{k+1|k}\mathbf{H}_o^T\boldsymbol{\Xi}(k+1)\left(\boldsymbol{\Xi}^T(k+1)\mathbf{H}_o\mathbf{P}_{k+1|k}\mathbf{H}_o^T\boldsymbol{\Xi}(k+1) + \boldsymbol{\Xi}^T(k+1)\mathbf{R}_o(k+1)\boldsymbol{\Xi}(k+1)\right)^{-1}\boldsymbol{\Xi}^T(k+1)\mathbf{H}_o\mathbf{P}_{k+1|k} \\ &= \mathbf{P}_{k+1|k} - \mathbf{P}_{k+1|k}\mathbf{H}_o^T\left(\mathbf{H}_o\mathbf{P}_{k+1|k}\mathbf{H}_o^T + \mathbf{R}_o(k+1)\right)^{-1}\mathbf{H}_o\mathbf{P}_{k+1|k} \end{aligned} \tag{25}$$

In order to derive the last expression, property $\boldsymbol{\Xi}^T(k+1) = \boldsymbol{\Xi}^{-1}(k+1)$ was employed. This property is a consequence of the definition of matrix $\boldsymbol{\Xi}(k+1)$ (cf. Eqs. (12) and (20) or (24)), and the fact that rotation matrices satisfy $C^T(\hat{\phi}_i) = C^{-1}(\hat{\phi}_i)$.

2.3 The Riccati Recursion

The metric we employ in order to characterize the positioning performance of CL is the covariance matrix of the robots' position estimates. By combining Eqs. (7) and (25) we derive the discrete-time Riccati recursion, that describes the time evolution of the covariance matrix:

$$\mathbf{P}_{k+2|k+1} = \mathbf{P}_{k+1|k} - \mathbf{P}_{k+1|k}\mathbf{H}_o^T\left(\mathbf{H}_o\mathbf{P}_{k+1|k}\mathbf{H}_o^T + \mathbf{R}_o(k+1)\right)^{-1}\mathbf{H}_o\mathbf{P}_{k+1|k} + \mathbf{Q}(k+1)$$

This recursion provides the value of the covariance matrix at each time step, right after the propagation phase of the EKF. To simplify the notation, we set $\mathbf{P}_k = \mathbf{P}_{k+1|k}$ and $\mathbf{P}_{k+1} = \mathbf{P}_{k+2|k+1}$, and therefore we can write

$$\mathbf{P}_{k+1} = \mathbf{P}_k - \mathbf{P}_k\mathbf{H}_o^T\left(\mathbf{H}_o\mathbf{P}_k\mathbf{H}_o^T + \mathbf{R}_o(k+1)\right)^{-1}\mathbf{H}_o\mathbf{P}_k + \mathbf{Q}(k+1) \tag{26}$$

We note that the matrices $\mathbf{Q}(k+1)$ and $\mathbf{R}_o(k+1)$ in this Riccati recursion are time varying, and this does not allow the derivation of any closed form expressions for the time evolution of $\mathbf{P}_k$, in the general case. We therefore have to resort to deriving *bounds* for the covariance of the CL position estimates. The following two lemmas are the basis of our analysis:

Lemma 2.1 *If* $\mathbf{R}_u$ *and* $\mathbf{Q}_u$ *are matrices such that* $\mathbf{R}_u \succeq \mathbf{R}_o(k)$ *and* $\mathbf{Q}_u \succeq \mathbf{Q}(k)$ *for all* $k \geq 0$*, then the solution to the Riccati recursion*

$$\mathbf{P}_{k+1}^u = \mathbf{P}_k^u - \mathbf{P}_k^u \mathbf{H}_o^T \left(\mathbf{H}_o \mathbf{P}_k^u \mathbf{H}_o^T + \mathbf{R}_u\right)^{-1} \mathbf{H}_o \mathbf{P}_k^u + \mathbf{Q}_u \tag{27}$$

with the initial condition $\mathbf{P}_0^u = \mathbf{P}_0$*, satisfies* $\mathbf{P}_k^u \succeq \mathbf{P}_k$ *for all* $k \geq 0$.

Lemma 2.2 *If* $\bar{\mathbf{R}}$ *and* $\bar{\mathbf{Q}}$ *are matrices such that* $\bar{\mathbf{R}} = E\{\mathbf{R}_o(k)\}$ *and* $\bar{\mathbf{Q}} = \{\mathbf{Q}(k)\}$ *for all* $k \geq 0$*, then the solution to the Riccati recursion*

$$\bar{\mathbf{P}}_{k+1} = \bar{\mathbf{P}}_k - \bar{\mathbf{P}}_k \mathbf{H}_o^T \left(\mathbf{H}_o \bar{\mathbf{P}}_k \mathbf{H}_o^T + \bar{\mathbf{R}}\right)^{-1} \mathbf{H}_o \bar{\mathbf{P}}_k + \bar{\mathbf{Q}} \tag{28}$$

with the initial condition $\bar{\mathbf{P}}_0 = \mathbf{P}_0$*, satisfies* $\bar{\mathbf{P}}_k \succeq E\{\mathbf{P}_k\}$ *for all* $k \geq 0$.

Essentially, Lemma 2.1 maintains that in order to derive an upper bound on the *worst-case* covariance matrix of the position estimates in CL, it suffices to derive *upper bounds* for the covariance matrices of the system and measurement noise, and to solve a *constant coefficient* Riccati recursion. Similarly, Lemma 2.2 states that an upper bound on the *expected* positioning uncertainty of CL is determined as the solution of a constant coefficient Riccati recursion, where the covariance matrices of the system and measurement noise have been replaced by their respective *average* values. The proofs for these lemmas are given in Appendices A and B respectively. In the remainder of this section, we derive appropriate upper bounds, as well as the average values of the matrices $\mathbf{Q}(k)$ and $\mathbf{R}_o(k)$ respectively.

- **Derivation of upper bounds for $\mathbf{Q}(t)$ and $\mathbf{R}_o(t)$**

In order to derive an upper bound for the covariance matrix $\mathbf{Q}(k)$ we recall that $\mathbf{Q}(k) = \mathbf{Diag}(Q_i(k))$, where

$$Q_i(k) = C(\hat{\phi}_i(k)) \begin{bmatrix} \delta t^2 \sigma_{V_i}^2 & 0 \\ 0 & \delta t^2 V_{m_i}^2(k) \sigma_{\phi_i}^2 \end{bmatrix} C^T(\hat{\phi}_i(k))$$

From the properties of rotation matrices it is known that $C^{-1}(\hat{\phi}_i(k)) = C^T(\hat{\phi}_i(k))$, and thus $Q_i(k)$ is related by a similarity transformation to the matrix

$$\begin{bmatrix} \delta t^2 \sigma_{V_i}^2 & 0 \\ 0 & \delta t^2 V_{m_i}^2(k) \sigma_{\phi_i}^2 \end{bmatrix}$$

which implies that the eigenvalues of $Q_i(k)$ are $\delta t^2 \sigma_{V_i}^2$ and $\delta t^2 V_{m_i}^2(k) \sigma_{\phi_i}^2$. We assume that the velocity of each robot is approximately constant, and equal to V_i, and denote

$$q_i = \max\left(\delta t^2 \sigma_{V_i}^2, \delta t^2 V_{m_i}^2(k) \sigma_{\phi_i}^2\right) \simeq \max\left(\delta t^2 \sigma_{V_i}^2, \delta t^2 V_i^2 \sigma_{\phi_i}^2\right) \tag{29}$$

This definition states that q_i is the largest eigenvalue of $Q_i(k)$, and therefore

$$Q_i(k) \preceq q_i I_{2\times 2} \Rightarrow \mathbf{Q}(k) \preceq \mathbf{Diag}(q_i I_{2\times 2}) = \mathbf{Q}_u \tag{30}$$

An upper bound on $\mathbf{R}_o(k)$ is obtained by considering each if its block diagonal elements, $\boldsymbol{R}_{o_i}(k)$. Referring to Eq. (18), we examine the terms $R_1(k)$, $R_2(k)$ and $R_3(k)$ separately: the term expressing the effect of the noise in the range measurements is

$$R_1(k) = \sigma_{\rho_i}^2 I_{2M_i \times 2M_i} - D_i(k)\,\mathrm{diag}\left(\frac{\sigma_{\rho_i}^2}{\hat{\rho}_{ij}^2}\right) D_i^T(k) \preceq \sigma_{\rho_i}^2 I_{2M_i \times 2M_i} \tag{31}$$

The last matrix inequality follows from the fact that the term being subtracted from $\sigma_{\rho_i}^2 I_{2M_i \times 2M_i}$ is a positive semi-definite matrix. The covariance term due to the noise in the bearing measurement is

$$\begin{aligned} R_2(k) &= \sigma_{\theta_i}^2 D_i(k) D_i^T(k) \\ &= \sigma_{\theta_i}^2 \mathbf{Diag}\left(\hat{\rho}_{ij}^2 \begin{bmatrix} \sin^2(\hat{\theta}_{ij}) & \sin(\hat{\theta}_{ij})\cos(\hat{\theta}_{ij}) \\ \sin(\hat{\theta}_{ij})\cos(\hat{\theta}_{ij}) & \cos^2(\hat{\theta}_{ij}) \end{bmatrix}\right) \\ &\preceq \sigma_{\theta_i}^2 \mathbf{Diag}\left(\hat{\rho}_{ij}^2 I_{2\times 2}\right) \\ &\preceq \sigma_{\theta_i}^2 \rho_o^2 I_{2M_i \times 2M_i} \end{aligned} \tag{32}$$

where ρ_o is the maximum range at which a measurement can occur, determined either by the characteristics of the robots' sensors or by the properties of the area in which the robots move. Finally, the covariance term due to the error in the orientation of the measuring robot is $R_3(k) = \sigma^2_{\phi_i} D_i(k) \mathbf{1}_{M_i \times M_i} D_i^T(k)$. Calculation of the eigenvalues of the matrices $\mathbf{1}_{M_i \times M_i}$ and $I_{M_i \times M_i}$ verifies that $\mathbf{1}_{M_i \times M_i} \preceq M_i I_{M_i \times M_i}$, and thus we can write $R_3(k) \preceq M_i \sigma^2_{\phi_i} D_i(k) D_i^T(k)$. By derivations analogous to those employed to yield an upper bound for $R_2(k)$, we can show that

$$R_3(k) \preceq M_i \sigma^2_{\phi_i} \rho_o^2 I_{2M_i \times 2M_i}$$

By combining this result with those of Eqs. (31), (32), we can write $\boldsymbol{R}_{o_i}(k) = R_1(k) + R_2(k) + R_3(k) \preceq \mathbf{R}_i^u$, where

$$\mathbf{R}_i^u = \left(\sigma^2_{\rho_i} + M_i \sigma^2_{\phi_i} \rho_o^2 + \sigma^2_{\theta_i} \rho_o^2\right) I_{2M_i \times 2M_i} = r_i I_{2M_i \times 2M_i} \tag{33}$$

with

$$r_i = \sigma^2_{\rho_i} + M_i \sigma^2_{\phi_i} \rho_o^2 + \sigma^2_{\theta_i} \rho_o^2 \tag{34}$$

Thus, we can write

$$\mathbf{R}_o(k) = \mathbf{Diag}(\boldsymbol{R}_{o_i}(k)) \preceq \mathbf{Diag}(r_i I_{M_i \times M_i}) = \mathbf{R}_u \tag{35}$$

- **Derivation of the Expected Values of $\mathbf{Q}(k)$ and $\mathbf{R}_o(k)$**

In order to derive the average value of $\mathbf{Q}(k)$ we note that

$$\begin{aligned} Q_i(k) &= C(\hat{\phi}_i(k)) \begin{bmatrix} \delta t^2 \sigma^2_{V_i} & 0 \\ 0 & \delta t^2 V^2_{m_i}(k) \sigma^2_{\phi_i} \end{bmatrix} C^T(\hat{\phi}_i(k)) \\ &= \delta t^2 \begin{bmatrix} \sigma^2_{V_i} \cos^2(\hat{\phi}_i) + V^2_{m_i}(k) \sigma^2_{\phi_i} \sin^2(\hat{\phi}_i) & \left(\sigma^2_{V_i} - V^2_{m_i}(k)\sigma^2_{\phi_i}\right) \sin(\hat{\phi}_i)\cos(\hat{\phi}_i) \\ \left(\sigma^2_{V_i} - V^2_{m_i}(k)\sigma^2_{\phi_i}\right) \sin(\hat{\phi}_i)\cos(\hat{\phi}_i) & \sigma^2_{V_i} \sin^2(\hat{\phi}_i) + V^2_{m_i}(k) \sigma^2_{\phi_i} \cos^2(\hat{\phi}_i) \end{bmatrix} \end{aligned}$$

and therefore, by averaging over all values of orientation, the expected value of $Q_i(k)$ is derived:

$$E\{Q_i(k)\} \;=\; \delta t^2 \frac{\sigma_V^2 + V_i^2 \sigma^2_{\phi_i}}{2} I_{2\times 2} \;=\; \bar{q}_i I_{2\times 2}$$

where

$$\bar{q}_i = \delta t^2 \frac{\sigma_V^2 + V_i^2 \sigma^2_{\phi_i}}{2}$$

Thus,

$$E\{\mathbf{Q}(k)\} \;=\; \mathbf{Diag}\left(E\{Q_i(k)\}\right) \;=\; \mathbf{Diag}(\bar{q}_i I_{2\times 2}) \;=\; \bar{\mathbf{Q}} \tag{36}$$

The average value of $\mathbf{R}_o(k)$ is derived by employing the property

$$E\{\mathbf{R}_o(k)\} \;=\; E\{\mathbf{Diag}(\boldsymbol{R}_{o_i}(k))\} \;=\; \mathbf{Diag}(E\{\boldsymbol{R}_{o_i}(k)\}) \tag{37}$$

We therefore see that the average values of the matrices $\boldsymbol{R}_{o_i}(k)$, $i = 1 \ldots N$ need to be determined. From Eq. (18) we note that evaluation of the average value of $\boldsymbol{R}_{o_i}(k)$ requires the computation of the expected values of the following terms:

$$T_1 = \frac{\widehat{\Delta p}_{ij} \widehat{\Delta p}_{ij}^T}{\hat{\rho}^2_{ij}}, \quad T_2 = \widehat{\Delta p}_{ij} \widehat{\Delta p}_{ij}^T, \quad \text{and} \quad T_3 = \widehat{\Delta p}_{ij} \widehat{\Delta p}_{i\ell}^T \tag{38}$$

for $j, \ell = 1 \ldots M_i$. The average value of T_1 is easily derived by employing the polar coordinate description of the vector $\widehat{\Delta p}_{ij}$ in terms of $\hat{\rho}_{ij}$ and $\hat{\theta}_{ij}$, which yields

$$\begin{aligned} T_1 &= \frac{\widehat{\Delta p}_{ij} \widehat{\Delta p}_{ij}^T}{\hat{\rho}^2_{ij}} \\ &= \frac{1}{\hat{\rho}^2_{ij}} \begin{bmatrix} \hat{\rho}^2_{ij} \cos^2(\hat{\theta}_{ij}) & \hat{\rho}^2_{ij} \sin(\hat{\theta}_{ij}) \cos(\hat{\theta}_{ij}) \\ \hat{\rho}^2_{ij} \sin(\hat{\theta}_{ij}) \cos(\hat{\theta}_{ij}) & \hat{\rho}^2_{ij} \sin^2(\hat{\theta}_{ij}) \end{bmatrix} \\ &= \begin{bmatrix} \cos^2(\hat{\theta}_{ij}) & \sin(\hat{\theta}_{ij}) \cos(\hat{\theta}_{ij}) \\ \sin(\hat{\theta}_{ij}) \cos(\hat{\theta}_{ij}) & \sin^2(\hat{\theta}_{ij}) \end{bmatrix} \end{aligned}$$

From the last expression we conclude that for any probability density function that guarantees a uniform distribution for the bearing angle of the measurements (i.e., any symmetric probability density function), the average value of the term T_1 is

$$E\{T_1\} \;=\; \frac{1}{2} I_{2\times 2}$$

In order to compute the expected value of the terms T_2 and T_3, we assume that the robots are located in a square arena of side α, and that their positions are described by uniformly distributed random variables in the interval $[-\alpha/2, \alpha/2]$. We can thus write

$$\begin{aligned}
E\{T_2\} = E\{\widehat{\Delta p}_{ij}\widehat{\Delta p}_{ij}^T\} &= E\left\{\begin{bmatrix} \widehat{\Delta x}_{ij}^2 & \widehat{\Delta x}_{ij}\widehat{\Delta y}_{ij} \\ \widehat{\Delta y}_{ij}\widehat{\Delta x}_{ij} & \widehat{\Delta y}_{ij}^2 \end{bmatrix}\right\} \\
&= \begin{bmatrix} E\{x_j^2 - 2x_i x_j + x_i^2\} & E\{x_j y_j - x_j y_i - x_i y_j + x_i y_i\} \\ E\{y_j x_j - y_j x_i - y_i x_j + y_i x_i\} & E\{y_j^2 - 2y_j y_i + y_i^2\} \end{bmatrix} \\
&= \begin{bmatrix} 2E\{x_i^2\} & 0 \\ 0 & 2E\{y_i^2\} \end{bmatrix} \\
&= \begin{bmatrix} \frac{\alpha^2}{6} & 0 \\ 0 & \frac{\alpha^2}{6} \end{bmatrix} \\
&= \frac{\alpha}{12} I_{2\times 2}
\end{aligned}$$

and similarly,

$$\begin{aligned}
E\{T_3\} = E\{\widehat{\Delta p}_{ij}\widehat{\Delta p}_{i\ell}^T\} &= E\left\{\begin{bmatrix} \widehat{\Delta x}_{ij}\widehat{\Delta x}_{i\ell} & \widehat{\Delta x}_{ij}\widehat{\Delta y}_{i\ell} \\ \widehat{\Delta y}_{ij}\widehat{\Delta x}_{i\ell} & \widehat{\Delta y}_{ij}\widehat{\Delta y}_{i\ell} \end{bmatrix}\right\} \\
&= \begin{bmatrix} E\{x_j x_\ell - x_i x_\ell - x_j x_i + x_i^2\} & E\{x_j y_\ell - x_j y_i - x_i y_\ell + x_i y_i\} \\ E\{y_j x_\ell - y_j x_i - y_i x_\ell + y_i x_i\} & E\{y_j y_\ell - y_i y_\ell - y_j y_i + y_i^2\} \end{bmatrix} \\
&= \begin{bmatrix} E\{x_i^2\} & 0 \\ 0 & E\{y_i^2\} \end{bmatrix} \\
&= \begin{bmatrix} \frac{\alpha^2}{12} & 0 \\ 0 & \frac{\alpha^2}{12} \end{bmatrix} \\
&= \frac{\alpha}{12} I_{2\times 2}
\end{aligned}$$

These results enable us to obtain the average value of the matrices $\boldsymbol{R}_{o_i}(k)$, $i = 1 \ldots N$. Employing the linearity of the expectation operator we obtain

$$\begin{aligned}
\bar{\mathbf{R}}_i &= E\{\boldsymbol{R}_{o_i}(k)\} \\
&= \begin{bmatrix} \left(\frac{1}{2}\sigma_{\rho_i}^2 + \frac{1}{6}\sigma_{\phi_i}^2 + \frac{1}{6}\sigma_{\theta_i}^2\right) I_{2\times 2} & \cdots & \frac{1}{12}\sigma_{\phi_i}^2 I_{2\times 2} \\ \vdots & \ddots & \vdots \\ \frac{1}{12}\sigma_{\phi_i}^2 I_{2\times 2} & \cdots & \left(\frac{1}{2}\sigma_{\rho_i}^2 + \frac{1}{6}\sigma_{\phi_i}^2 + \frac{1}{6}\sigma_{\theta_i}^2\right) I_{2\times 2} \end{bmatrix} \\
&= \left(\frac{1}{2}\sigma_{\rho_i}^2 + \frac{1}{12}\sigma_{\phi_i}^2 + \frac{1}{6}\sigma_{\theta_i}^2\right) I_{2M_i\times 2M_i} + \frac{1}{12}\sigma_{\phi_i}^2 \left(\mathbf{1}_{M_i\times M_i} \otimes I_{2\times 2}\right)
\end{aligned}$$

The average value of $\mathbf{R}_o(k)$ is therefore

$$\bar{\mathbf{R}} \;=\; E\{\mathbf{R}_o(k)\} \;=\; \mathbf{Diag}(\bar{\mathbf{R}}_i) \tag{39}$$

2.4 Evaluation of the Upper Bounds at Steady State

Lemmas 2.1 and 2.2 allow the evaluation of upper bounds on the worst case uncertainty and on the average uncertainty of the position estimates in CL, at *any* time instant after the deployment of the robot team. This can be achieved, for example, by numerical evaluation of the solution to the recursions in Eqs. (27) and (28) respectively. For many applications, it is of interest however, to study the steady-state behavior of the positioning uncertainty in CL. For this reason, we now derive the steady-state values of the solutions to the recursions (27) and (28). By "steady-state values" we refer to the values of the covariance matrix after a sufficient time has elapsed, enough for the the initial transient phenomena in the solutions to subside. The steady state solutions are derived by evaluating the limit of $\mathbf{P}_k^u$ and $\bar{\mathbf{P}}_k$ as $k \to \infty$.

We note at this point that the Riccati recursions of Eqs. (27) and (28) essentially describe the time evolution of the covariance of the position estimates in two hypothetical CL scenarios, where the system model is a Linear Time Invariant (LTI) one. Therefore, the problem of computing the upper bounds on the steady state positioning uncertainty in CL reduces to the problem of *determining the steady state covariance matrix for a LTI CL system model*.

To avoid redundant derivations, in the following we will solve for the steady state solution of the following Riccati recursion:

$$\mathbf{P}_{k+1}^s = \mathbf{P}_k^s - \mathbf{P}_k^s \mathbf{H}_o'^T \left(\mathbf{H}_o' \mathbf{P}_k^s \mathbf{H}_o'^T + \mathbf{R}_s\right)^{-1} \mathbf{H}_o' \mathbf{P}_k^s + \mathbf{Q}_s \tag{40}$$

After deriving the steady state solution of this recursion, we employ the substitutions

$$\mathbf{R}_s \to \mathbf{R}_u, \qquad \mathbf{Q}_s \to \mathbf{Q}_u$$

and

$$\mathbf{R}_s \to \bar{\mathbf{R}}\,, \qquad \mathbf{Q}_s \to \bar{\mathbf{Q}}$$

in order to obtain the steady state solutions of the Riccati recursions of Lemmas (2.1) and (2.2) respectively.

We first note that the Riccati recursion in Eq. (40) can be reformulated as follows, by use of the matrix inversion lemma (cf. Appendix H):

$$\begin{aligned}\mathbf{P}_{k+1}^s &= \mathbf{P}_k^s - \mathbf{P}_k^s \mathbf{H}_o^T \left(\mathbf{H}_o \mathbf{P}_k^s \mathbf{H}_o^T + \mathbf{R}_s\right)^{-1} \mathbf{H}_o \mathbf{P}_k^s + \mathbf{Q}_s \\ &= \mathbf{P}_k^s \left(I_{2N\times 2N} + \mathbf{H}_o^T \mathbf{R}_s^{-1} \mathbf{H}_o \mathbf{P}_k^s\right)^{-1} + \mathbf{Q}_s \end{aligned} \tag{41}$$

The derivations are simplified by defining the *normalized* covariance matrix as

$$\mathbf{P}_{n_k} = \mathbf{Q}_s^{-1/2} \mathbf{P}_k^s \mathbf{Q}_s^{-1/2} \tag{42}$$

Pre- and post-multiplying Eq. (41) by $\mathbf{Q}_s^{-1/2}$, and simple algebraic manipulation yields

$$\mathbf{P}_{n_{k+1}} = \mathbf{P}_{n_k} \left(I_{2N\times 2N} + \mathbf{C}_s \mathbf{P}_{n_k}\right)^{-1} + I_{2N\times 2N} \tag{43}$$

where

$$\mathbf{C}_s = \mathbf{Q}_s^{1/2} \mathbf{H}_o^T \mathbf{R}_s^{-1} \mathbf{H}_o \mathbf{Q}_s^{1/2}$$

Note that the only parameter in the Riccati recursion (43) is the matrix $\mathbf{C}_s$, which contains the main parameters that characterize the localization performance of the robotic team. The eigenvalues of this matrix, which are studied in Appendix E, are in close relation with the type and number of exteroceptive measurements recorded by the robots of the team, and determine the properties of the upper bound on the steady-state positioning uncertainty. To further simplify the derivations, we denote the Singular Value Decomposition (SVD) of $\mathbf{C}_s$ as

$$\mathbf{C}_s = \mathbf{U}_s \,\mathrm{diag}(\lambda_i) \mathbf{U}_s^T = \mathbf{U}_s \Lambda \mathbf{U}_s^T$$

and substituting in Eq. (43) we obtain[2]

$$\begin{aligned}\mathbf{P}_{n_{k+1}} &= \mathbf{P}_{n_k} \left(I + \mathbf{U}_s \Lambda \mathbf{U}_s^T \mathbf{P}_{n_k}\right)^{-1} + I \Rightarrow \\ \mathbf{U}_s^T \mathbf{P}_{n_{k+1}} \mathbf{U}_s &= \mathbf{U}_s^T \mathbf{P}_{n_k} \mathbf{U}_s \mathbf{U}_s^T \left(I + \mathbf{U}_s \Lambda \mathbf{U}_s^T \mathbf{P}_{n_k}\right)^{-1} \mathbf{U}_s + I \Rightarrow \\ \mathbf{U}_s^T \mathbf{P}_{n_{k+1}} \mathbf{U}_s &= \mathbf{U}_s^T \mathbf{P}_{n_k} \mathbf{U}_s \left(I + \Lambda \mathbf{U}_s^T \mathbf{P}_{n_k} \mathbf{U}_s\right)^{-1} + I \end{aligned}$$

[2] To make the notation less cumbersome, we hereafter omit the dimension index from the identity matrices, whenever their dimension is equal to the dimension of the state covariance matrix. I.e., from this point on, $I = I_{2N\times 2N}$.

We define

$$\mathbf{P}_{nn_k} = \mathbf{U}_s^T \mathbf{P}_{n_k} \mathbf{U}_s \quad (44)$$

and we obtain the recursion

$$\mathbf{P}_{nn_{k+1}} = \mathbf{P}_{nn_{k+1}} \left(I + \Lambda \mathbf{P}_{nn_{k+1}} \right)^{-1} + I \quad (45)$$

This form of the recursion is simpler, since now the only parameter is the diagonal matrix of the eigenvalues of $\mathbf{C}_s$.

We hereafter present the derivation of the steady-state solution for $\mathbf{P}_{nn_{k+1}}$, based on the availability of absolute positioning information:

2.4.1 Observable system

We first study the case in which at least one of the robots has access to absolute position measurements. In this case the system is observable [10], and therefore the covariance of the robot's position estimates remains bounded at steady state. For this case, it is shown in Appendix E that $\mathrm{rank}(\mathbf{C}_s) = 2N$, and therefore all the singular values of $\mathbf{C}_s$ are positive.

Since we are dealing with an observable system, the solution to Eq. (45) will converge to a constant value at steady state, determined by solving the Discrete Algebraic Riccati Equation (DARE):

$$\mathbf{P}_{nn_{ss}} = \mathbf{P}_{nn_{ss}} \left(I + \Lambda \mathbf{P}_{nn_{ss}} \right)^{-1} + I$$

Since the system is both controllable and observable, the solution of the above DARE is unique [11]. Therefore, we can "guess" a solution, and if it satisfies the DARE, we can be assured that this is the only possible solution. We now assume a diagonal form for $\mathbf{P}_{nn_{ss}}$. In that case, all the matrices in the above DARE are diagonal, and thus we obtain the following set of $2N$ independent equations:

$$P_{nn_{ss}}(i,i) = \frac{P_{nn_{ss}}(i,i)}{1 + \lambda_i P_{nn_{ss}}(i,i)} + 1, \quad i = 1 \ldots 2N \quad (46)$$

Whose solution is given by

$$P_{nn_{ss}}(i,i) = \frac{1}{2} + \sqrt{\frac{1}{4} + \frac{1}{\lambda_i}}$$

By substitution of this result in Eqs. (44) and (42), we obtain the steady state solution to the Riccati recursion (40):

$$\mathbf{P}_{ss}^s = \mathbf{Q}_s^{1/2} \mathbf{U}_s \, \mathrm{diag}\left(\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{1}{\lambda_i}} \right) \mathbf{U}_s^T \mathbf{Q}_s^{1/2} \quad (47)$$

Finally, from this result, by setting

$$\mathbf{R}_s \to \mathbf{R}_u, \qquad \mathbf{Q}_s \to \mathbf{Q}_u$$

and

$$\mathbf{R}_s \to \bar{\mathbf{R}}\,, \qquad \mathbf{Q}_s \to \bar{\mathbf{Q}}$$

we can derive the following lemmas:

Lemma 2.3 *The steady state covariance of the position estimates for a team of robots performing CL, when at least one robot has access to absolute positioning information is bounded above by the matrix*

$$\mathbf{P}_{ss}^u = \mathbf{Q}_u^{1/2} \mathbf{U}_u \, \mathrm{diag}\left(\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{1}{\lambda_{u_i}}} \right) \mathbf{U}_u^T \mathbf{Q}_u^{1/2} \quad (48)$$

where we have denoted the singular value decomposition of $\mathbf{C}_u = \mathbf{Q}_u^{1/2} \mathbf{H}_o^T \mathbf{R}_u^{-1} \mathbf{H}_o \mathbf{Q}_u^{1/2}$ *as* $\mathbf{C}_u = \mathbf{U}_u \, \mathrm{diag}(\lambda_{u_i}) \mathbf{U}_u^T$.

Lemma 2.4 *The expected steady state covariance of the position estimates for a team of robots performing CL, when at least one robot has access to absolute positioning information is bounded above by the matrix*

$$\bar{\mathbf{P}}_{ss} \quad = \quad \bar{\mathbf{Q}}^{1/2}\bar{\mathbf{U}}\,\mathrm{diag}\left(\frac{1}{2}+\sqrt{\frac{1}{4}+\frac{1}{\bar{\lambda}_i}}\right)\bar{\mathbf{U}}^T\bar{\mathbf{Q}}^{1/2} \tag{49}$$

where we have denoted the singular value decomposition of $\bar{\mathbf{C}} = \bar{\mathbf{Q}}^{1/2}\mathbf{H}_o^T\bar{\mathbf{R}}^{-1}\mathbf{H}_o\bar{\mathbf{Q}}^{1/2}$ *as* $\bar{\mathbf{C}} = \bar{\mathbf{U}}\,\mathrm{diag}(\bar{\lambda}_i)\bar{\mathbf{U}}^T$.

At this point we should note that the upper bounds on the steady-state uncertainty depend on the topology of the RPMG and the accuracy of the proprioceptive and exteroceptive sensors of the robots. However, the steady-state uncertainty is independent of the initial covariance of the robots, which comes as no surprise, since the system is observable.

2.4.2 Unobservable System

If none of the robots has access to absolute position measurements, the system is unobservable from a Control Theoretic point of view. In Appendix E it is shown that in this case $\mathrm{rank}(\mathbf{C}_s) = 2N-2$, which implies that $\mathbf{C}_s$ has two singular values equal to zero. This fact somewhat complicates the derivations, as now the steady-state solution to Eq. (45) depends on the initial uncertainty of the robots' position estimates.

We first consider the situation in which the initial covariance matrix is equal to zero, i.e. $\mathbf{P}_0 = \mathbf{0}_{2N\times 2N}$. We denote the solution to Eq. (45) by $\mathbf{P}^{(0)}_{nn_k}$ in this case, and it is easy to see that $\mathbf{P}^{(0)}_{nn_0} = \mathbf{0}_{2N\times 2N}$. As a result, for $k=0$ the right-hand side of Eq. (45) is a diagonal matrix. By a simple induction argument, we can show that the solution to this recursion with zero initial condition retains a diagonal form for all $k \geq 0$. Addressing each of the diagonal elements individually, we observe that for the first $2N-2$ elements, which correspond to the nonzero singular values, we obtain the recursions

$$\mathbf{P}^{(0)}_{nn_{k+1}}(i,i) = \mathbf{P}^{(0)}_{nn_k}(i,i)\left(1+\lambda_i\mathbf{P}^{(0)}_{nn_k}(i,i)\right)^{-1}+1, \quad i = 1\ldots 2N-2 \tag{50}$$

while for the last two diagonal elements we obtain

$$\mathbf{P}^{(0)}_{nn_{k+1}}(i,i) = \mathbf{P}^{(0)}_{nn_k}(i,i)+1, \quad i = 2N-1, 2N$$

The steady-state solution for the first $2N-2$ elements is derived by solving $2N-2$ independent scalar equations of the form

$$\mathbf{P}^{(0)}_{nn_{ss}}(i,i) \quad = \quad \frac{\mathbf{P}^{(0)}_{nn_{ss}}(i,i)}{1+\lambda_i\mathbf{P}^{(0)}_{nn_{ss}}(i,i)}+1, i = 1\ldots 2N \tag{51}$$

which have the same structure as in Eq. (46) (the eigenvalues λ_i will, in general, be different). Therefore the asymptotic solution for $\mathbf{P}_{nn_k}$ is given by

$$\mathbf{P}^{(0)}_{nn_{ss}}(k) = \begin{bmatrix} \mathrm{diag}_{2N-2}\left(\frac{1}{2}+\sqrt{\frac{1}{4}+\frac{1}{\lambda_i}}\right) & \mathbf{0}_{(2N-2)\times 2} \\ \mathbf{0}_{2\times(2N-2)} & kI_{2\times 2}\end{bmatrix} \tag{52}$$

From the last expression we see that when the initial value for $\mathbf{P}_{nn_k}$ is equal to zero, at steady state the rate of increase of the matrix $\mathbf{P}_{nn_k}$ is given by

$$\mathbf{D} = \mathbf{P}^{(0)}_{nn_{ss}}(k+1) - \mathbf{P}^{(0)}_{nn_{ss}}(k) = \begin{bmatrix} \mathbf{0}_{(2N-2)\times(2N-2)} & \mathbf{0}_{(2N-2)\times 2} \\ \mathbf{0}_{2\times(2N-2)} & I_{2\times 2}\end{bmatrix}$$

Since Eq. (45) describes the time evolution of the covariance in a LTI system, we do not expect the *rate* of increase at steady state to depend on the initial conditions. With this in mind, we will now introduce a change of variables, that will facilitate the derivation of the steady state solution of Eq. (45) for arbitrary initial conditions. We set

$$\mathbf{P}_{nn_k} \quad = \quad \widetilde{\mathbf{P}}_k + k\mathbf{D} \tag{53}$$

and substitution in Eq. (45) yields

$$\begin{aligned}
\widetilde{\mathbf{P}}_{k+1} + (k+1)\mathbf{D} &= \left(\widetilde{\mathbf{P}}_k + k\mathbf{D}\right)\left(I + \Lambda\left(\widetilde{\mathbf{P}}_k + k\mathbf{D}\right)\right)^{-1} + I \Rightarrow \\
\widetilde{\mathbf{P}}_{k+1} + (k+1)\mathbf{D} &= \left(\widetilde{\mathbf{P}}_k + k\mathbf{D}\right)\left(I + \Lambda\widetilde{\mathbf{P}}_k + k\Lambda\mathbf{D}\right)^{-1} + I \Rightarrow \\
\widetilde{\mathbf{P}}_{k+1} + (k+1)\mathbf{D} &= \left(\widetilde{\mathbf{P}}_k + k\mathbf{D}\right)\left(I + \Lambda\widetilde{\mathbf{P}}_k\right)^{-1} + I \Rightarrow \\
\widetilde{\mathbf{P}}_k + (k+1)\mathbf{D} &= \widetilde{\mathbf{P}}_k\left(I + \Lambda\widetilde{\mathbf{P}}_k\right)^{-1} + k\mathbf{D}\left(I + \Lambda\widetilde{\mathbf{P}}_k\right)^{-1} + I
\end{aligned}$$

Where we have used the fact that since the 2 smallest eigenvalues of $\mathbf{C}_s$ equal zero, we have $\Lambda\mathbf{D} = \mathbf{0}_{2N\times 2N}$. By application of the matrix inversion lemma in the second term of the last expression we obtain

$$\begin{aligned}
\widetilde{\mathbf{P}}_k + (k+1)\mathbf{D} &= \widetilde{\mathbf{P}}_k\left(I + \Lambda\widetilde{\mathbf{P}}_k\right)^{-1} + k\mathbf{D}\left(I - \Lambda\left(I + \widetilde{\mathbf{P}}_k\Lambda\right)^{-1}\widetilde{\mathbf{P}}_k\right) + I \Rightarrow \\
\widetilde{\mathbf{P}}_{k+1} + (k+1)\mathbf{D} &= \widetilde{\mathbf{P}}_k\left(I + \Lambda\widetilde{\mathbf{P}}_k\right)^{-1} + k\mathbf{D} + I
\end{aligned}$$

where the result $\Lambda\mathbf{D} = \mathbf{0}_{2N\times 2N}$ has been employed once more. Finally, from the last expression we obtain

$$\begin{aligned}
\widetilde{\mathbf{P}}_{k+1} &= \widetilde{\mathbf{P}}_k\left(I + \Lambda\widetilde{\mathbf{P}}_k\right)^{-1} + I - \mathbf{D} \Rightarrow \\
\widetilde{\mathbf{P}}_{k+1} &= \widetilde{\mathbf{P}}_k\left(I + \Lambda\widetilde{\mathbf{P}}_k\right)^{-1} + \mathbf{D}'
\end{aligned} \tag{54}$$

where

$$\mathbf{D}' = I - \mathbf{D} = \begin{bmatrix} I_{(2N-2)\times(2N-2)} & \mathbf{0}_{(2N-2)\times 2} \\ \mathbf{0}_{2\times(2N-2)} & \mathbf{0}_{2\times 2} \end{bmatrix}$$

Our objective now is to determine the steady-state solution of the Riccati recursion (54), for any initial covariance matrix $\widetilde{\mathbf{P}}_0$. For this purpose we employ the following result, which is proven in [11] (Section 8.6, Lemmas 8.6.2 and 8.6.3):

Lemma 2.5 *Suppose $P_k^{(0)}$ is the solution to the discrete-time Riccati recursion*

$$P_{k+1} = FP_kF^T + GQG^T - (FP_kH^T + GS)(HP_kH^T + R)^{-1}(FP_kH^T + GS)^T, \tag{55}$$

with initial value $P_0 = 0$. Then the solution to the Riccati recursion with the same $\{F, G, H\}$ and $\{Q, R, S\}$ matrices, but with an arbitrary initial condition Π_0 is defined by the identity

$$P_{k+1} - P_{k+1}^{(0)} = \Phi_p^{(0)}(k+1, 0)\left(I + \Pi_0\mathcal{O}_k^{(0)}\right)^{-1}\Pi_0\Phi_p^{(0)}(k+1,0)^T$$

where $\Phi_p^{(0)}(k+1,0)$ is given by

$$\Phi_p^{(0)}(k+1,0) = (F - K_pH)^{k+1}(I + PJ_{k+1})$$

and

$$\mathcal{O}_k^{(0)} = J_{k+1}$$

In these expressions P is any solution to the Discrete Algebraic Riccati Equation (DARE)

$$P = FPF^T + GQG^T - (FPH^T + GS)(HPH^T + R)^{-1}(FPH^T + GS)^T,$$

$K_p = \left(FPH^T + GS\right)\left(R + HPH^T\right)^{-1}$ *and J_k denotes the solution to the* dual *Riccati recursion with zero initial condition, which, in the case $S = 0$, is written as*

$$J_{k+1} = FJ_kF^T + H^TR^{-1}H - F^TJ_kG(Q^{-1} + G^TJ_kG)^{-1}J_kF, \quad J_0 = 0$$

To apply this lemma, we first reformulate Eq. (54) as follows:

$$\begin{aligned}
\widetilde{\mathbf{P}}_{k+1} &= \widetilde{\mathbf{P}}_k\left(I+\Lambda\widetilde{\mathbf{P}}_k\right)^{-1}+\mathbf{D}'\Rightarrow\\
\widetilde{\mathbf{P}}_{k+1} &= \widetilde{\mathbf{P}}_k\left(I+\sqrt{\Lambda}\left(\sqrt{\Lambda}\widetilde{\mathbf{P}}_k\right)\right)^{-1}+\mathbf{D}'\Rightarrow\\
\widetilde{\mathbf{P}}_{k+1} &= \widetilde{\mathbf{P}}_k\left(I+\sqrt{\Lambda}\left(I+\sqrt{\Lambda}\widetilde{\mathbf{P}}_k\sqrt{\Lambda}\right)^{-1}\sqrt{\Lambda}\widetilde{\mathbf{P}}_k\right)+\mathbf{D}'\Rightarrow\\
\widetilde{\mathbf{P}}_{k+1} &= \widetilde{\mathbf{P}}_k+\widetilde{\mathbf{P}}_k\sqrt{\Lambda}\left(I+\sqrt{\Lambda}\widetilde{\mathbf{P}}_k\sqrt{\Lambda}\right)^{-1}\sqrt{\Lambda}\widetilde{\mathbf{P}}_k+\mathbf{D}'
\end{aligned}$$

where $\sqrt{\Lambda}=\operatorname{diag}\left(\sqrt{\lambda_i}\right)$. Introducing the substitutions

$$P_k\leftrightarrow\widetilde{\mathbf{P}}_k,\quad G\leftrightarrow\mathbf{G}=\begin{bmatrix}I_{(2N-2)\times(2N-2)}\\ \mathbf{0}_{2\times(2N-2)}\end{bmatrix},\quad Q\leftrightarrow I_{(2N-2)\times(2N-2)},\quad H\leftrightarrow\sqrt{\Lambda},\quad R\leftrightarrow I,\quad S\leftrightarrow\mathbf{0}_{2\times(2N+2)}$$

allows us to specialize Lemma 2.5 to our problem as follows:

Lemma 2.6 *Suppose $\widetilde{\mathbf{P}}_k^{(0)}$ is the solution to the Riccati recursion*

$$\begin{aligned}
\widetilde{\mathbf{P}}_{k+1} &= \widetilde{\mathbf{P}}_k\left(I+\Lambda\widetilde{\mathbf{P}}_k\right)^{-1}+\mathbf{D}' &(56)\\
&= \widetilde{\mathbf{P}}_k+\widetilde{\mathbf{P}}_k\sqrt{\Lambda}\left(I+\sqrt{\Lambda}\widetilde{\mathbf{P}}_k\sqrt{\Lambda}\right)^{-1}\sqrt{\Lambda}\widetilde{\mathbf{P}}_k+\mathbf{D}' &(57)
\end{aligned}$$

with zero initial condition. Then the solution to this recursion when the initial covariance matrix is an arbitrary positive semidefinite matrix $\widetilde{\mathbf{P}}_0$, is defined by the relation

$$\widetilde{\mathbf{P}}_{k+1}-\widetilde{\mathbf{P}}_{k+1}^{(0)}=\Phi_p^{(0)}(k+1,0)\left(I+\widetilde{\mathbf{P}}_0\mathbf{J}_{k+1}\right)^{-1}\widetilde{\mathbf{P}}_0\Phi_p^{(0)}(k+1,0)^T \qquad (58)$$

where

$$\Phi_p^{(0)}(k+1,0)=\left(I-\mathbf{P}\sqrt{\Lambda}\left(I+\sqrt{\Lambda}\mathbf{P}\sqrt{\Lambda}\right)^{-1}\sqrt{\Lambda}\right)^{k+1}\left(I+\mathbf{P}\mathbf{J}_{k+1}\right) \qquad (59)$$

In these expressions $\mathbf{P}$ is any solution to the Discrete Algebraic Riccati Equation (DARE)

$$\mathbf{P}=\mathbf{P}-\mathbf{P}\sqrt{\Lambda}\left(I+\sqrt{\Lambda}\mathbf{P}\sqrt{\Lambda}\right)^{-1}\sqrt{\Lambda}\mathbf{P}+\mathbf{D}' \qquad (60)$$

and $\mathbf{J}_k$ denotes the solution to the dual *Riccati recursion with zero initial condition:*

$$\mathbf{J}_{k+1}=\mathbf{J}_k+\Lambda-\mathbf{J}_k\mathbf{G}\left(I_{(2N-2)\times(2N-2)}+\mathbf{G}^T\mathbf{J}_k\mathbf{G}\right)^{-1}\mathbf{G}^T\mathbf{J}_k,\quad \mathbf{J}_0=\mathbf{0}_{2N\times 2N} \qquad (61)$$

We now apply this lemma to derive the steady-state value of $\widetilde{\mathbf{P}}_k$, when the initial covariance or the robots' position estimates is an arbitrary positive semidefinite matrix $\mathbf{P}_0$, in which case we have

$$\widetilde{\mathbf{P}}_0 \;=\; \mathbf{P}_{nn_0}-0\cdot\mathbf{D} \;=\; \mathbf{U}_s^T\mathbf{P}_{n_0}\mathbf{U}_s \;=\; \mathbf{U}_s^T\mathbf{Q}_s^{-1/2}\mathbf{P}_0\mathbf{Q}_s^{-1/2}\mathbf{U}_s \qquad (62)$$

In the following, we seek to derive the steady-state solution of $\widetilde{\mathbf{P}}_k$, and therefore we will evaluate the results of Lemma 2.6 after sufficient time, i.e., as $k\to\infty$. We first note that the steady-state solution to the recursion in Eq. (57) with zero initial condition can be directly derived by the definition of $\widetilde{\mathbf{P}}_k$ in Eq. (53):

$$\begin{aligned}
\widetilde{\mathbf{P}}_{ss}^{(0)}(k) &= \mathbf{P}_{nn_{ss}}^{(0)}(k)-k\mathbf{D}\\
&= \begin{bmatrix}\operatorname{diag}_{2N-2}\left(\frac{1}{2}+\sqrt{\frac{1}{4}+\frac{1}{\lambda_i}}\right) & \mathbf{0}_{(2N-2)\times 2}\\ \mathbf{0}_{2\times(2N-2)} & kI_{2\times 2}\end{bmatrix}-k\mathbf{D}\\
&= \begin{bmatrix}\operatorname{diag}_{2N-2}\left(\frac{1}{2}+\sqrt{\frac{1}{4}+\frac{1}{\lambda_i}}\right) & \mathbf{0}_{(2N-2)\times 2}\\ \mathbf{0}_{2\times(2N-2)} & \mathbf{0}_{2\times 2}\end{bmatrix} &(63)
\end{aligned}$$

Substitution of $\widetilde{\mathbf{P}}_{ss}^{(0)}$ for $\mathbf{P}$ in Eq. (60) verifies that $\widetilde{\mathbf{P}}_{ss}^{(0)}$ is a solution of the DARE, and therefore we have

$$\begin{aligned} \Phi_p^{(0)}(k+1,0) &= \left(I - \widetilde{\mathbf{P}}_{ss}^{(0)}\sqrt{\Lambda}\left(I+\sqrt{\Lambda}\widetilde{\mathbf{P}}_{ss}^{(0)}\sqrt{\Lambda}\right)^{-1}\sqrt{\Lambda}\right)^{k+1}\left(I+\widetilde{\mathbf{P}}_{ss}^{(0)}\mathbf{J}_{k+1}\right) \\ &= \left(I+\widetilde{\mathbf{P}}_{ss}^{(0)}\Lambda\right)^{-(k+1)}\left(I+\widetilde{\mathbf{P}}_{ss}^{(0)}\mathbf{J}_{k+1}\right) \\ &= \begin{bmatrix} \mathrm{diag}_{2N-2}\left(1+\frac{\lambda_i}{2}+\lambda_i\sqrt{\frac{1}{4}+\frac{1}{\lambda_i}}\right) & \mathbf{0}_{(2N-2)\times 2} \\ \mathbf{0}_{2\times(2N-2)} & I_{2\times 2}\end{bmatrix}^{-(k+1)}\left(I+\widetilde{\mathbf{P}}_{ss}^{(0)}\mathbf{J}_{k+1}\right) \end{aligned} \tag{64}$$

where we have applied the matrix inversion lemma to simplify the expression.

The next step is to derive the solution of the dual Riccati recursion (61) as $k \to \infty$. Note that since the initial condition of this recursion is zero, at $k = 0$ the right hand side of Eq. (61) is a diagonal matrix. By induction, it is simple to show that $\mathbf{J}_k$ will retain its diagonal structure for all $k \geq 0$, and therefore the solution to the recursion is obtained by solving a set of independent scalar recursions, for the diagonal elements $\mathbf{J}_k(i,i), i = 1\dots 2N$. These recursions are given by

$$\mathbf{J}_{k+1}(i,i) = \mathbf{J}_k(i,i) + \lambda_i - \frac{\mathbf{J}_k(i,i)^2}{1+\mathbf{J}_k(i,i)}, \qquad i = 1\dots 2N-2 \tag{65}$$

while the elements $\mathbf{J}_k(2N-1,2N-1)$ and $\mathbf{J}_k(2N,2N)$ remain equal to zero for all time. By evaluating the steady state solution of these recursions (i.e., by requiring that $\mathbf{J}_{k+1}(i,i) = \mathbf{J}_k(i,i)$, and solving the resulting equations) we obtain the following solution for $\mathbf{J}_k$ at steady state:

$$\mathbf{J}_{ss} = \begin{bmatrix} \mathrm{diag}_{2N-2}\left(\frac{\lambda_i}{2}+\sqrt{\frac{\lambda_i^2}{4}+\lambda_i}\right) & \mathbf{0}_{(2N-2)\times 2} \\ \mathbf{0}_{2\times(2N-2)} & \mathbf{0}_{2\times 2}\end{bmatrix} \tag{66}$$

We can now compute the steady-state value of the quantity $\Phi_p^{(0)}(k+1,0)$. From Eq. (64) we obtain

$$\begin{aligned} \lim_{k\to\infty}\Phi_p^{(0)}(k+1,0) &= \lim_{k\to\infty}\begin{bmatrix} \mathrm{diag}_{2N-2}\left(1+\frac{\lambda_i}{2}+\lambda_i\sqrt{\frac{1}{4}+\frac{1}{\lambda_i}}\right) & \mathbf{0}_{(2N-2)\times 2} \\ \mathbf{0}_{2\times(2N-2)} & I_{2\times 2}\end{bmatrix}^{-(k+1)}\left(I+\widetilde{\mathbf{P}}_{ss}^{(0)}\mathbf{J}_{k+1}\right) \\ &= \begin{bmatrix} \lim_{k\to\infty}\mathrm{diag}_{2N-2}\left(1+\frac{\lambda_i}{2}+\lambda_i\sqrt{\frac{1}{4}+\frac{1}{\lambda_i}}\right)^{-(k+1)} & \mathbf{0}_{(2N-2)\times 2} \\ \mathbf{0}_{2\times(2N-2)} & I_{2\times 2}\end{bmatrix}\left(I+\widetilde{\mathbf{P}}_{ss}^{(0)}\mathbf{J}_{ss}\right) \\ &= \begin{bmatrix} \mathbf{0}_{(2N-2)\times(2N-2)} & \mathbf{0}_{(2N-2)\times 2} \\ \mathbf{0}_{2\times(2N-2)} & I_{2\times 2}\end{bmatrix}\left(I+\widetilde{\mathbf{P}}_{ss}^{(0)}\mathbf{J}_{ss}\right) \end{aligned}$$

Where we have used the fact that

$$\left(1+\frac{\lambda_i}{2}+\lambda_i\sqrt{\frac{1}{4}+\frac{1}{\lambda_i}}\right) > 1 \Rightarrow \lim_{k\to\infty}\left(1+\frac{\lambda_i}{2}+\lambda_i\sqrt{\frac{1}{4}+\frac{1}{\lambda_i}}\right)^{-(k+1)} = 0$$

Furthermore, substitution for $\widetilde{\mathbf{P}}_{ss}^{(0)}$ and $\mathbf{J}_{ss}$ from Eqs. (63) and (66), yields

$$\lim_{k\to\infty}\Phi_p^{(0)}(k+1,0) = \begin{bmatrix} \mathbf{0}_{(2N-2)\times(2N-2)} & \mathbf{0}_{(2N-2)\times 2} \\ \mathbf{0}_{2\times(2N-2)} & I_{2\times 2}\end{bmatrix} = \mathbf{D}$$

Using this result, Eq. (58) yields

$$\begin{aligned} \lim_{k\to\infty}\left(\widetilde{\mathbf{P}}_{k+1} - \widetilde{\mathbf{P}}_{k+1}^{(0)}\right) &= \lim_{k\to\infty}\left(\Phi_p^{(0)}(k+1,0)\left(I+\widetilde{\mathbf{P}}_0\mathbf{J}_{k+1}\right)^{-1}\widetilde{\mathbf{P}}_0\Phi_p^{(0)}(k+1,0)^T\right) \\ &= \mathbf{D}\left(I+\widetilde{\mathbf{P}}_0\mathbf{J}_{ss}\right)^{-1}\widetilde{\mathbf{P}}_0\mathbf{D}^T \end{aligned}$$

and therefore

$$\widetilde{\mathbf{P}}_{ss} = \lim_{k\to\infty} \widetilde{\mathbf{P}}_{k+1} = \widetilde{\mathbf{P}}_{ss}^{(0)} + \mathbf{D}\left(I + \widetilde{\mathbf{P}}_0 \mathbf{J}_{ss}\right)^{-1} \widetilde{\mathbf{P}}_0 \mathbf{D}^T$$

This result allows us to evaluate the steady-state solution of Eq. (41), for the case when none of the robots has access to absolute position information. Using Eq. (53), we obtain

$$\mathbf{P}_{nn_{ss}} = \widetilde{\mathbf{P}}_{ss}^{(0)} + \mathbf{D}\left(I + \widetilde{\mathbf{P}}_0 \mathbf{J}_{ss}\right)^{-1} \widetilde{\mathbf{P}}_0 \mathbf{D}^T + k\mathbf{D}$$

and substitution in Eq. (44) yields

$$\mathbf{P}_{n_{ss}} = \mathbf{U}_s \left(\widetilde{\mathbf{P}}_{ss}^{(0)} + \mathbf{D}\left(I + \widetilde{\mathbf{P}}_0 \mathbf{J}_{ss}\right)^{-1} \widetilde{\mathbf{P}}_0 \mathbf{D}^T + k\mathbf{D}\right) \mathbf{U}_s^T$$

Finally, substitution in Eq. (42) leads to

$$\mathbf{P}_{ss}^s = \mathbf{Q}_s^{1/2} \mathbf{U}_s \left(\widetilde{\mathbf{P}}_{ss}^{(0)} + \mathbf{D}\left(I + \widetilde{\mathbf{P}}_0 \mathbf{J}_{ss}\right)^{-1} \widetilde{\mathbf{P}}_0 \mathbf{D}^T + k\mathbf{D}\right) \mathbf{U}_s^T \mathbf{Q}_s^{1/2} \tag{67}$$

We now treat each of the terms in the last expression independently, to produce a simpler expression. The term that contributes with a constant rate of increase in $\mathbf{P}_{ss}^s$ is given by

$$\begin{aligned}
\mathbf{P}_r(k) &= k\mathbf{Q}_s^{1/2} \mathbf{U}_s \mathbf{D} \mathbf{U}_s^T \mathbf{Q}_s^{1/2} \\
&= k\mathbf{Q}_s^{1/2} \mathbf{U}_s \begin{bmatrix} \mathbf{0}_{(2N-2)\times(2N-2)} & \mathbf{0}_{(2N-2)\times 2} \\ \mathbf{0}_{2\times(2N-2)} & I_{2\times 2} \end{bmatrix} \mathbf{U}_s^T \mathbf{Q}_s^{1/2} \\
&= k\mathbf{Q}_s^{1/2} \left(\mathbf{U}_{2N-1} \mathbf{U}_{2N-1}^T + \mathbf{U}_{2N} \mathbf{U}_{2N}^T\right) \mathbf{Q}_s^{1/2}
\end{aligned}$$

where $\mathbf{U}_{2N-1}$ and $\mathbf{U}_{2N}$ are the singular vectors of $\mathbf{C}_s$ corresponding to the zero singular values. Using the expressions from Eqs. (176) and (177), and carrying out the algebra, we obtain

$$\mathbf{P}_r(k) = k q_{s_T} \mathbf{1}_{N\times N} \otimes I_{2\times 2} \tag{68}$$

The term of $\mathbf{P}_{ss}^s$ expressing the effect of the initial uncertainty is given by

$$\begin{aligned}
\mathbf{P}_{\text{init}} &= \mathbf{Q}_s^{1/2} \mathbf{U}_s \mathbf{D} \left(I + \widetilde{\mathbf{P}}_0 \mathbf{J}_{ss}\right)^{-1} \widetilde{\mathbf{P}}_0 \mathbf{D}^T \mathbf{U}_s^T \mathbf{Q}_s^{1/2} \\
&= \mathbf{Q}_s^{1/2} \mathbf{U}_s \mathbf{D} \mathbf{U}_s^T \mathbf{U}_s \left(I + \mathbf{U}_s^T \mathbf{Q}_s^{-1/2} \mathbf{P}_0 \mathbf{Q}_s^{-1/2} \mathbf{U}_s \mathbf{J}_{ss}\right)^{-1} \mathbf{U}_s^T \mathbf{Q}_s^{-1/2} \mathbf{P}_0 \mathbf{Q}_s^{-1/2} \mathbf{U}_s \mathbf{D}^T \mathbf{U}_s^T \mathbf{Q}_s^{1/2}
\end{aligned}$$

where we have used Eq. (62). We now note that

$$\mathbf{Q}_s^{1/2} \mathbf{U}_s \mathbf{D} \mathbf{U}_s^T \mathbf{Q}_s^{1/2} = q_{s_T} \mathbf{1}_{N\times N} \otimes I_{2\times 2}$$

and thus

$$\begin{aligned}
\mathbf{P}_{\text{init}} &= q_{s_T}^2 \left(\mathbf{1}_{N\times N} \otimes I_{2\times 2}\right) \mathbf{Q}_s^{-1/2} \mathbf{U}_s \left(I + \mathbf{U}_s^T \mathbf{Q}_s^{-1/2} \mathbf{P}_0 \mathbf{Q}_s^{-1/2} \mathbf{U}_s \mathbf{J}_{ss}\right)^{-1} \mathbf{U}_s^T \mathbf{Q}_s^{-1/2} \mathbf{P}_0 \mathbf{Q}_s^{-1} \left(\mathbf{1}_{N\times N} \otimes I_{2\times 2}\right) \\
&= q_{s_T}^2 \left(\mathbf{1}_{N\times N} \otimes I_{2\times 2}\right) \mathbf{Q}_s^{-1/2} \left(I + \mathbf{Q}_s^{-1/2} \mathbf{P}_0 \mathbf{Q}_s^{-1/2} \mathbf{U}_s \mathbf{J}_{ss} \mathbf{U}_s^T\right)^{-1} \mathbf{Q}_s^{-1/2} \mathbf{P}_0 \mathbf{Q}_s^{-1} \left(\mathbf{1}_{N\times N} \otimes I_{2\times 2}\right) \qquad (69) \\
&= q_{s_T}^2 \left(\mathbf{1}_{N\times N} \otimes I_{2\times 2}\right) \mathbf{Q}_s^{-1/2} \left(\mathbf{Q}_s^{1/2} + \mathbf{P}_0 \mathbf{Q}_s^{-1/2} \mathbf{U}_s \mathbf{J}_{ss} \mathbf{U}_s^T\right)^{-1} \mathbf{P}_0 \mathbf{Q}_s^{-1} \left(\mathbf{1}_{N\times N} \otimes I_{2\times 2}\right) \\
&= q_{s_T}^2 \left(\mathbf{1}_{N\times N} \otimes I_{2\times 2}\right) \mathbf{Q}_s^{-1} \left(I + \mathbf{P}_0 \mathbf{Q}_s^{-1/2} \mathbf{U}_s \mathbf{J}_{ss} \mathbf{U}_s^T \mathbf{Q}_s^{-1/2}\right)^{-1} \mathbf{P}_0 \mathbf{Q}_s^{-1} \left(\mathbf{1}_{N\times N} \otimes I_{2\times 2}\right)
\end{aligned}$$

We denote

$$h(\lambda_i) = \frac{\lambda_i}{2} + \sqrt{\frac{\lambda_i^2}{4} + \lambda_i}$$

and thus

$$\mathbf{U}_s \mathbf{J}_{ss} \mathbf{U}_s^T = \mathbf{U}_s \operatorname{diag}(h(\lambda_i)) \mathbf{U}_s^T = h(\mathbf{C}_s)$$

With this notation, we can write

$$\mathbf{P}_{\text{init}} = q_{s_T}^2 (\mathbf{1}_{N\times N} \otimes I_{2\times 2}) \mathbf{Q}_s^{-1} \left(I + \mathbf{P}_0 \mathbf{Q}_s^{-1/2} h(\mathbf{C}_s) \mathbf{Q}_s^{-1/2} \right)^{-1} \mathbf{P}_0 \mathbf{Q}_s^{-1} (\mathbf{1}_{N\times N} \otimes I_{2\times 2})$$

From the properties of the Kronecker product we obtain

$$\mathbf{1}_{N\times N} \otimes I_{2\times 2} = (\mathbf{1}_{N\times 1} \mathbf{1}_{1\times N}) \otimes I_{2\times 2} = (\mathbf{1}_{N\times 1} \otimes I_{2\times 2})(\mathbf{1}_{1\times N} \otimes I_{2\times 2})$$

Additionally, we note that if W is a $2N \times 2N$ matrix, then

$$\begin{aligned}
(\mathbf{1}_{N\times 1} \otimes I_{2\times 2}) W (\mathbf{1}_{1\times N} \otimes I_{2\times 2}) &= (\mathbf{1}_{N\times 1} \otimes I_{2\times 2})(\mathbf{1}_{1\times N} \otimes I_{2\times 2}) W (\mathbf{1}_{N\times 1} \otimes I_{2\times 2})(\mathbf{1}_{1\times N} \otimes I_{2\times 2}) \\
&= (\mathbf{1}_{N\times 1} \otimes I_{2\times 2}) \Big((\mathbf{1}_{1\times N} \otimes I_{2\times 2}) W (\mathbf{1}_{N\times 1} \otimes I_{2\times 2}) \Big) (\mathbf{1}_{1\times N} \otimes I_{2\times 2}) \\
&= (\mathbf{1}_{N\times N} \otimes I_{2\times 2}) \otimes \Big((\mathbf{1}_{1\times N} \otimes I_{2\times 2}) W (\mathbf{1}_{N\times 1} \otimes I_{2\times 2}) \Big)
\end{aligned}$$

Using these results, we see that $\mathbf{P}_{\text{init}}$ can be written as

$$\mathbf{P}_{\text{init}} = q_{s_T}^2 (\mathbf{1}_{N\times N} \otimes I_{2\times 2}) \otimes \begin{bmatrix} \alpha & \beta \\ \beta & \delta \end{bmatrix}$$

where

$$\begin{aligned}
\begin{bmatrix} \alpha & \beta \\ \beta & \delta \end{bmatrix} &= (\mathbf{1}_{1\times N} \otimes I_{2\times 2}) \mathbf{Q}_s^{-1} \left(I + \mathbf{P}_0 \mathbf{Q}_s^{-1/2} h(\mathbf{C}_s) \mathbf{Q}_s^{-1/2} \right)^{-1} \mathbf{P}_0 \mathbf{Q}_s^{-1} (\mathbf{1}_{N\times 1} \otimes I_{2\times 2}) \\
&= (\mathbf{1}_{1\times N} \otimes I_{2\times 2}) W (\mathbf{1}_{N\times 1} \otimes I_{2\times 2})
\end{aligned} \tag{70}$$

with

$$W = \mathbf{Q}_s^{-1} \left(I + \mathbf{P}_0 \mathbf{Q}_s^{-1/2} h(\mathbf{C}_s) \mathbf{Q}_s^{-1/2} \right)^{-1} \mathbf{P}_0 \mathbf{Q}_s^{-1} = [w_{ij}]$$

From this expression, we conclude that $\alpha = \sum_{i,j \text{ odd}} w_{ij}$ ($\delta = \sum_{i,j \text{ even}} w_{ij}$) is the sum of all elements of W with two odd (even) indices and $\beta = \sum_{i \text{ odd,j even}} w_{ij}$ is the sum of all elements of W with an odd row index and an even column index.

To summarize, we have shown that the steady state solution of the Riccati recursion (41) when the system is not observable, is given by

$$\begin{aligned}
\mathbf{P}_{ss}^s &= k q_{s_T} \mathbf{1}_{N\times N} \otimes I_{2\times 2} + \mathbf{Q}_s^{1/2} \mathbf{U}_s \begin{bmatrix} \operatorname{diag}_{2N-2}\left(\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{1}{\lambda_i}} \right) & \mathbf{0}_{(2N-2)\times 2} \\ \mathbf{0}_{2\times(2N-2)} & \mathbf{0}_{2\times 2} \end{bmatrix} \mathbf{U}_s^T \mathbf{Q}_s^{1/2} \\
&+ q_{s_T}^2 (\mathbf{1}_{N\times N} \otimes I_{2\times 2}) \otimes \begin{bmatrix} \alpha & \beta \\ \beta & \delta \end{bmatrix}
\end{aligned} \tag{71}$$

Finally, from this result, by setting

$$\mathbf{R}_s \rightarrow \mathbf{R}_u, \qquad \mathbf{Q}_s \rightarrow \mathbf{Q}_u$$

and

$$\mathbf{R}_s \rightarrow \bar{\mathbf{R}}, \qquad \mathbf{Q}_s \rightarrow \bar{\mathbf{Q}}$$

we can derive the following lemmas:

Lemma 2.7 *The steady state covariance of the position estimates for a team of robots performing CL, when none of the robots has access to absolute positioning information, and the initial covariance of the robots' position estimates is $\mathbf{P}_0$, is bounded above by the matrix*

$$\begin{aligned}
\mathbf{P}_{ss}^u &= k q_{u_T} \mathbf{1}_{N\times N} \otimes I_{2\times 2} + \mathbf{Q}_u^{1/2} \mathbf{U}_u \begin{bmatrix} \operatorname{diag}_{2N-2}\left(\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{1}{\lambda_{u_i}}} \right) & \mathbf{0}_{(2N-2)\times 2} \\ \mathbf{0}_{2\times(2N-2)} & \mathbf{0}_{2\times 2} \end{bmatrix} \mathbf{U}_u^T \mathbf{Q}_u^{1/2} \\
&+ q_{u_T}^2 (\mathbf{1}_{N\times N} \otimes I_{2\times 2}) \otimes \begin{bmatrix} \alpha_u & \beta_u \\ \beta_u & \delta_u \end{bmatrix}
\end{aligned} \tag{72}$$

where we have denoted the singular value decomposition of $\mathbf{C}_u = \mathbf{Q}_u^{1/2}\mathbf{H}_o^T\mathbf{R}_u^{-1}\mathbf{H}_o\mathbf{Q}_u^{1/2}$ *as* $\mathbf{C}_u = \mathbf{U}_u \operatorname{diag}(\lambda_{u_i})\mathbf{U}_u^T$*, and* q_{u_T} *is defined by*

$$\frac{1}{q_{u_T}} = \sum_{i=1}^{N} \frac{1}{q_{u_i}}$$

For computing the values of α_u*,* β_u *and* δ_u*, we define*

$$W = \mathbf{Q}_u^{-1}\left(I + \mathbf{P}_0\mathbf{Q}_u^{-1/2}h(\mathbf{C}_u)\mathbf{Q}_u^{-1/2}\right)^{-1}\mathbf{P}_0\mathbf{Q}_u^{-1} = [w_{ij}]$$

where

$$h(\mathbf{C}_u) = \mathbf{U}_u \operatorname{diag}\left(\frac{\lambda_{u_i}}{2} + \sqrt{\frac{\lambda_{u_i}^2}{4} + \lambda_{u_i}}\right)\mathbf{U}_u^T$$

With this notation, $\alpha_u = \sum_{i,j \text{ odd}} w_{ij}$ *(*$\delta_u = \sum_{i,j \text{ even}} w_{ij}$*) is the sum of all elements of* W *with two odd (even) indices and* $\beta_u = \sum_{i \text{ odd,j even}} w_{ij}$ *is the sum of all elements of* W *with an odd row index and an even column index.*

Lemma 2.8 *The expected steady state covariance of the position estimates for a team of robots performing CL, when none of the robots has access to absolute positioning information, and the initial covariance of the robots' position estimates is* $\mathbf{P}_0$*, is bounded above by the matrix*

$$\begin{aligned}\bar{\mathbf{P}}_{ss} &= k\bar{q}_T\mathbf{1}_{N\times N}\otimes I_{2\times 2} + \bar{\mathbf{Q}}^{1/2}\bar{\mathbf{U}}\begin{bmatrix}\operatorname{diag}_{2N-2}\left(\frac{1}{2}+\sqrt{\frac{1}{4}+\frac{1}{\bar{\lambda}_i}}\right) & \mathbf{0}_{(2N-2)\times 2}\\ \mathbf{0}_{2\times(2N-2)} & \mathbf{0}_{2\times 2}\end{bmatrix}\bar{\mathbf{U}}^T\bar{\mathbf{Q}}^{1/2}\\ &+ \bar{q}_T^2\left(\mathbf{1}_{N\times N}\otimes I_{2\times 2}\right)\otimes\begin{bmatrix}\bar{\alpha} & \bar{\beta}\\ \bar{\beta} & \bar{\delta}\end{bmatrix}\end{aligned} \qquad (73)$$

where we have denoted the singular value decomposition of $\bar{\mathbf{C}} = \bar{\mathbf{Q}}^{1/2}\mathbf{H}_o^T\bar{\mathbf{R}}^{-1}\mathbf{H}_o\bar{\mathbf{Q}}^{1/2}$ *as* $\bar{\mathbf{C}} = \bar{\mathbf{U}}\operatorname{diag}(\bar{\lambda}_i)\bar{\mathbf{U}}^T$*, and* $\bar{q}_T$ *is defined by*

$$\frac{1}{\bar{q}_T} = \sum_{i=1}^{N}\frac{1}{\bar{q}_i}$$

For computing the values of $\bar{\alpha}$*,* $\bar{\beta}$ *and* $\bar{\delta}$*, we define*

$$\bar{W} = \bar{\mathbf{Q}}^{-1}\left(I + \mathbf{P}_0\bar{\mathbf{Q}}^{-1/2}h(\bar{\mathbf{C}})\bar{\mathbf{Q}}^{-1/2}\right)^{-1}\mathbf{P}_0\bar{\mathbf{Q}}^{-1} = [\bar{w}_{ij}]$$

where

$$h(\bar{\mathbf{C}}) = \bar{\mathbf{U}}\operatorname{diag}\left(\frac{\bar{\lambda}_i}{2} + \sqrt{\frac{\bar{\lambda}_i^2}{4} + \bar{\lambda}_i}\right)\bar{\mathbf{U}}^T$$

With this notation, $\bar{\alpha} = \sum_{i,j \text{ odd}} \bar{w}_{ij}$ *(*$\bar{\delta} = \sum_{i,j \text{ even}} \bar{w}_{ij}$*) is the sum of all elements of* W *with two odd (even) indices and* $\beta = \sum_{i \text{ odd,j even}} \bar{w}_{ij}$ *is the sum of all elements of* $\bar{W}$ *with an odd row index and an even column index.*

Several observations can be made with respect to the above results. We note that the upper bounds comprise of three terms, the first of which contributes with a *constant rate* of uncertainty increase. The second term is a constant term, whose value depends on the *topology* of the RPMG and the *accuracy* of the sensors on the robots. Finally, the third term is a constant term that describes the effect of the *initial uncertainty* on the steady-state covariance. It also depends on the noise characteristics of the sensors of the robots, as well as the RPMG topology. The fact that the steady-state bound depends on the initial uncertainty is a consequence of the fact that the system is *not* observable, and therefore initial errors in the estimates for the robots' positions cannot be fully compensated for.

It is clear that the most important term in the bounds is the one that corresponds to a *constant rate* of uncertainty increase. After sufficient time, this term will always dominate the remaining ones, and will largely determine the positioning performance of the team. A striking observation is that q_{u_T} and $\bar{q}_T$ are *independent* of both the topology of the RPMG and of the precision of the robots' relative position measurements. This quantity depends solely on the number of robots in the team, and the accuracy of the robots' Dead Reckoning capabilities. An intuitive interpretation of this result is that the primary factor determining the rate of uncertainty increase is the rate at which uncertainty is injected in the unobservable subspace of the system. Since the number, or the accuracy, of the relative position measurements does not alter this subspace, we should expect no change in the rate of uncertainty increase, as a result of changes in the information contributed by the exteroceptive measurements.

2.5 RPMG Reconfigurations

In the preceding analysis, it is assumed that the topology of the graph describing the relative position measurements between robots does not change. However, this may be difficult to implement in a realistic scenario. For example, due to the robots' motion or because of obstacles in the environment, some robots may not be able to measure their relative positions. Additionally, robot teams often need to allocate computational and communication resources to mission-specific tasks and this may force them to reduce the number of measurements they process for localization purposes. Consequently, it is of considerable interest to study the effects of changes in the topology of the RPMG on the localization accuracy of the team.

Consider the following scenario: At the initial stage of the deployment of a robotic team, the RPMG has a dense topology $\mathcal{T}_A$, e.g., the complete graph shown in Fig. 8(a), and retains this topology until some time instant t_1, when it assumes a sparser topology $\mathcal{T}_B$, e.g., the ring graph shown in Fig. 8(b). This sparse topology may even be an *empty graph*, i.e., the case in which the robots localize independently, based only on odometry. Subsequent topology changes are assumed to occur at time instants $t_i, i = 1 \ldots n-1$, and finally, at time instant t_n, the RPMG returns to its initial, dense topology, $\mathcal{T}_A$. Assuming that the time intervals (t_{i-1}, t_i) are of sufficient duration for the transient phenomena in the time evolution of uncertainty to subside, the following lemma applies:

Lemma 2.9 *After a sequence of RPMG reconfigurations and once the RPMG resumes its initial topology, the upper bounds on the positioning uncertainty of the robots at steady state are* identical *to the ones the robot team would have if no RPMG reconfigurations had taken place.*

Proof For the purposes of this proof, we will use the result of Eq. (71). For convenience, we will express this equation with respect to the normalized covariance matrix $\mathbf{P}_{n_k} = \mathbf{Q}_s^{-1/2}\mathbf{P}_k^s\mathbf{Q}_s^{-1/2}$. In particular, we have

$$\begin{aligned}
\mathbf{P}_{n_{ss}} &= \mathbf{Q}_s^{-1/2}\mathbf{P}_{ss}^s\mathbf{Q}_s^{-1/2} \\
&= kq_{s_T}\mathbf{Q}_s^{-1/2}\left(\mathbf{1}_{N\times N}\otimes I_{2\times 2}\right)\mathbf{Q}_s^{-1/2} + \mathbf{U}_s \begin{bmatrix} \mathrm{diag}_{2N-2}\left(\frac{1}{2}+\sqrt{\frac{1}{4}+\frac{1}{\lambda_i}}\right) & \mathbf{0}_{(2N-2)\times 2} \\ \mathbf{0}_{2\times(2N-2)} & \mathbf{0}_{2\times 2}\end{bmatrix}\mathbf{U}_s^T \\
&+ q_{s_T}^2\mathbf{Q}_s^{-1/2}\left(\mathbf{1}_{N\times N}\otimes I_{2\times 2}\right)\otimes\begin{bmatrix}\alpha & \beta \\ \beta & \delta\end{bmatrix}\mathbf{Q}_s^{-1/2}
\end{aligned}$$

Employing the results of Eqs. (176) and (177), we can see that the column vectors of the matrix

$$\mathbf{V} = \sqrt{q_{s_T}}\mathbf{Q}_s^{-1/2}\mathbf{1}_{N\times 1}\otimes I_{2\times 2}$$

are the two basis vectors of the nullspace of $\mathbf{C}_s$. For this matrix we have

$$\mathbf{V}\mathbf{V}^T = q_{s_T}\mathbf{Q}_s^{-1/2}\left(\mathbf{1}_{N\times 1}\otimes I_{2\times 2}\right)\left(\mathbf{1}_{1\times N}\otimes I_{2\times 2}\right)\mathbf{Q}_s^{-1/2} = q_{s_T}\mathbf{Q}_s^{-1/2}\left(\mathbf{1}_{N\times N}\otimes I_{2\times 2}\right)\mathbf{Q}_s^{-1/2}$$

and therefore we can write

$$\mathbf{P}_{n_{ss}} = k\mathbf{V}\mathbf{V}^T + \mathbf{U}_s \begin{bmatrix} \mathrm{diag}_{2N-2}\left(\frac{1}{2}+\sqrt{\frac{1}{4}+\frac{1}{\lambda_i}}\right) & \mathbf{0}_{(2N-2)\times 2} \\ \mathbf{0}_{2\times(2N-2)} & \mathbf{0}_{2\times 2}\end{bmatrix}\mathbf{U}_s^T \tag{74}$$

$$+ \; q_{s_T}^2\mathbf{Q}_s^{-1/2}\left(\mathbf{1}_{N\times N}\otimes I_{2\times 2}\right)\otimes\begin{bmatrix}\alpha & \beta \\ \beta & \delta\end{bmatrix}\mathbf{Q}_s^{-1/2} \tag{75}$$

Moreover, the quantity that expresses the effect of the initial uncertainty can be expressed equivalently as (cf. Eq. (69)):

$$\begin{aligned}
\mathbf{P}_{\mathrm{init}} &= q_{s_T}^2\left(\mathbf{1}_{N\times N}\otimes I_{2\times 2}\right)\otimes\begin{bmatrix}\alpha & \beta \\ \beta & \delta\end{bmatrix} \\
&= q_{s_T}^2\left(\mathbf{1}_{N\times N}\otimes I_{2\times 2}\right)\mathbf{Q}_s^{-1/2}\left(I+\mathbf{Q}_s^{-1/2}\mathbf{P}_0\mathbf{Q}_s^{-1/2}\mathbf{U}_s\mathbf{J}_{ss}\mathbf{U}_s^T\right)^{-1}\mathbf{Q}_s^{-1/2}\mathbf{P}_0\mathbf{Q}_s^{-1}\left(\mathbf{1}_{N\times N}\otimes I_{2\times 2}\right) \\
&= q_{s_T}^2\left(\mathbf{1}_{N\times N}\otimes I_{2\times 2}\right)\mathbf{Q}_s^{-1/2}\left(I+\mathbf{P}_{n_0}h(\mathbf{C}_s)\right)^{-1}\mathbf{P}_{n_0}\mathbf{Q}_s^{-1/2}\left(\mathbf{1}_{N\times N}\otimes I_{2\times 2}\right) \\
&= \mathbf{Q}_s^{1/2}\mathbf{V}\mathbf{V}^T\left(I+\mathbf{P}_{n_0}h(\mathbf{C}_s)\right)^{-1}\mathbf{P}_{n_0}\mathbf{V}\mathbf{V}^T\mathbf{Q}_s^{1/2}
\end{aligned}$$

Therefore Eq. (74) is equivalently written as

$$\begin{aligned}\mathbf{P}_{n_{ss}} &= k\mathbf{V}\mathbf{V}^T + \mathbf{U}_s \begin{bmatrix} \mathrm{diag}_{2N-2}\left(\frac{1}{2}+\sqrt{\frac{1}{4}+\frac{1}{\lambda_i}}\right) & \mathbf{0}_{(2N-2)\times 2} \\ \mathbf{0}_{2\times(2N-2)} & \mathbf{0}_{2\times 2}\end{bmatrix} \mathbf{U}_s^T \\ &+ \mathbf{V}\mathbf{V}^T\left(I+\mathbf{P}_{n_0}h(\mathbf{C}_s)\right)^{-1}\mathbf{P}_{n_0}\mathbf{V}\mathbf{V}^T \end{aligned} \quad (76)$$

Introducing the notation

$$f(\lambda_i) = \frac{1}{2}+\sqrt{\frac{1}{4}+\frac{1}{\lambda_i}}$$

further simplifies the preceding expression, to yield:

$$\begin{aligned}\mathbf{P}_{n_{ss}} &= k\mathbf{V}\mathbf{V}^T + \mathbf{U}_s \begin{bmatrix} \mathrm{diag}_{2N-2}\left(f(\lambda_i)\right) & \mathbf{0}_{(2N-2)\times 2} \\ \mathbf{0}_{2\times(2N-2)} & \mathbf{0}_{2\times 2}\end{bmatrix} \mathbf{U}_s^T + \mathbf{V}\mathbf{V}^T\left(I+\mathbf{P}_{n_0}h(\mathbf{C}_s)\right)^{-1}\mathbf{P}_{n_0}\mathbf{V}\mathbf{V}^T \\ &= \mathbf{V}\left(kI_{2\times 2}+\mathbf{V}^T\left(I+\mathbf{P}_{n_0}h(\mathbf{C}_s)\right)^{-1}\mathbf{P}_{n_0}\mathbf{V}\right)\mathbf{V}^T + \mathbf{U}_s \begin{bmatrix} \mathrm{diag}_{2N-2}\left(f(\lambda_i)\right) & \mathbf{0}_{(2N-2)\times 2} \\ \mathbf{0}_{2\times(2N-2)} & \mathbf{0}_{2\times 2}\end{bmatrix} \mathbf{U}_s^T \\ &= \mathbf{U}_s \begin{bmatrix} \mathrm{diag}_{2N-2}\left(f(\lambda_i)\right) & \mathbf{0}_{(2N-2)\times 2} \\ \mathbf{0}_{2\times(2N-2)} & kI_{2\times 2}+\mathbf{V}^T\left(I+\mathbf{P}_{n_0}h(\mathbf{C}_s)\right)^{-1}\mathbf{P}_{n_0}\mathbf{V}\end{bmatrix} \mathbf{U}_s^T \end{aligned} \quad (77)$$

where we have employed the fact that the matrix of singular vectors $\mathbf{U}_s$ can be partitioned as

$$\mathbf{U}_s = [\mathbf{S} \quad \mathbf{V}]$$

with $\mathbf{S}$ being the $2N \times (2N-2)$ matrix of singular vectors corresponding to the nonzero singular values of $\mathbf{C}_s$.

Assuming that the RPMG remains in the topology $\mathcal{T}_A$ for the time-step interval $[0, t_1]$, and that this interval is of sufficient duration for the covariance to reach steady state, then at time-step t_1 the normalized covariance matrix is given by

$$\mathbf{P}_{n_{ss}}(t_1) = \mathbf{U}_A \begin{bmatrix} \mathrm{diag}_{2N-2}\left(f(\lambda_{A_i})\right) & \mathbf{0}_{(2N-2)\times 2} \\ \mathbf{0}_{2\times(2N-2)} & t_1 I_{2\times 2}+M_A\end{bmatrix} \mathbf{U}_A^T \quad (78)$$

where

$$M_A = \mathbf{V}^T\left(I+\mathbf{P}_{n_0}h(\mathbf{C}_A)\right)^{-1}\mathbf{P}_{n_0}\mathbf{V}$$

In these expressions the quantities that depend on the RPMG topology $\mathcal{T}_A$ have been denoted by the subscript A. It is important to note that the basis vectors of the nullspace of the matrix $\mathbf{C}_s$ are *independent* of the topology of the RPMG. This essentially is a consequence of the fact that the unobservable subspace remains the same, regardless of the topology of the RPMG.

During the second phase, the RPMG remains in topology $\mathcal{T}_B$ for the time interval $[t_1, t_2]$. Thus, if steady state is reached, at time step t_2, the normalized covariance is given by

$$\mathbf{P}_{n_{ss}}(t_2) = \mathbf{U}_B \begin{bmatrix} \mathrm{diag}_{2N-2}\left(f(\lambda_{B_i})\right) & \mathbf{0}_{(2N-2)\times 2} \\ \mathbf{0}_{2\times(2N-2)} & (t_2-t_1) I_{2\times 2}+M_B\end{bmatrix} \mathbf{U}_B^T \quad (79)$$

where

$$M_B = \mathbf{V}^T\left(I+\mathbf{P}_{n_{ss}}(t_1)h(\mathbf{C}_B)\right)^{-1}\mathbf{P}_{n_{ss}}(t_1)\mathbf{V}$$

We will now derive a simpler expression for M_B. We start by applying the matrix inversion lemma, to obtain

$$\begin{aligned} M_B &= \mathbf{V}^T\left(I+\mathbf{P}_{n_{ss}}(t_1)h(\mathbf{C}_B)\right)^{-1}\mathbf{P}_{n_{ss}}(t_1)\mathbf{V} \\ &= \mathbf{V}^T\mathbf{P}_{n_{ss}}(t_1)\mathbf{V} - \mathbf{V}^T\mathbf{P}_{n_{ss}}(t_1)\left(I+h(\mathbf{C}_B)\mathbf{P}_{n_{ss}}(t_1)\right)^{-1}h(\mathbf{C}_B)\mathbf{P}_{n_{ss}}(t_1)\mathbf{V}\end{aligned}$$

We now study the matrix product $Z = h(\mathbf{C}_B)\mathbf{P}_{n_{ss}}(t_1)\mathbf{V}$, that appears in the last equation. We have

$$\begin{aligned} Z &= \mathbf{U}_B\,\mathrm{diag}\left(h(\lambda_{B_i})\right)\mathbf{U}_B^T\mathbf{U}_A \begin{bmatrix} \mathrm{diag}_{2N-2}\left(f(\lambda_{A_i})\right) & \mathbf{0}_{(2N-2)\times 2} \\ \mathbf{0}_{2\times(2N-2)} & t_1 I_{2\times 2}+M_A\end{bmatrix} \mathbf{U}_A^T\mathbf{V} \\ &= \mathbf{U}_B\,\mathrm{diag}\left(h(\lambda_{B_i})\right)\begin{bmatrix}\mathbf{S}_B^T \\ \mathbf{V}^T\end{bmatrix}\left[\mathbf{S}_A \quad \mathbf{V}\right] \begin{bmatrix} \mathrm{diag}_{2N-2}\left(f(\lambda_{A_i})\right) & \mathbf{0}_{(2N-2)\times 2} \\ \mathbf{0}_{2\times(2N-2)} & t_1 I_{2\times 2}+M_A\end{bmatrix} \begin{bmatrix}\mathbf{S}_A^T \\ \mathbf{V}^T\end{bmatrix}\mathbf{V} \\ &= \mathbf{U}_B\,\mathrm{diag}\left(h(\lambda_{B_i})\right)\begin{bmatrix}\mathbf{S}_B^T\mathbf{S}_A & \mathbf{0}_{(2N-2)\times 2} \\ \mathbf{0}_{2\times(2N-2)} & I_{2\times 2}\end{bmatrix} \begin{bmatrix} \mathrm{diag}_{2N-2}\left(f(\lambda_{A_i})\right) & \mathbf{0}_{(2N-2)\times 2} \\ \mathbf{0}_{2\times(2N-2)} & t_1 I_{2\times 2}+M_A\end{bmatrix} \begin{bmatrix}\mathbf{0}_{(2N-2)\times 2} \\ I_{2\times 2}\end{bmatrix}\end{aligned}$$

In the last line we have used the fact that $\mathbf{S}_A^T\mathbf{V} = \mathbf{S}_B^T\mathbf{V} = \mathbf{0}_{(2N-2)\times 2}$, which results from the orthogonality of the columns of the matrices $\mathbf{U}_A$ and $\mathbf{U}_B$. We also note that the last two diagonal elements of $\mathrm{diag}\,(h(\lambda_{B_i}))$ (i.e., the ones that correspond to the zero eigenvalues) are equal to zero. Thus Z can be written as

$$\begin{aligned}
Z &= \mathbf{U}_B \begin{bmatrix} \mathrm{diag}_{2N-2}\,(h(\lambda_{B_i}))\,\mathbf{S}_B^T\mathbf{S}_A & \mathbf{0}_{(2N-2)\times 2} \\ \mathbf{0}_{2\times(2N-2)} & \mathbf{0}_{2\times 2} \end{bmatrix} \begin{bmatrix} \mathrm{diag}_{2N-2}\,(f(\lambda_{A_i})) & \mathbf{0}_{(2N-2)\times 2} \\ \mathbf{0}_{2\times(2N-2)} & t_1 I_{2\times 2} + M_A \end{bmatrix} \begin{bmatrix} \mathbf{S}_A^T \\ \mathbf{V}^T \end{bmatrix} \mathbf{V} \\
&= \mathbf{U}_B \begin{bmatrix} \mathrm{diag}_{2N-2}\,(h(\lambda_{B_i}))\,\mathbf{S}_B^T\mathbf{S}_A\,\mathrm{diag}_{2N-2}\,(f(\lambda_{A_i})) & \mathbf{0}_{(2N-2)\times 2} \\ \mathbf{0}_{2\times(2N-2)} & \mathbf{0}_{2\times 2} \end{bmatrix} \begin{bmatrix} \mathbf{0}_{(2N-2)\times 2} \\ I_{2\times 2} \end{bmatrix} \\
&= \mathbf{U}_B \begin{bmatrix} \mathrm{diag}_{2N-2}\,(h(\lambda_{B_i}))\,\mathbf{S}_B^T\mathbf{S}_A\,\mathrm{diag}_{2N-2}\,(f(\lambda_{A_i})) & \mathbf{0}_{(2N-2)\times 2} \\ \mathbf{0}_{2\times(2N-2)} & \mathbf{0}_{2\times 2} \end{bmatrix} \begin{bmatrix} \mathbf{0}_{(2N-2)\times 2} \\ I_{2\times 2} \end{bmatrix} \\
&= \mathbf{0}_{(2N-2)\times 2}
\end{aligned}$$

Using this result, M_B can be written as

$$\begin{aligned}
M_B &= \mathbf{V}^T\mathbf{P}_{n_{ss}}(t_1)\mathbf{V} \\
&= \mathbf{V}^T\mathbf{U}_A \begin{bmatrix} \mathrm{diag}_{2N-2}\,(f(\lambda_{A_i})) & \mathbf{0}_{(2N-2)\times 2} \\ \mathbf{0}_{2\times(2N-2)} & t_1 I_{2\times 2} + M_A \end{bmatrix} \mathbf{U}_A^T\mathbf{V} \\
&= \mathbf{V}^T \begin{bmatrix} \mathbf{S}_A & \mathbf{V} \end{bmatrix} \begin{bmatrix} \mathrm{diag}_{2N-2}\,(f(\lambda_{A_i})) & \mathbf{0}_{(2N-2)\times 2} \\ \mathbf{0}_{2\times(2N-2)} & t_1 I_{2\times 2} + M_A \end{bmatrix} \begin{bmatrix} \mathbf{S}_A^T \\ \mathbf{V}^T \end{bmatrix} \mathbf{V} \\
&= \begin{bmatrix} \mathbf{0}_{2\times(2N-2)} & I_{2\times 2} \end{bmatrix} \begin{bmatrix} \mathrm{diag}_{2N-2}\,(f(\lambda_{A_i})) & \mathbf{0}_{(2N-2)\times 2} \\ \mathbf{0}_{2\times(2N-2)} & t_1 I_{2\times 2} + M_A \end{bmatrix} \begin{bmatrix} \mathbf{0}_{(2N-2)\times 2} \\ I_{2\times 2} \end{bmatrix} \\
&= t_1 I_{2\times 2} + M_A
\end{aligned}$$

Substitution of this result in Eq. (79) yields

$$\mathbf{P}_{n_{ss}}(t_2) = \mathbf{U}_B \begin{bmatrix} \mathrm{diag}_{2N-2}\,(f(\lambda_{B_i})) & \mathbf{0}_{(2N-2)\times 2} \\ \mathbf{0}_{2\times(2N-2)} & t_2 I_{2\times 2} + M_A \end{bmatrix} \mathbf{U}_B^T \tag{80}$$

From the last expression, we conclude that the term describing the effect of the initial uncertainty is *the same* for both topologies $\mathcal{T}_A$ and $\mathcal{T}_B$.

If at time step t_2 the RPMG assumes its initial topology, $\mathcal{T}_A$, once again, then by a similar proof we can show that the value of the normalized covariance at some time $t_3 > t_2$ is given by

$$\mathbf{P}_{n_{ss}}(t_3) = \mathbf{U}_A \begin{bmatrix} \mathrm{diag}_{2N-2}\,(f(\lambda_{A_i})) & \mathbf{0}_{(2N-2)\times 2} \\ \mathbf{0}_{2\times(2N-2)} & t_3 I_{2\times 2} + M_A \end{bmatrix} \mathbf{U}_A^T \tag{81}$$

However, we can see that this result is the same one that would result from use of Eq. (77) if *no reconfigurations* had occurred. We have thus proved the lemma for the case where the intermediate topology $\mathcal{T}_B$ is a connected one.

If during the intermediate phase the robots localize based only on odometry, then during this time interval their covariance bounds are propagated by

$$\mathbf{P}_{k+1}^s = \mathbf{P}_k^s + \mathbf{Q}_s$$

or, expressed using the normalized covariance,

$$\mathbf{P}_{n_{k+1}} = \mathbf{P}_{n_k} + I_{2N\times 2N}$$

Thus, at time step t_2 we would have

$$\begin{aligned}
\mathbf{P}_{n_{\mathrm{od}}}(t_2) &= \mathbf{P}_{n_{ss}}(t_1) + (t_2 - t_1) I_{2N\times 2N} \\
&= \mathbf{U}_A \begin{bmatrix} \mathrm{diag}_{2N-2}\,(f(\lambda_{A_i})) & \mathbf{0}_{(2N-2)\times 2} \\ \mathbf{0}_{2\times(2N-2)} & t_1 I_{2\times 2} + M_A \end{bmatrix} \mathbf{U}_A^T + (t_2 - t_1) I_{2N\times 2N} \\
&= \mathbf{U}_A \begin{bmatrix} \mathrm{diag}_{2N-2}\,(f(\lambda_{A_i})) + (t_2 - t_1) I_{(2N-2)\times(2N-2)} & \mathbf{0}_{(2N-2)\times 2} \\ \mathbf{0}_{2\times(2N-2)} & t_2 I_{2\times 2} + M_A \end{bmatrix} \mathbf{U}_A^T
\end{aligned} \tag{82}$$

By comparison of this result with the result of Eq. (80) we observe that the basic structure of the covariance matrix remains the same. By a proof analogous to the one presented in the preceding analysis, we can show that Eq. (81) holds without change.

Up to this point, we have proven the lemma for the situations where the intermediate topology, $\mathcal{T}_B$, is either a connected or an empty graph. To show that the lemma also holds for any other possible topology $\mathcal{T}_B$ (for example, the case in which only some of the robots localize using odometry, while a subgroup or robots can still record relative position measurements) we study the Riccati recursion that describes the normalized covariance during the intermediate phase. This recursion can be written as (cf. Eq. (89)):

$$\mathbf{P}_{n_{k+1}} = \left(\mathbf{P}_{n_k}^{-1} + \mathbf{C}_s\right)^{-1} + I_{2N\times 2N} \tag{83}$$

where

$$\mathbf{C}_s = \mathbf{Q}_s^{1/2}\mathbf{H}_o^T\mathbf{R}_s^{-1}\mathbf{H}_o\mathbf{Q}_s^{1/2}$$

If the RPMG is not a connected one, then the matrix $\mathbf{H}_o^T\mathbf{R}_s^{-1}\mathbf{H}_o$, which expresses the information provided by the exteroceptive measurements, will satisfy the relation:

$$\mathbf{H}_o^T\mathbf{R}_s^{-1}\mathbf{H}_o \prec \mathbf{H}_o'^T\mathbf{R}_s'^{-1}\mathbf{H}_o'$$

where the matrices $\mathbf{H}_o'^T$ and $\mathbf{R}_s'$ correspond to a *connected* RPMG that contains the original one as a subgraph. As a consequence, the original matrix $\mathbf{C}_s$ and the one corresponding to the connected graph will satisfy $\mathbf{C}_s \prec \mathbf{C}_s'$. However, it is easy to show that the right hand side of Eq. (83) is a matrix-decreasing function of $\mathbf{C}_s$, which means that

$$\mathbf{C}_s \prec \mathbf{C}_s' \Rightarrow \mathbf{P}_{n_{k+1}} \succeq \mathbf{P}'_{n_{k+1}}$$

Additionally, in Appendix A it is shown that the right hand side of the Riccati recursion is a matrix-increasing function of the covariance matrix $\mathbf{P}_{n_k}$. Using these two results, and applying induction, in a fashion similar to the one presented in Appendix A, we can show that the value of the covariance matrix at time t_2 will satisfy $\mathbf{P}_{n_{ss}}(t_2) \preceq \mathbf{P}'_{n_{ss}}(t_2)$.

In Appendix I it is shown that the matrix

$$M_A = \mathbf{V}^T\left(I + \mathbf{P}_{n_{ss}}(t_2)h(\mathbf{C}_A)\right)^{-1}\mathbf{P}_{n_{ss}}(t_2)\mathbf{V}$$

is a matrix-increasing function of the covariance $\mathbf{P}_{n_{ss}}(t_2)$. This means that

$$\mathbf{P}_{n_{ss}}(t_2) \preceq \mathbf{P}'_{n_{ss}}(t_2) \Rightarrow M_A \preceq M_A'$$

But we have seen that

$$\mathbf{P}_{n_{ss}}(t_2) \preceq \mathbf{P}'_{n_{ss}}(t_2) \preceq \mathbf{P}_{n_{\mathrm{od}}}(t_2)$$

and that for the matrices $\mathbf{P}_{n_{ss}}(t_2)$ and $\mathbf{P}_{n_{\mathrm{od}}}(t_2)$ the value of M_A is *the same*. Thus, we conclude that for any possible topology, the value of M_A will be identical to the one derived for the case of a connected graph and for the case of Dead Reckoning. This implies that the lemma holds for any possible intermediate topology $\mathcal{T}_B$.

■

This is a significant result due to its important implications. Consider the scenario where the robots of a team, during a phase of their mission, are forced to receive and process a small number of measurements, or even resort to mere Dead Reckoning, due to communication or sensor failures, or because CPU and bandwidth resources are required by other tasks of higher priority. During this interval, a reduced amount of positioning information is available to the robots (sparse RPMG topology) and as a result the performance of CL will temporarily deteriorate. However, once the initial, dense RPMG topology is restored, the team's positioning performance will have sustained *no degradation*. Furthermore, Lemma 2.9 indicates that a dense topology for the RPMG during the initial phase of the deployment of a robot team has a long-term effect on the localization performance of the team. Specifically, if during the initial deployment, the robots leverage their communication and computational resources to support a dense RPMG, this will improve their positioning accuracy at the beginning of CL. Later on, and as the robots focus on mission-specific and other time-critical tasks, they will have to rely on sparser RPMGs as resources dictate. However, when at a subsequent time instant the RPMG resumes its initial, dense topology, the above lemma guarantees that the maximum expected uncertainty will be *identical* to the one that would arise if the dense RPMG topology was retained throughout the run of the robots.

3 Continuous-Time Analysis

3.1 Motivation

In the previous section, the analysis was presented in discrete time, under the assumption that all the measurements (exteroceptive and proprioceptive) are available at the same frequency. However, in practice different sensors usually have different sampling rates. In order to address this problem, we present in this section a continuous-time analysis of the performance of CL. For a robot team with a set of sensors each of which has a given accuracy and a given sampling rate (in general different for each sensor), we can construct a continuous-time system model, in which the covariance of the position estimates will be identical to the covariance of the position estimates in the actual, discrete-time system.

Assuming that the odometric measurements of robot i are available every δt_i seconds, and that the standard deviations of velocity and orientation errors in discrete time are $\sigma_{V_{d_i}}$ and $\sigma_{\phi_{d_i}}$ respectively, then selecting

$$\sigma_{V_{c_i}} = \sqrt{\delta t_i}\sigma_{V_{d_i}}, \text{ and } \sigma_{\phi_{c_i}} = \sqrt{\delta t_i}\sigma_{\phi_{d_i}} \tag{84}$$

yields an "equivalent" continuous time system model, in the sense that for both systems the the rate of influx of uncertainty due to system noise is identical. The proof of this result is given in Appendix G.

Similarly, if exteroceptive measurements whose covariance matrix is R_{d_i} are available every $\delta t'_i$ seconds for the i-th robot, then the covariance matrix function of the measurements in the equivalent continuous time system model is $(R_{d_i}\delta t'_i)\,\delta(t-\tau)$, where $\delta(t-\tau)$ is the Dirac delta function [12]. The factor $\delta t'_i$ can be seen as a normalizing factor to ensure that the information influx in the system due to the exteroceptive measurements is appropriately scaled with the sampling frequency of these measurements.

In the following sections, the continuous-time analysis is presented. Since many readers are not familiar with the continuous-time EKF and the continuous-time Riccati equation, we first present an analysis for a hypothetical scenario of a team or robots localizing in a one-dimensional environment. In this simple case the main results of the derivations can be exhibited more clearly, and a more intuitive understanding can be developed.

3.2 Motion in 1D

A group of N robots moving in 1D uses proprioceptive measurements (e.g. velocity) to propagate their state estimates. The continuous-time state propagation equation for this system is written as (cf. Appendix C):

$$\dot{x}(t) = v(t) + w(t)$$

where $x(t)$ is a vector containing the positions of the robots, $v(t)$ is the input (here the velocities of the robots), and $w(t)$ is the noise in the measurements of these velocities. By comparison with Eq. (167) we see that $F(t) = \mathbf{0}_{N\times N}$, and also $B(t) = G(t) = I_{N\times N}$, the identity matrix[3]. $w(t)$ is assumed to be white zero-mean Gaussian, with constant covariance matrix Q. Since the noise processes that corrupt the measurements of different robots are independent, Q is a diagonal matrix, $Q = \mathrm{diag}(q_i)$, where q_i is the covariance of the noise affecting the measurements of the ith robot.

The robots are also equipped with exteroceptive sensors that allow them to measure: (i) their relative position, and (ii) their absolute position. We note that in this formulation absolute position measurements are *not* required, but availability of such measurements greatly improves localization performance. The measurement model for the exteroceptive measurements is:

$$z(t) = Hx(t) + n(t) \tag{85}$$

where H is the measurement matrix, relating the measurements with the current state of the system, and n is the noise in the measurement, assumed white zero-mean Gaussian, with covariance matrix R.

In order to determine the behavior of the covariance, we need to study the eigenvalues λ_i^2 of the matrix $C = Q^{1/2}H^TR^{-1}HQ^{1/2}$. The types of measurements performed by the robots play a significant role in determining these eigenvalues. We are here concerned with 2 different types of exteroceptive sensor measurements: (i) relative position measurements, i.e., measurements of the difference of the positions of two robots, and (ii) absolute position measurements. The measurement matrix can be written as

$$H = \begin{bmatrix} H_{IJ} \\ H_0 \end{bmatrix} \tag{86}$$

[3]Throughout this Technical Report $I_{m\times m}$ denotes the $m\times m$ identity matrix, $\mathbf{1}_{m\times n}$ denotes the $m\times n$ matrix of ones, and $\mathbf{0}_{m\times n}$ denotes the $m\times n$ matrix of zeros.

where H_{IJ} and H_0 are submatrices that correspond to the relative and absolute position measurements respectively.

These two matrices have a special structure. Specifically, matrix H_0 is composed of a set of submatrices H_{0i} that contain a "1" at the ith column, corresponding to the robot i receiving an absolute position measurement, i.e.

$$H_0 = \begin{bmatrix} \vdots \\ H_{0_i} \\ \vdots \end{bmatrix}$$

with

$$H_{0i} = \begin{bmatrix} 0 & \dots & 1 & \dots & 0 \end{bmatrix}$$

On the other hand, matrix H_{IJ} comprises of rows, each of which corresponds to a relative position measurement, or equivalently, to one edge of the RPMG. The row associated with the measurement between robots i and j contains a "-1", at the column, corresponding to the robot i registering the relative position measurement, and a "1" at the column that corresponds to the robot j which is observed in the measurement, i.e.

$$H_{IJ} = \begin{bmatrix} H_{kl} \\ \vdots \\ H_{mn} \end{bmatrix}$$

with

$$H_{ij} = \begin{bmatrix} 0 & \dots & -1 & \dots & 1 & \dots & 0 \end{bmatrix}$$

The matrix so defined is identical with the *incidence matrix* of the RPMG, when this is viewed as an unweighted graph. In Appendix E.1 it is shown that when the robots do not receive absolute position measurements (in which case the measurement matrix H equals H_{IJ}), C has exactly one eigenvalue equal to zero. Contrary to that, when at least one robot receives absolute positioning information, all eigenvalues are greater than zero.

Using $F = \mathbf{0}_{N\times N}$, $B = I_{N\times N}$, and $G = I_{N\times N}$, the continuous time Riccati equation that describes the time evolution of the covariance for the position estimates of the robots is written as (cf. Eq. (169)):

$$\dot{P} = Q - PH^T R^{-1} HP \tag{87}$$

For the solution of this matrix differential equation the standard methodology involving the decomposition of $P(t)$ into two matrices, and forming the Hamiltonian matrix is employed [13]. The solution is described in what follows.

In order to facilitate the derivations, we first define as P_n the *normalized covariance*

$$P_n = Q^{-1/2} P Q^{-1/2} \Rightarrow P = Q^{1/2} P_n Q^{1/2} \tag{88}$$

Substitution in Eq. (87) yields

$$\begin{aligned} Q^{1/2}\dot{P}_n Q^{1/2} &= Q - Q^{1/2} P_n Q^{1/2} H^T R^{-1} H Q^{1/2} P_n Q^{1/2} \Rightarrow \\ \dot{P}_n &= I_{N\times N} - P_n Q^{1/2} H^T R^{-1} H Q^{1/2} P_n \end{aligned}$$

We introduce the matrix $C = Q^{1/2} H^T R^{-1} H Q^{1/2}$, and the previous equation is simplified to:

$$\dot{P}_n = I_{N\times N} - P_n C P_n \tag{89}$$

The solution to this equation is found by substituting

$$P_n = AB^{-1} \tag{90}$$

Note that since

$$BB^{-1} = I_{N\times N}$$

it is

$$\begin{aligned}
\frac{d}{dt}\left(BB^{-1}\right) = \mathbf{0}_{N\times N} \quad \Rightarrow \\
\dot{B}B^{-1} + B\frac{d}{dt}(B^{-1}) = \mathbf{0}_{N\times N} \quad \Rightarrow \\
\frac{d}{dt}(B^{-1}) = -B^{-1}\dot{B}B^{-1}
\end{aligned}$$

Substituting in Eq. (90) we have

$$\dot{P}_n = \dot{A}B^{-1} - AB^{-1}\dot{B}B^{-1} \tag{91}$$

Using Eqs. (90) and (91), Eq. (89) can be written as:

$$\dot{A}B^{-1} - AB^{-1}\dot{B}B^{-1} = I_{N\times N} - AB^{-1}CAB^{-1}$$

Multiplying both sides by B we have

$$\dot{A} - AB^{-1}\dot{B} = B - AB^{-1}CA$$

Separating the nonlinear from the linear terms and noting that

$$\begin{aligned}
-AB^{-1}\dot{B} &= -AB^{-1}CA \Rightarrow \\
\dot{B} &= CA
\end{aligned}$$

we can decompose the Riccati in the following two equations:

$$\begin{aligned}
\dot{A} &= B \\
\dot{B} &= CA
\end{aligned}$$

or in a matrix form

$$\begin{bmatrix} \dot{B} \\ \dot{A} \end{bmatrix} = \begin{bmatrix} \mathbf{0}_{N\times N} & C \\ I_{N\times N} & \mathbf{0}_{N\times N} \end{bmatrix} \begin{bmatrix} B \\ A \end{bmatrix} \tag{92}$$

Where the matrix

$$\mathcal{H} = \begin{bmatrix} \mathbf{0}_{N\times N} & C \\ I_{N\times N} & \mathbf{0}_{N\times N} \end{bmatrix} \tag{93}$$

is the Hamiltonian of this system. The general solution of Eq. (92) is given by

$$\begin{bmatrix} B(t) \\ A(t) \end{bmatrix} = e^{\mathcal{H}t} \begin{bmatrix} B(0) \\ A(0) \end{bmatrix} \tag{94}$$

where $A(0)$ and $B(0)$ are the initial values for these matrices. These are selected so that the identity $P_n(0) = A(0)B^{-1}(0)$ holds, i.e., $A(0) = P_n(0)$ and $B(0) = I_{N\times N}$. Employing Taylor series expansion for computing the exponential of the Hamiltonian matrix yields:

$$\begin{aligned}
e^{\mathcal{H}t} &= I_{N\times N} + \mathcal{H}t + \frac{\mathcal{H}^2t^2}{2!} + \frac{\mathcal{H}^3t^3}{3!} + \cdots = \\
&= \begin{bmatrix} I_{N\times N} + C\frac{t^2}{2!} + C^2\frac{t^4}{4!} + C^3\frac{t^6}{6!} + \cdots & C\frac{t}{1!} + C^2\frac{t^3}{3!} + C^3\frac{t^5}{5!} + \cdots \\ \frac{t}{1!}I_{N\times N} + C\frac{t^3}{3!} + C^2\frac{t^5}{5!} + \cdots & I_{N\times N} + C\frac{t^2}{2!} + C^2\frac{t^4}{4!} + C^3\frac{t^6}{6!} + \cdots \end{bmatrix}
\end{aligned}$$

In order to derive a simpler expression for this relation, the Singular Value Decomposition of C is employed. That is, we write $C = U\Lambda U^T$ where U is an orthonormal matrix containing the singular vectors of C, and Λ is a diagonal matrix whose diagonal elements are the eigenvalues of C. Since $C = Q^{1/2}H^TR^{-1}HQ^{1/2}$ is a symmetric positive semidefinite matrix, its eigenvalues, λ_i^2, are real and nonnegative (and equal the squares of the singular values λ_i of $Q^{1/2}H^TR^{-1/2}$), i.e. $\Lambda = \mathrm{diag}(\lambda_i^2)$. We also note that since for symmetric positive semidefinite matrices, such as C, the singular values are identical to the eigenvalues, and the eigenvectors to the singular vectors, these terms can be used interchangeably.

We now manipulate each of the submatrices comprising $e^{\mathcal{H}t}$ separately. The 2 diagonal submatrices are equal to each other, and are given by:

$$\begin{aligned}
e^{\mathcal{H}t}(1,1) = e^{\mathcal{H}t}(2,2) &= I_{N\times N} + C\frac{t^2}{2!} + C^2\frac{t^4}{4!} + C^3\frac{t^6}{6!} + \cdots \\
&= I_{N\times N} + U\Lambda U^T\frac{t^2}{2!} + (U\Lambda U^T)^2\frac{t^4}{4!} + (U\Lambda U^T)^3\frac{t^6}{6!} + \cdots \\
&= I_{N\times N} + U\Lambda U^T\frac{t^2}{2!} + U\Lambda^2 U^T\frac{t^4}{4!} + U\Lambda^3 U^T\frac{t^6}{6!} + \cdots \\
&= U(I_{N\times N} + \Lambda\frac{t^2}{2!} + \Lambda^2\frac{t^4}{4!} + \Lambda^3\frac{t^6}{6!} + \cdots)U^T \\
&= U(I_{N\times N} + (\Lambda^{1/2})^2\frac{t^2}{2!} + (\Lambda^{1/2})^4\frac{t^4}{4!} + (\Lambda^{1/2})^6\frac{t^6}{6!} + \cdots)U^T \\
&= U\,\mathrm{diag}\left(1 + \lambda_i^2\frac{t^2}{2!} + \lambda_i^4\frac{t^4}{4!} + \lambda_i^6\frac{t^6}{6!} + \cdots\right)U^T \\
&= \frac{1}{2}U\,\mathrm{diag}\left(e^{\lambda_i t} + e^{-\lambda_i t}\right)U^T
\end{aligned}$$

To obtain the last expression, the result from Appendix D was used. The upper right submatrix of $e^{\mathcal{H}t}$ is given by:

$$\begin{aligned}
e^{\mathcal{H}t}(1,2) &= C\frac{t}{1!} + C^2\frac{t^3}{3!} + C^3\frac{t^5}{5!} + \cdots \\
&= U\Lambda U^T(I_{N\times N}t + U\Lambda U^T\frac{t^3}{3!} + U\Lambda^2 U^T\frac{t^5}{5!} + \cdots) \\
&= U\Lambda U^T U(t + \Lambda\frac{t^3}{3!} + \Lambda^2\frac{t^5}{5!} + \cdots)U^T \\
&= U\Lambda^{1/2}(\Lambda^{1/2}t + \Lambda^{3/2}\frac{t^3}{3!} + \Lambda^{5/2}\frac{t^5}{5!} + \cdots)U^T \\
&= U\,\mathrm{diag}\left(\lambda_i(\lambda_i t + \lambda_i^3\frac{t^3}{3!} + \lambda_i^5\frac{t^5}{5!} + \cdots)\right)U^T \\
&= \frac{1}{2}U\,\mathrm{diag}\left(\lambda_i(e^{\lambda_i t} - e^{-\lambda_i t})\right)U^T
\end{aligned}$$

We treat the lower left submatrix of $e^{\mathcal{H}t}$ in a similar manner:

$$\begin{aligned}
e^{\mathcal{H}t}(2,1) &= \frac{t}{1!}I_{N\times N} + C\frac{t^3}{3!} + C^2\frac{t^5}{5!} + \cdots \\
&= tI_{N\times N} + U\Lambda U^T\frac{t^3}{3!} + U\Lambda^2 U^T\frac{t^5}{5!} + \cdots \\
&= U(tI_{N\times N} + \Lambda\frac{t^3}{3!} + \Lambda^2\frac{t^5}{5!} + \cdots)U^T \\
&= U\Lambda^{-1/2}(t\Lambda^{1/2} + \Lambda^{3/2}\frac{t^3}{3!} + \Lambda^{5/2}\frac{t^5}{5!} + \cdots)U^T \\
&= U\,\mathrm{diag}\left(\frac{1}{\lambda_i}(\lambda_i t + \lambda_i^3\frac{t^3}{3!} + \lambda_i^5\frac{t^5}{5!} + \cdots)\right)U^T \\
&= \frac{1}{2}U\,\mathrm{diag}\left(\frac{e^{\lambda_i t} - e^{-\lambda_i t}}{\lambda_i}\right)U^T
\end{aligned}$$

In Appendix E.1 it is shown, that when none of the robots receives absolute position measurements, the smallest eigenvalue of C is equal to zero. Thus, in this case the quantity $\mathrm{diag}\left(\frac{e^{\lambda_i t}-e^{-\lambda_i t}}{\lambda_i}\right)$ that appears in the last expression presents a problem, since the eigenvalues appear in the denominator. Note that the quantity being divided by the zero eigenvalue is also equal to zero ($e^{0t} - e^{-0t} = 0$) and therefore the above expression is actually undefined. However, in Appendix F it is proven formally that the quantity under consideration exists, and is given by

$$e^{\mathcal{H}t}(2,1) = \frac{1}{2} U \begin{bmatrix} \mathrm{diag}_{N-1}\left(\frac{e^{\lambda_i t} - e^{-\lambda_i t}}{\lambda_i}\right) & \mathbf{0}_{2\times(N-1)} \\ \mathbf{0}_{(N-1)\times 1} & 2t \end{bmatrix} U^T \tag{95}$$

This expression is quite cumbersome, and its use would make the resulting formulas unappealing and difficult to understand. We will therefore continue to use the initial, less strict notation in the following, bearing in mind that its true meaning is given by this last expression.

Combining th previous results, the following expression for $e^{\mathcal{H}t}$ is derived:

$$e^{\mathcal{H}t} = \begin{bmatrix} \frac{1}{2} U \,\mathrm{diag}\left(e^{\lambda_i t} + e^{-\lambda_i t}\right) U^T & \frac{1}{2} U \,\mathrm{diag}\left(\lambda_i (e^{\lambda_i t} - e^{-\lambda_i t})\right) U^T \\ \frac{1}{2} U \,\mathrm{diag}\left(\frac{e^{\lambda_i t} - e^{-\lambda_i t}}{\lambda_i}\right) U^T & \frac{1}{2} U \,\mathrm{diag}\left(e^{\lambda_i t} + e^{-\lambda_i t}\right) U^T \end{bmatrix} \tag{96}$$

Substituting for $e^{\mathcal{H}t}$ in (94) and using the initial values $A(0) = P_n(0)$, $B(0) = I_{N\times N}$, yields:

$$\begin{bmatrix} B(t) \\ A(t) \end{bmatrix} = e^{\mathcal{H}t} \begin{bmatrix} I_{N\times N} \\ P_n(0) \end{bmatrix} = \begin{bmatrix} \frac{1}{2} U \,\mathrm{diag}\left(e^{\lambda_i t} + e^{-\lambda_i t}\right) U^T + \frac{1}{2} U \,\mathrm{diag}\left(\lambda_i (e^{\lambda_i t} - e^{-\lambda_i t})\right) U^T P_n(0) \\ \frac{1}{2} U \,\mathrm{diag}\left(\frac{e^{\lambda_i t} - e^{-\lambda_i t}}{\lambda_i}\right) U^T + \frac{1}{2} U \,\mathrm{diag}\left(e^{\lambda_i t} + e^{-\lambda_i t}\right) U^T P_n(0) \end{bmatrix}$$

Thus, using Eq. (90) the solution for the normalized covariance becomes:

$$\begin{aligned} P_n(t) &= \left(\frac{1}{2} U \,\mathrm{diag}\left(\frac{e^{\lambda_i t} - e^{-\lambda_i t}}{\lambda_i}\right) U^T + \frac{1}{2} U \,\mathrm{diag}\left(e^{\lambda_i t} + e^{-\lambda_i t}\right) U^T P_n(0)\right) \times \\ &\times \left(\frac{1}{2} U diag\left(e^{\lambda_i t} + e^{-\lambda_i t}\right) U^T + \frac{1}{2} U \,\mathrm{diag}\left(\lambda_i (e^{\lambda_i t} - e^{-\lambda_i t})\right) U^T P_n(0)\right)^{-1} \\ &= U \left(\mathrm{diag}\left(\frac{e^{\lambda_i t} - e^{-\lambda_i t}}{\lambda_i}\right) + \mathrm{diag}\left(e^{\lambda_i t} + e^{-\lambda_i t}\right) U^T P_n(0) U\right) \times \\ &\times \left(\mathrm{diag}\left(e^{\lambda_i t} + e^{-\lambda_i t}\right) + \mathrm{diag}\left(\lambda_i (e^{\lambda_i t} - e^{-\lambda_i t})\right) U^T P_n(0) U\right)^{-1} U^T \\ &= U \; (K(t) + L(t) P_0) \; (L(t) + \Lambda K(t) P_0)^{-1} \; U^T \end{aligned} \tag{97}$$

Where we have denoted

$$K(t) = \mathrm{diag}\left(\frac{e^{\lambda_i t} - e^{-\lambda_i t}}{\lambda_i}\right)$$

$$L(t) = \mathrm{diag}\left(e^{\lambda_i t} + e^{-\lambda_i t}\right)$$

and

$$P_0 = U^T P_n(0) U$$

We will now show that $P_n(t)$ can be written as

$$P_n(t) = U \left(K(t) L(t)^{-1} + M(t)\right) U^T \tag{98}$$

where $M(t)$ is a matrix to be specified. For notation simplicity, we drop the time arguments from $K(t)$, $L(t)$ and $M(t)$ in the following. From Eqs. (97) and (98) we have:

$$\begin{aligned} P_n(t) = U(K + LP_0)(L + \Lambda K P_0)^{-1} U^T &= U\left(KL^{-1} + M\right) U^T \Rightarrow \\ (K + LP_0)(L + \Lambda K P_0)^{-1} &= \left(KL^{-1} + M\right) \Rightarrow \\ K + LP_0 &= \left(KL^{-1} + M\right)(L + \Lambda K P_0) \Rightarrow \\ K + LP_0 &= K + L^{-1} K^2 \Lambda P_0 + M(L + \Lambda K P_0) \Rightarrow \\ \left(L - L^{-1} K^2 \Lambda\right) P_0 &= M(L + \Lambda K P_0) \Rightarrow \\ M &= \left(L - L^{-1} K^2 \Lambda\right) P_0 (L + \Lambda K P_0)^{-1} \end{aligned}$$

We note that :

$$
\begin{aligned}
L - L^{-1}K^2\Lambda &= L\left(I_{N\times N} - L^{-2}K^2\Lambda\right) \\
&= \mathrm{diag}\left(e^{\lambda_i t} + e^{-\lambda_i t}\right)\left(I_{N\times N} - \mathrm{diag}\left(e^{\lambda_i t} + e^{-\lambda_i t}\right)^{-2}\mathrm{diag}\left(\frac{e^{\lambda_i t} - e^{-\lambda_i t}}{\lambda_i}\right)^2 \mathrm{diag}(\lambda_i^2)\right) \\
&= \mathrm{diag}\left(e^{\lambda_i t} + e^{-\lambda_i t}\right)\left(I_{N\times N} - \mathrm{diag}\left(\frac{e^{\lambda_i t} - e^{-\lambda_i t}}{e^{\lambda_i t} + e^{-\lambda_i t}}\right)^2\right) \\
&= \mathrm{diag}\left(e^{\lambda_i t} + e^{-\lambda_i t}\right)\left(\mathrm{diag}\left(\frac{(e^{\lambda_i t} + e^{-\lambda_i t})^2 - (e^{\lambda_i t} - e^{-\lambda_i t})^2}{e^{\lambda_i t} + e^{-\lambda_i t}}\right)^2\right) \\
&= \mathrm{diag}\left(\frac{(e^{\lambda_i t} + e^{-\lambda_i t})^2 - (e^{\lambda_i t} - e^{-\lambda_i t})^2}{e^{\lambda_i t} + e^{-\lambda_i t}}\right) \\
&= \mathrm{diag}\left(\frac{4}{e^{\lambda_i t} + e^{-\lambda_i t}}\right) \\
&= 4L^{-1}
\end{aligned}
$$

Thus M can be written as

$$
\begin{aligned}
M &= \left(L - L^{-1}K^2\Lambda\right)P_0(L + \Lambda K P_0)^{-1} \\
&= 4L^{-1}P_0(L + \Lambda K P_0)^{-1} \\
&= 4L^{-1}P_0\left(I_{N\times N} + \Lambda K L^{-1}P_0\right)^{-1}L^{-1}
\end{aligned}
$$

and substitution in (98) yields:

$$
P_n(t) = U\left(KL^{-1} + 4L^{-1}P_0\left(I_{N\times N} + \Lambda K L^{-1}P_0\right)^{-1}L^{-1}\right)U^T \tag{99}
$$

To determine the behavior of the covariance at steady state, we compute the limit of the above quantity as time goes to infinity. We identify two different cases, based on the availability of absolute position measurements.

3.2.1 Steady-State Covariance - Observable system

When at least one of the robots receives absolute position measurements, all the eigenvalues of C will be positive, and thus at steady state (i.e., as $t \to \infty$), we obtain:

$$
\begin{aligned}
\lim_{t\to\infty} KL^{-1} &= \lim_{t\to\infty}\mathrm{diag}\left(\frac{e^{\lambda_i t} - e^{-\lambda_i t}}{\lambda_i}\right)\mathrm{diag}\left(e^{\lambda_i t} + e^{-\lambda_i t}\right)^{-1} \\
&= \mathrm{diag}\left(\frac{1}{\lambda_i}\right)
\end{aligned}
$$

And also

$$
\begin{aligned}
\lim_{t\to\infty} 4L^{-1}P_0\left(I_{N\times N} + \Lambda K L^{-1}P_0\right)^{-1}L^{-1} &= \lim_{t\to\infty}\mathrm{diag}\left(e^{\lambda_i t} + e^{-\lambda_i t}\right)^{-1}(I_{N\times N} + \Lambda^{1/2}P_0)^{-1}\mathrm{diag}\left(e^{\lambda_i t} + e^{-\lambda_i t}\right)^{-1} \\
&= \mathbf{0}_{N\times N}
\end{aligned}
$$

Using the above two results, we see that

$$
\begin{aligned}
\lim_{t\to\infty} P_n(t) &= \lim_{t\to\infty} U\left(KL^{-1} + 4L^{-1}P_0\left(I_{N\times N} + \Lambda K L^{-1}P_0\right)^{-1}L^{-1}\right)U^T \\
&= U\,\mathrm{diag}\left(\frac{1}{\lambda_i}\right)U^T
\end{aligned}
$$

and therefore the steady state covariance for the position estimates of the robots will be

$$
P_{ss}(t) \;=\; Q^{1/2}U\,\mathrm{diag}\left(\frac{1}{\lambda_i}\right)U^TQ^{1/2} \;=\; Q^{1/2}\sqrt{C^{-1}}Q^{1/2} \tag{100}
$$

where $\sqrt{C^{-1}} = U\Lambda^{-1/2}U^T$ is the matrix square root of C^{-1}, which always exists since the eigenvalues of C are positive. Notice that when at least one robot receives absolute position measurements, the steady state uncertainty depends on the topology of the RPMG (affecting C), and the uncertainty of proprioceptive and exteroceptive measurements, represented by Q and R (which is embedded in C).

In order to gain more insight on how the measurement accuracy and the graph topology affect the steady state localization uncertainty, we consider the simple case in which $Q = qI$, and $R = rI$, i.e., a homogeneous robot group. In this case, it is trivial to show that the expression for the steady state covariance reduces to

$$P_{ss} = \sqrt{qr}\sqrt{(H^T H)^{-1}} \tag{101}$$

Since in this equation the effects of the graph topology and the measurement covariances are decoupled, we can see more clearly the effect of the accuracy of the sensors on the steady state localization uncertainty.

3.2.2 Steady-State Covariance - Unobservable system

When none of the robots receives absolute positioning measurements, the smallest (N-th) singular value of C equals zero. Using the expression in Eq. (95), we can write

$$\begin{aligned}
KL^{-1} &= \operatorname{diag}\left(\frac{e^{\lambda_i t} - e^{-\lambda_i t}}{\lambda_i}\right)\operatorname{diag}\left(e^{\lambda_i t} + e^{-\lambda_i t}\right)^{-1} \\
&= \begin{bmatrix} \operatorname{diag}_{N-1}\left(\frac{e^{\lambda_i t}-e^{-\lambda_i t}}{\lambda_i\left(e^{\lambda_i t}+e^{-\lambda_i t}\right)}\right) & \mathbf{0}_{N-1\times 1} \\ \mathbf{0}_{1\times N-1} & \frac{2t}{e^{\lambda_i 0}+e^{-\lambda_i 0}} \end{bmatrix} \\
&= \begin{bmatrix} \operatorname{diag}_{N-1}\left(\frac{e^{\lambda_i t}-e^{-\lambda_i t}}{\lambda_i\left(e^{\lambda_i t}+e^{-\lambda_i t}\right)}\right) & \mathbf{0}_{N-1\times 1} \\ \mathbf{0}_{1\times N-1} & 0 \end{bmatrix} + \begin{bmatrix} \mathbf{0}_{(N-1)\times(N-1)} & \mathbf{0}_{(N-1)\times 1} \\ \mathbf{0}_{1\times(N-1)} & t \end{bmatrix}
\end{aligned} \tag{102}$$

Taking the limit of the first term of this expression yields:

$$\lim_{t\to\infty}\begin{bmatrix} \operatorname{diag}_{N-1}\left(\frac{e^{\lambda_i t}-e^{-\lambda_i t}}{\lambda_i\left(e^{\lambda_i t}+e^{-\lambda_i t}\right)}\right) & \mathbf{0}_{N-1\times 1} \\ \mathbf{0}_{1\times N-1} & 0 \end{bmatrix} = \begin{bmatrix} \operatorname{diag}_{N-1}\left(\frac{1}{\lambda_i}\right) & \mathbf{0}_{(N-1)\times 1} \\ \mathbf{0}_{1\times(N-1)} & 0 \end{bmatrix}$$

And thus at steady state, the term KL^{-1} becomes

$$\begin{bmatrix} \operatorname{diag}_{N-1}\left(\frac{1}{\lambda_i}\right) & \mathbf{0}_{(N-1)\times 1} \\ \mathbf{0}_{1\times(N-1)} & t \end{bmatrix} \tag{103}$$

In the last line, we have again used t as the bottom right element of the matrix, to point out that this element contributes with a constant rate of increase of uncertainty. We also note that

$$\begin{aligned}
&\lim_{t\to\infty} 4L^{-1}P_0\left(I_{N\times N} + \Lambda KL^{-1}P_0\right)^{-1}L^{-1} = \\
&\lim_{t\to\infty} 4\begin{bmatrix} \operatorname{diag}_{N-1}\left(\frac{1}{e^{\lambda_i t}+e^{-\lambda_i t}}\right) & \mathbf{0}_{(N-1)\times 1} \\ \mathbf{0}_{1\times(N-1)} & 1/2 \end{bmatrix} P_0\left(I_{N\times N} + \Lambda KL^{-1}P_0\right)^{-1}\begin{bmatrix} \operatorname{diag}_{N-1}\left(\frac{1}{e^{\lambda_i t}+e^{-\lambda_i t}}\right) & \mathbf{0}_{(N-1)\times 1} \\ \mathbf{0}_{1\times(N-1)} & 1/2 \end{bmatrix} \\
&\quad = \begin{bmatrix} \mathbf{0}_{(N-1)\times(N-1)} & \mathbf{0}_{(N-1)\times 1} \\ \mathbf{0}_{1\times(N-1)} & 1 \end{bmatrix} P_0\left(I_{N\times N} + \Lambda^{1/2}P_0\right)^{-1}\begin{bmatrix} \mathbf{0}_{(N-1)\times(N-1)} & \mathbf{0}_{(N-1)\times 1} \\ \mathbf{0}_{1\times(N-1)} & 1 \end{bmatrix}
\end{aligned}$$

By denoting

$$P_0\left(I_{N\times N} + \Lambda^{1/2}P_0\right)^{-1} = \begin{bmatrix} m_{11} & m_{12} & \dots & m_{1N} \\ \vdots & \vdots & \vdots & \vdots \\ m_{N1} & m_{N2} & \dots & m_{NN} \end{bmatrix}$$

we can write

$$
\begin{aligned}
&\lim_{t\to\infty} 4L^{-1}P_0\left(I_{N\times N}+\Lambda K L^{-1}P_0\right)^{-1}L^{-1} = \\
&= \begin{bmatrix} \mathbf{0}_{(N-1)\times(N-1)} & \mathbf{0}_{(N-1)\times 1} \\ \mathbf{0}_{1\times(N-1)} & 1 \end{bmatrix} \begin{bmatrix} m_{11} & m_{12} & \ldots & m_{1N} \\ \vdots & \vdots & \vdots & \vdots \\ m_{N1} & m_{N2} & \ldots & m_{NN} \end{bmatrix} \begin{bmatrix} \mathbf{0}_{(N-1)\times(N-1)} & \mathbf{0}_{(N-1)\times 1} \\ \mathbf{0}_{1\times(N-1)} & 1 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ m_{N1} & m_{N2} & \ldots & m_{NN} \end{bmatrix} \begin{bmatrix} \mathbf{0}_{(N-1)\times(N-1)} & \mathbf{0}_{(N-1)\times 1} \\ \mathbf{0}_{1\times(N-1)} & 1 \end{bmatrix} \\
&= \begin{bmatrix} \mathbf{0}_{(N-1)\times(N-1)} & \mathbf{0}_{(N-1)\times 1} \\ \mathbf{0}_{1\times(N-1)} & m_{NN} \end{bmatrix} \qquad (104)
\end{aligned}
$$

Thus only the last element of the last row of $P_0\left(I_{N\times N}+\Lambda^{1/2}P_0\right)^{-1}$ is needed. Recalling that $P_0 = U^T P_n(0)U$, we can write

$$
\begin{aligned}
M &= U^T P_n(0)U\left(I_{N\times N}+\Lambda^{1/2}U^T P_n(0)U\right)^{-1} \\
&= U^T P_n(0)U\left(I_{N\times N}+\Lambda^{1/2}U^T P_n(0)U\right)^{-1}U^T U \\
&= U^T P_n(0)\left(U^T\right)^{-1}\left(I_{N\times N}+\Lambda^{1/2}U^T P_n(0)U\right)^{-1}U^T U \\
&= U^T P_n(0)\left(U^T+\Lambda^{1/2}U^T P_n(0)UU^T\right)^{-1}U^{-1}U \\
&= U^T P_n(0)\left(UU^T+U\Lambda^{1/2}U^T P_n(0)\right)^{-1}U \\
&= U^T P_n(0)\left(I_{N\times N}+\sqrt{C}P_n(0)\right)^{-1}U \\
&= \begin{bmatrix} U^T_{1:N-1} \\ U^T_N \end{bmatrix} P_n(0)\left(I_{N\times N}+\sqrt{C}P_n(0)\right)^{-1}\begin{bmatrix} U_{1:N-1} & U_N \end{bmatrix} \\
&= \begin{bmatrix} U^T_{1:N-1}P_n(0)\left(I_{N\times N}+\sqrt{C}P_n(0)\right)^{-1}U_{1:N-1} & U^T_{1:N-1}P_n(0)\left(I_{N\times N}+\sqrt{C}P_n(0)\right)^{-1}U_N \\ U^T_N P_n(0)\left(I_{N\times N}+\sqrt{C}P_n(0)\right)^{-1}U_{1:N-1} & U^T_N P_n(0)\left(I_{N\times N}+\sqrt{C}P_n(0)\right)^{-1}U_N \end{bmatrix}
\end{aligned}
$$

In the above, $U_{1:N-1}$ is a matrix of dimensions $N\times(N-1)$, consisting of the $N-1$ singular vectors of C corresponding to the nonzero singular values. From the above expression, we obtain m_{NN}:

$$
m_{NN} = U^T_N P_n(0)\left(I_{N\times N}+\sqrt{C}P_n(0)\right)^{-1}U_N \qquad (105)
$$

In Appendix E.1 it is shown that $U_N = \sqrt{q_{total}}Q^{-1/2}\mathbf{1}_{N\times 1}$, where

$$
\frac{1}{q_{total}} = \sum_{i=1}^{N}\frac{1}{q_i} \qquad (106)
$$

Substitution in Eq. (105) yields:

$$
\begin{aligned}
m_{NN} &= q_{total}\mathbf{1}^T_{N\times 1}Q^{-1/2}P_n(0)\left(I_{N\times N}+\sqrt{C}P_n(0)\right)^{-1}Q^{-1/2}\mathbf{1}_{N\times 1} \\
&= q_{total}\mathbf{1}^T_{N\times 1}Q^{-1}P(0)Q^{-1/2}\left(I_{N\times N}+\sqrt{C}Q^{-1/2}P(0)Q^{-1/2}\right)^{-1}Q^{-1/2}\mathbf{1}_{N\times 1}
\end{aligned}
$$

We denote $W = q_{total}Q^{-1}P(0)Q^{-1/2}\left(I+\sqrt{C}Q^{-1/2}P(0)Q^{-1/2}\right)^{-1}Q^{-1/2} = [w_{ij}]$, and the above relation reduces to

$$m_{NN} = \sum_{i,j} w_{ij} \tag{107}$$

That is, m_{NN} is the sum of all elements of the matrix W. Using this result, and the result of Eq. (103), we have the following expression for the normalized steady state uncertainty:

$$P_n(t) = U\left(\begin{bmatrix} \mathrm{diag}_{N-1}\left(\frac{1}{\lambda_i}\right) & \mathbf{0}_{(N-1)\times 1} \\ \mathbf{0}_{1\times(N-1)} & t \end{bmatrix} + \begin{bmatrix} \mathbf{0}_{(N-1)\times(N-1)} & \mathbf{0}_{(N-1)\times 1} \\ \mathbf{0}_{1\times(N-1)} & m_{NN} \end{bmatrix}\right)U^T$$

Thus the actual uncertainty at steady state is:

$$\begin{aligned} P(t) &= Q^{1/2}U\left(\begin{bmatrix} \mathrm{diag}_{N-1}\left(\frac{1}{\lambda_i}\right) & \mathbf{0}_{(N-1)\times 1} \\ \mathbf{0}_{1\times(N-1)} & t \end{bmatrix} + \begin{bmatrix} \mathbf{0}_{(N-1)\times(N-1)} & \mathbf{0}_{(N-1)\times 1} \\ \mathbf{0}_{1\times(N-1)} & m_{NN} \end{bmatrix}\right)U^T Q^{1/2} \\ &= Q^{1/2}U\begin{bmatrix} \mathrm{diag}_{N-1}\left(\frac{1}{\lambda_i}\right) & \mathbf{0}_{(N-1)\times 1} \\ \mathbf{0}_{1\times(N-1)} & 0 \end{bmatrix}U^T Q^{1/2} + Q^{1/2}U\begin{bmatrix} \mathbf{0}_{(N-1)\times(N-1)} & \mathbf{0}_{(N-1)\times 1} \\ \mathbf{0}_{1\times(N-1)} & t+m_{NN} \end{bmatrix}U^T Q^{1/2} \\ &= P_1 + P_2(t) \end{aligned} \tag{108}$$

In the above relation, the term P_1 is a constant term, that is independent of the initial uncertainty of the robots. The term $P_2(t)$ can be written as

$$\begin{aligned} P_2(t) &= Q^{1/2}U\begin{bmatrix} \mathbf{0}_{(N-1)\times(N-1)} & \mathbf{0}_{(N-1)\times 1} \\ \mathbf{0}_{1\times(N-1)} & t+m_{NN} \end{bmatrix}U^T Q^{1/2} \\ &= (t+m_{NN})\, Q^{1/2}U_N U_N^T Q^{1/2} \\ &= \left(t + \sum_{i,j}[w_{ij}]\right) q_{total}\ \mathbf{1}_{N\times N} \\ &= tq_{total}\mathbf{1}_{N\times N} + \sum_{i,j}[w_{ij}]\ q_{total}\mathbf{1}_{N\times N} \end{aligned}$$

We have thus proven the following lemma:

Lemma 3.1 *For a group of N robots moving in 1D and performing cooperative localization, their positional uncertainty at steady state grows linearly with respect to time, and is given by*

$$P_{ss}(t) = Q^{1/2}U\begin{bmatrix} \mathrm{diag}_{N-1}\left(\frac{1}{\lambda_i}\right) & \mathbf{0}_{(N-1)\times 1} \\ \mathbf{0}_{1\times(N-1)} & 0 \end{bmatrix}U^T Q^{1/2} + \sum_{i,j}[w_{ij}]\ q_{total}\ \mathbf{1}_{N\times N} + t\ q_{total}\ \mathbf{1}_{N\times N}$$

Where

$$\frac{1}{q_{total}} = \sum_{i=1}^{N}\frac{1}{q_i}$$

and

$$W = q_{total}Q^{-1}P(0)Q^{-1/2}\left(I_{N\times N}+\sqrt{C}Q^{-1/2}P(0)Q^{-1/2}\right)^{-1}Q^{-1/2} = [w_{ij}]$$

It is worth noting that the rate at which the uncertainty grows is q_{total}, and is *identical* for all the robots in the group, and *independent* of the topology of the RPMG. We also note that from Eq. (106) it follows that

$$\frac{1}{q_{total}} = \sum_{i=1}^{N}\frac{1}{q_i} \geq \frac{1}{\max_i(q_i)} \Rightarrow q_{total} \leq \max_i(q_i) \tag{109}$$

In the absence of relative position measurements the uncertainty of each robot grows linearly with time, and is given by $P_i = q_i\, t$. Therefore, during CL the rate of uncertainty increase is smaller than the rate of increase the robot with the best odometry sensors would have, in the absence of relative position measurements.

The constant term of the steady state covariance depends of the network topology (affecting the eigenvectors and eigenvalues of C), the initial uncertainty and the accuracy of all the measurements performed by the robots. The effects of the number of robots, the initial uncertainty, and the network topology, become more evident in Section 3.2.4, where simulation results are presented.

3.2.3 Reconfigurations of the RPMG

The preceding analysis assumes that the topology of the graph describing the relative position measurements between robots does not change. However, in a realistic scenario this may not be the case. In practice the topology of the RPMG may change as the robots move in space (see also Section 3.4), and therefore the study of the effects of RPMG reconfigurations on the positioning accuracy is of considerable interest.

In this section the following scenario is examined: At the initial stage of the deployment of a robotic team (Phase 1), the RPMG has a topology A, e.g., the complete graph shown in Fig. 8(a), and retains this topology until time instant t_1, when it assumes a different topology B (Phase 2). This topology may be a connected one, e.g., the ring graph shown in Fig. 8(b), or even an empty graph topology, i.e. the case in which all the robots localize independently. We will show that consideration of both cases leads to the same result. Finally, at a second time instant t_2, the RPMG assumes the initial topology, A, once again (Phase 3). We assume that the time intervals $(0, t_1),\ (t_1, t_2)$ are of enough duration in order for the transient phenomena in the time evolution of uncertainty to subside. For this scenario, the following lemma applies:

Lemma 3.2 *The steady state uncertainty of the robots after the RPMG has resumed its initial topology is identical to the steady state uncertainty that would occur if no RPMG reconfigurations had taken place. This implies that these reconfigurations inflict no loss of positioning accuracy at steady state.*

Proof Assuming that the time interval $(0, t_1)$ is of enough duration for the steady state results to apply, at time t_1 the normalized covariance is given by Eq. (108) as:

$$P_n(t_1) = U_A \left(\begin{bmatrix} \mathrm{diag}_{N-1}\left(\frac{1}{\lambda_{Ai}}\right) & \mathbf{0}_{(N-1)\times 1} \\ \mathbf{0}_{1\times(N-1)} & t_1 + m_{NN_A} \end{bmatrix} \right) U_A^T \tag{110}$$

where

$$m_{NN_A} \quad = \quad U_N^T P_n(0) \left(I_{N\times N} + \sqrt{C_A} P_n(0) \right)^{-1} U_N \tag{111}$$

In the above relations $P_n(0)$ is the initial normalized covariance, and the subscript A has been appended to denote quantities related to the RPMG topology A. Since U_N, the eigenvector of C associated with the zero eigenvalue, is independent of the topology of the RPMG, no additional subscript need be appended to it.

We first consider the case in which the RPMG assumes a connected topology B at time t_1. At a later time instant t_2, after sufficient time has passed from the change of the RPMG topology, the normalized covariance matrix is given by

$$P_n(t_2) = U_B \begin{bmatrix} \mathrm{diag}_{N-1}\left(\frac{1}{\lambda_{Bi}}\right) & \mathbf{0}_{(N-1)\times 1} \\ \mathbf{0}_{1\times(N-1)} & (t_2 - t_1) + m_{NN_B} \end{bmatrix} U_B^T \tag{112}$$

where

$$m_{NN_B} \quad = \quad U_N^T P_n(t_1) \left(I_{N\times N} + \sqrt{C_B} P_n(t_1) \right)^{-1} U_N \tag{113}$$

In order to simplify the last expression, we first employ the matrix inversion lemma (Appendix H):

$$\begin{aligned} m_{NN_B} &= U_N^T P_n(t_1) \left(I_{N\times N} + \sqrt{C_B} P_n(t_1) \right)^{-1} U_N \\ &= U_N^T P_n(t_1) \left(I_{N\times N} - \sqrt{C_B} \left(I_{N\times N} + P_n(t_1)\sqrt{C_B} \right)^{-1} P_n(t_1) \right) U_N \\ &= U_N^T P_n(t_1) U_N - U_N^T P_n(t_1) \sqrt{C_B} \left(I_{N\times N} + P_n(t_1)\sqrt{C_B} \right)^{-1} P_n(t_1) U_N \end{aligned}$$

and substitution from Eq. (110) yields:

$$\begin{aligned}
m_{NN_B} &= U_N^T U_A \begin{bmatrix} \mathrm{diag}_{N-1}\left(\frac{1}{\lambda_{Ai}}\right) & \mathbf{0}_{(N-1)\times 1} \\ \mathbf{0}_{1\times(N-1)} & t_1 + m_{NN_A} \end{bmatrix} U_A^T U_N - \\
&- U_N^T U_A \begin{bmatrix} \mathrm{diag}_{N-1}\left(\frac{1}{\lambda_{Ai}}\right) & \mathbf{0}_{(N-1)\times 1} \\ \mathbf{0}_{1\times(N-1)} & t_1 + m_{NN_A} \end{bmatrix} U_A^T \sqrt{C_B}\left(I_{N\times N} + P_n(t_1)\sqrt{C_B}\right)^{-1} P_n(t_1) U_N \\
&= [0\ 0\ 0\ \cdots\ 1] \begin{bmatrix} \mathrm{diag}_{N-1}\left(\frac{1}{\lambda_{Ai}}\right) & \mathbf{0}_{(N-1)\times 1} \\ \mathbf{0}_{1\times(N-1)} & t_1 + m_{NN_A} \end{bmatrix} U_A^T U_N - \\
&- [0\ 0\ 0\ \cdots\ 1] \begin{bmatrix} \mathrm{diag}_{N-1}\left(\frac{1}{\lambda_{Ai}}\right) & \mathbf{0}_{(N-1)\times 1} \\ \mathbf{0}_{1\times(N-1)} & t_1 + m_{NN_A} \end{bmatrix} U_A^T \sqrt{C_B}\left(I_{N\times N} + P_n(t_1)\sqrt{C_B}\right)^{-1} P_n(t_1) U_N \\
&= (t_1 + m_{NN_A})[0\ 0\ 0\ \cdots\ 1] U_A^T U_N - \\
&- (t_1 + m_{NN_A})[0\ 0\ 0\ \cdots\ 1] U_A^T \sqrt{C_B}\left(I_{N\times N} + P_n(t_1)\sqrt{C_B}\right)^{-1} P_n(t_1) U_N \\
&= (t_1 + m_{NN_A}) U_N^T U_N - (t_1 + m_{NN_A}) U_N^T \sqrt{C_B}\left(I_{N\times N} + P_n(t_1)\sqrt{C_B}\right)^{-1} P_n(t_1) U_N \\
&= (t_1 + m_{NN_A}) - (t_1 + m_{NN_A}) U_N^T U_B \Lambda_B^{1/2} U_B^T \left(I_{N\times N} + P_n(t_1)\sqrt{C_B}\right)^{-1} P_n(t_1) U_N \\
&= (t_1 + m_{NN_A}) - (t_1 + m_{NN_A})[0\ 0\ 0\ \cdots\ 1]\Lambda_B^{1/2} U_B^T \left(I_{N\times N} + P_n(t_1)\sqrt{C_B}\right)^{-1} P_n(t_1) U_N \\
&= (t_1 + m_{NN_A})
\end{aligned} \tag{114}$$

In the last line we have used the fact that the smallest eigenvalue of C_B is zero, and thus the product $[0\ 0\ 0\ \cdots\ 1]\Lambda_B^{1/2}$ yields a zero $1 \times N$ vector. Substitution in Eq. (112) yields the uncertainty for the robot team at time t_2:

$$P_n(t_2) = U_B \begin{bmatrix} \mathrm{diag}_{N-1}\left(\frac{1}{\lambda_{Bi}}\right) & \mathbf{0}_{(N-1)\times 1} \\ \mathbf{0}_{1\times(N-1)} & t_2 + m_{NN_A} \end{bmatrix} U_B^T \tag{115}$$

From the last expression we conclude that the steady state covariance term due to the initial uncertainty of the robots, M_{NN}, is equal to m_{NN_A} for *both* topologies A and B. At time t_2 the RPMG assumes topology A again, and by following the same steps, it is straightforward to show that the steady state normalized covariance matrix will be, in analogy with Eq. (115),

$$P_n(t) = U_A \begin{bmatrix} \mathrm{diag}_{N-1}\left(\frac{1}{\lambda_{Ai}}\right) & \mathbf{0}_{(N-1)\times 1} \\ \mathbf{0}_{1\times(N-1)} & t + m_{NN_A} \end{bmatrix} U_A^T \tag{116}$$

By comparison of the last expression with that in Eq. (110), we observe that the result is identical to the result that would be derived if the RPMG had undergone *no reconfigurations*. In the following we show that the same property holds for the case in which during Phase 2, the robots localize independently, without performing relative position measurements.

If no relative position information is utilized, then the Riccati equation describing the time evolution of covariance is simply $\dot{P} = Q$, or $\dot{P}_n = I_{N\times N}$. Therefore, if the normalized covariance matrix at time t_1 is given by Eq. (110), then at time t_2 we have

$$\begin{aligned}
P_n(t_2)' &= U_A \left(\begin{bmatrix} \mathrm{diag}_{N-1}\left(\frac{1}{\lambda_{Ai}}\right) & \mathbf{0}_{(N-1)\times 1} \\ \mathbf{0}_{1\times(N-1)} & t_1 + m_{NN_A} \end{bmatrix} \right) U_A^T + (t_2 - t_1) I_{N\times N} \\
&= U_A \left(\begin{bmatrix} \mathrm{diag}_{N-1}\left(\frac{1}{\lambda_{Ai}}\right) + (t_2 - t_1) & \mathbf{0}_{(N-1)\times 1} \\ \mathbf{0}_{1\times(N-1)} & t_2 + m_{NN_A} \end{bmatrix} \right) U_A^T
\end{aligned} \tag{117}$$

At time t_2 the RMPG resumes topology A, and therefore at steady state, the normalized covariance matrix will be

$$P_n(t) = U_A \left(\begin{bmatrix} \mathrm{diag}_{N-1}\left(\frac{1}{\lambda_{Ai}}\right) & \mathbf{0}_{(N-1)\times 1} \\ \mathbf{0}_{1\times(N-1)} & (t - t_2) + m'_{NN_A} \end{bmatrix} \right) U_A^T \tag{118}$$

where

$$
\begin{aligned}
m'_{NN_A} &= U_N^T P_n(t_2)' \left(I_{N\times N} + \sqrt{C_A} P_n(t_2)'\right)^{-1} U_N \\
&= U_N^T P_n(t_2)' \left(I_{N\times N} - \sqrt{C_A}\left(I_{N\times N} + P_n(t_2)'\sqrt{C_A}\right)^{-1} P_n(t_2)'\right) U_N \\
&= U_N^T P_n(t_2)' U_N - U_N^T P_n(t_2)'\sqrt{C_A}\left(I_{N\times N} + P_n(t_2)'\sqrt{C_A}\right)^{-1} P_n(t_2)' U_N
\end{aligned}
$$

By comparison of the last result, where $P_n(t_2)'$ is given in Eq. (117), with the expressions in Eqs. (114) and (112), it becomes evident that following steps analogous to the derivation in Eq. (114) yields

$$
m'_{NN_A} = t_2 + m_{NN_A} \tag{119}
$$

and therefore, Eq. (118) becomes

$$
P_n(t) = U_A \left(\begin{bmatrix} \mathrm{diag}_{N-1}\left(\frac{1}{\lambda_{Ai}}\right) & \mathbf{0}_{(N-1)\times 1} \\ \mathbf{0}_{1\times(N-1)} & t + m_{NN_A} \end{bmatrix} \right) U_A^T
$$

which is identical to the expression in Eq. (116).

■

Clearly, the above results can be extended to the case of more than one intermediate phases of the RPMG topology. The preceding analysis shows that when the relative position information available to the robots is temporarily reduced, or even when relative measurements are not performed by any robot for a finite time interval, then after the initial RPMG topology is restored, the accuracy of the position estimates for the robots will have sustained *no degradation*. Additionally, we observe that the steady state covariance term attributed to the initial uncertainty depends *only* on the first topology of the RPMG, regardless of the subsequent topologies. This result implies that it is beneficial to employ a dense RPMG topology during the initial stage of the deployment of a robot team. In this way the resulting m_{NN} term will be small, and this will benefit localization of the robots for any topology the RPMG assumes in later stages.

3.2.4 Simulation Results

In this section we present simulation results that validate the preceding theoretical analysis. Initially a heterogeneous team of 5 robots moving in 1D is considered. The covariance of the measurements provided by the proprioceptive and exteroceptive sensors of the robots, q_i and r_i respectively, as well as the uncertainty about the initial positions of the robots were assigned different values for each robot in the group. Fig. 1 presents the evolution of the covariance for each of the five robots. It becomes clear that the rate of increase is the same for *all* robots in the team. After the initial transient phase, the uncertainty of each robot grows linearly, with the constant offset for the uncertainty being larger for robots that receive measurements of poor accuracy.

Fig. 2 shows the effect of different RPMG topologies on the steady state positional uncertainty of the robots. To preserve figure clarity, a homogeneous robot group is considered in this case. The group consists of five robots, and the plot shows the evolution of the uncertainty for four different RPMG topologies. In each of the RPMGs considered, each robot of the group measures the relative position of a number of robots, and this number is the degree of connectivity (d) for each node of the RPMG. In the plots shown in Fig. 2 this degree ranges from 1 to 4. We may observe that the rate of increase of uncertainty is *identical* for all RPMG topologies considered, even though these vary significantly, from a ring graph (when the $d = 1$) to a fully connected graph (when $d = 4$). The only effect of RPMG topology is, as evident from the figure, on the constant term of the steady state uncertainty.

In Fig. 3 the effect of the number of robots on the rate of uncertainty growth is presented. Robot groups of 1 (i.e. a single robot performing Dead Reckoning) to 6 robots are considered, and the RPMG topology is fully connected, in all cases. It is clear that an increase in the number of the robots that cooperate results in an improvement of the accuracy of localization for all the robots. However, it should be noted that this improvement follows a law of diminishing return, i.e. the gains from adding one robot to the group are less significant for large robot groups.

The last set of figures demonstrates the effects of RPMG reconfigurations. In Figs. 4(a) and 4(b) a heterogeneous team of 5 robots is considered. Initially the robots perform cooperative localization with a fully connected RPMG

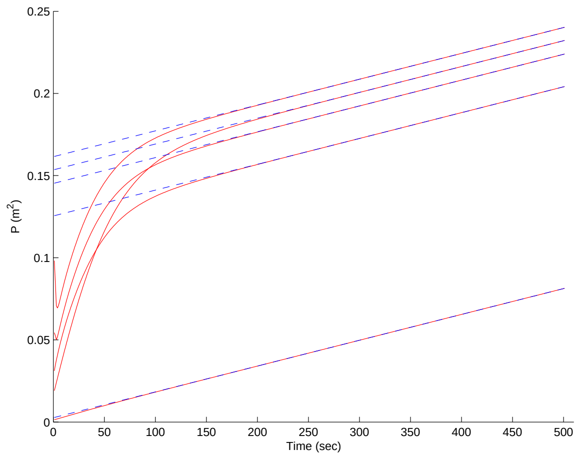


Figure 1: True vs. theoretical covariance for a heterogeneous team of 5 robots. Solid lines correspond to the true, and dashed lines to the theoretical uncertainty.

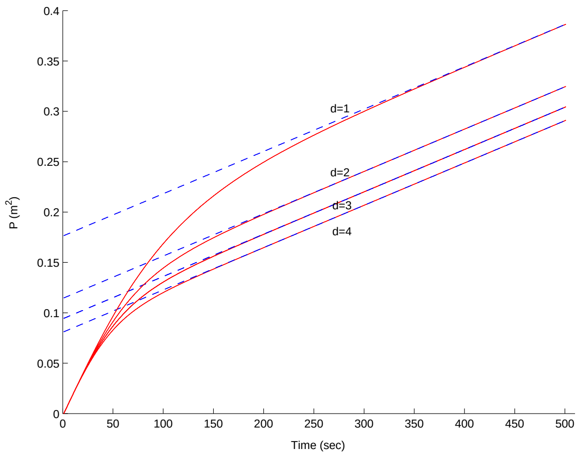


Figure 2: True vs. theoretical covariance for a homogeneous robot group, and four different RPMG topologies. d denotes the the degree of connectivity of each node of the graph.

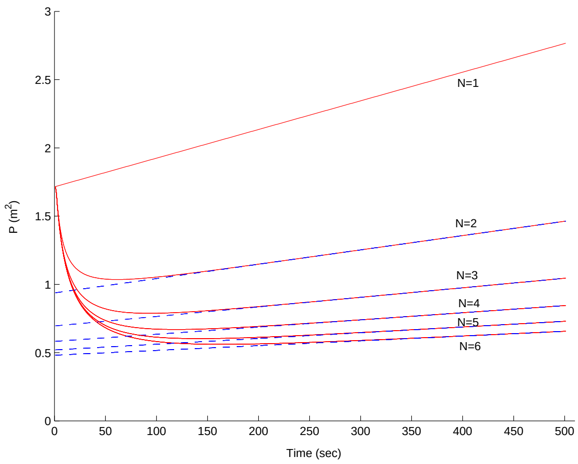
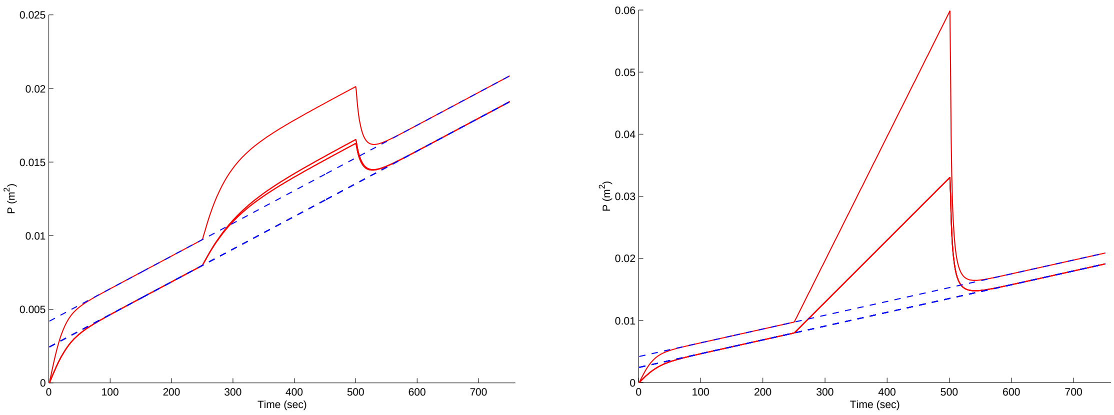


Figure 3: True vs. theoretical covariance for a homogeneous robot groups, of different sizes. The RPMG has been considered fully connected for these simulations.



(a) Ring intermediate topology

(b) No relative position measurements recorded between 250sec and 500sec

Figure 4: The effects of RPMG reconfigurations for a heterogeneous team of 5 robots.

topology, at $t_1 = 250$sec the topology changes to a different one, and at $t_2 = 500$sec, the graph resumes its initial, dense topology. The solid lines show the simulation results, while the dashed ones represent the theoretically computed steady state covariance, evaluated for the first graph topology. The intermediate topology for Fig. 4(a) is a ring graph, while in Fig. 4(b) in the time interval (t_1, t_2) no relative position measurements are recorded. For both of these cases, we observe that the steady state covariance of the position estimates in the last phase is identical to the covariance that would result, had no RPMG reconfigurations taken place. Thus the theoretical results of the previous section are validated.

3.3 Cooperative Localization in 2D

We now turn our attention to the more practical case of mobile robots moving in two dimensions. The difference compared to the one-dimensional case, presented in Section 3.2, is that the coefficients of the Riccati equation are now time varying, and a closed form solution for the covariance cannot be obtained. We thus provide upper bounds for the steady state uncertainty, in a manner analogous to the discrete-time case.

3.3.1 Position propagation

We first study the influx of uncertainty to the system, due to the motion of the robots. The continuous time kinematic equations for the ith robot of the team are

$$\dot{x}_i(t) = V_i(t)\cos(\phi_i(t)) \tag{120}$$

$$\dot{y}_i(t) = V_i(t)\sin(\phi_i(t)) \tag{121}$$

where $V_i(t)$ and $\omega_i(t)$ are the linear and rotational velocity of the robot at time t. Using measurements from the robot's proprioceptive sensors, and the estimates for the robot's orientation, we can write the following set of equations for propagating the estimate of the robot's position:

$$\begin{aligned} \dot{\hat{x}}_i(t) &= V_{m_i}(t)\cos(\hat{\phi}_i(t)) \\ \dot{\hat{y}}_i(t) &= V_{m_i}(t)\sin(\hat{\phi}_i(t)) \end{aligned}$$

where $V_{m_i}(t) = V_i(t) - w_{V_i}(t)$ are the measurements of the translational velocity of the robot, contaminated by a white zero-mean Gaussian noise process, whose covariance function is $\sigma^2_{V_i}\delta(t-\tau)$. In the previous expressions, $\hat{\phi}_i(t)$ is the estimate of the robot's orientation at time t. The errors in the orientation estimates, $\widetilde{\phi}_i(t) = \phi_i(t) - \hat{\phi}_i(t)$ are modeled by a white zero-mean Gaussian noise process, whose variance, $\sigma^2_{\phi_i} = E\{\widetilde{\phi_i}^2\}$ is bounded. The variance of the noise in the velocity measurements, as well as in the orientation estimates, is determined as discussed in Section 3.1.

By linearizing Eqs. (120), (121), the position error propagation equations for the robot can be written as

$$\begin{aligned} \begin{bmatrix} \dot{\tilde{x}}_i(t) \\ \dot{\tilde{y}}_i(t) \end{bmatrix} &= \begin{bmatrix} \cos(\hat{\phi}_i(t)) & -V_{m_i}(t)\sin(\hat{\phi}_i(t)) \\ \sin(\hat{\phi}_i(t)) & V_{m_i}(t)\cos(\hat{\phi}_i(t)) \end{bmatrix} \begin{bmatrix} w_{V_i}(t) \\ \widetilde{\phi}_i(t) \end{bmatrix} \\ \Leftrightarrow \dot{\widetilde{X}}_i(t) &= F_i(t)\widetilde{X}_i(t) + G_{c_i}(t)W_i(t) \end{aligned} \tag{122}$$

where $F_i(t) = \mathbf{0}_{2\times 2}$, and

$$Q_{c_i}(t,\tau) = E\{W_i(t)W_i^T(\tau)\} = \begin{bmatrix} \sigma^2_{V_i} & 0 \\ 0 & \sigma^2_{\phi_i} \end{bmatrix} \delta(t-\tau) \tag{123}$$

is the covariance function of all sources of uncertainty, i.e., the errors in velocity measurements and the errors in the orientation estimates. The matrix $G_{c_i}Q_{c_i}G^T_{c_i}$ that describes the influx of uncertainty to the system due to noise in the robot's odometry and orientation estimates, is given by

$$G_{c_i}Q_{c_i}G^T_{c_i} = \begin{bmatrix} \sigma^2_V\cos^2(\hat{\phi}(t)) + \sigma^2_\phi V^2_m(t)\sin^2(\hat{\phi}(t)) & \left(\sigma^2_V - \sigma^2_\phi V^2_m(t)\right)\sin(\hat{\phi}(t))\cos(\hat{\phi}(t)) \\ \left(\sigma^2_V - \sigma^2_\phi V^2_m(t)\right)\sin(\hat{\phi}(t))\cos(\hat{\phi}(t)) & \sigma^2_V\sin^2(\hat{\phi}(t)) + \sigma^2_\phi V^2_m(t)\cos^2(\hat{\phi}(t)) \end{bmatrix}$$

It becomes clear that this is a time varying matrix, depending on the robot's velocity and orientation. Using this matrix in the Riccati equation that describes the time evolution of the position uncertainty of robot i would preclude the possibility of deriving a closed for solution for the covariance in the general case.

In the derivation of the upper bound for the uncertainty of the robots' position estimates it is useful to employ the average value of the matrix in Eq. (124) (cf. Section 3.3.3). This value is computed by averaging over all values of the orientation of the robot, and is easily shown to be

$$\bar{Q}_i = E\{G_{c_i}Q_{c_i}G^T_{c_i}\} = \frac{\sigma^2_{V_i} + \sigma^2_{\phi_i}V^2}{2} I_{2\times 2} = q_i I_{2\times 2} \tag{124}$$

When no relative positioning information is available the covariance of each robot is propagated using only odometric information, and the covariance of the ith robot is described by the Riccati equation $\dot{P}_i = G_{c_i}Q_{c_i}G^T_{c_i}$. It is easy to verify that the trace of $G_{c_i}Q_{c_i}G^T_{c_i}$ is equal to $2q_i$, thus $\operatorname{trace}\dot{P}_i = 2q_i$. Under the realistic assumption that on average the covariance of the position estimates along the two coordinate axes is equal, $\bar{P}_{xx_i}(t) = \bar{P}_{yy_i}(t)$, we can write

$$\dot{\bar{P}}_{xx_i}(t) = \dot{\bar{P}}_{yy_i}(t) = q_i \tag{125}$$

i.e., uncertainty grows linearly with time at a rate of q_i (cf. Fig (7)). This rate depends on the accuracy of both the odometry and the orientation estimates of the robot, as well as on its velocity. Eq. (125) shows that if the robots of the team have different sensor noise characteristics, and they all localize independently, the rate of uncertainty increase for each of them will differ. This result should be contrasted with the case in which the robots perform relative position measurements, presented in Section 3.3.5.

3.3.2 Exteroceptive Measurement Model

The description of the exteroceptive measurement model is identical with the one presented in Section 2.2 for the discrete-time case. The only difference is that the time-step arguments $(k+1)$ are now substituted by time arguments (t), and the covariance matrices are evaluated using the variance computed for the equivalent system model, as explained in Section 3.1. Additionally, the upper bound, $\mathbf{R}_u$, for the matrix $\mathbf{R}_o(t)$ is identical to the one derived for the discrete time, in Eq. (35). To avoid redundant derivations, we do not present the analysis here, and simply state the final result. The quantity of interest for the continuous time analysis is the matrix expressing the total information available to the estimator at each step, given by:

$$\mathbf{H}^T(t)\mathbf{R}^{-1}(t)\mathbf{H}(t) = \sum_{i=1}^{N} \boldsymbol{H}_i^T(t)\boldsymbol{R}_i^{-1}(t)\boldsymbol{H}_i(t)$$

where $\boldsymbol{H}_i^T(t)\boldsymbol{R}_i^{-1}(t)\boldsymbol{H}_i(t)$ is the information provided by the measurements performed by robot i. We have $\boldsymbol{H}_i(t) = \boldsymbol{\Xi}_{\hat{\phi}_i}^T(t)\boldsymbol{H}_{o_i}$, and $\boldsymbol{R}_i(t) = \boldsymbol{\Xi}_{\hat{\phi}_i}(t)\boldsymbol{R}_{o_i}(t)\boldsymbol{\Xi}_{\hat{\phi}_i}(t)$ where all the quantities appearing in these expressions have been defined in Section 2.2. We can thus write

$$\begin{aligned}\mathbf{H}^T(t)\mathbf{R}^{-1}(t)\mathbf{H}(t) &= \sum_{i=1}^{N}\boldsymbol{H}_i^T(t)\boldsymbol{R}_i^{-1}(t)\boldsymbol{H}_i(t)\\ &= \sum_{i=1}^{N}\boldsymbol{H}_{o_i}^T\boldsymbol{\Xi}_{\hat{\phi}_i}(t)\boldsymbol{\Xi}_{\hat{\phi}_i}^T(t)\boldsymbol{R}_{o_i}^{-1}(t)\boldsymbol{\Xi}_{\hat{\phi}_i}(t)\boldsymbol{\Xi}_{\hat{\phi}_i}^T(t)\boldsymbol{H}_{o_i}\\ &= \sum_{i=1}^{N}\boldsymbol{H}_{o_i}^T\boldsymbol{R}_{o_i}^{-1}(t)\boldsymbol{H}_{o_i} \qquad (126)\end{aligned}$$

3.3.3 Bounds for the Uncertainty at Steady State

In this section we formulate the Riccati equation for the error state covariance of the robot team and outline the steps that yield an upper bound for its solution. The error state vector for the entire robot team is defined as the stacked vector containing the position error vectors of all N robots (cf. Eq. (122)), i.e. a vector of dimension $2N$. Since the proprioceptive measurements of the N robots are uncorrelated, the matrix G^TQG for the Riccati equation is

$$G^TQG = \mathbf{Diag}(G_{c_i}^TQ_{c_i}G_{c_i}) \qquad (127)$$

where $\mathbf{Diag}(G_{c_i}^TQ_{c_i}G_{c_i})$ is a block diagonal matrix, whose nonzero submatrices are the system noise covariances of each of the robots of the team. Thus, noting that the state transition matrix in continuous time is $F = \mathbf{0}_{2N\times 2N}$, and substituting from the previous expression and Eq. (126) into Eq. (87), yields the following Riccati equation for the covariance of the position estimates:

$$\dot{\mathbf{P}} = \mathbf{Diag}(G_{c_i}^T(t)Q_{c_i}(t)G_{c_i}(t)) - \mathbf{P}\,\sum_i\boldsymbol{H}_{o_i}^T\boldsymbol{R}_{o_i}^{-1}(t)\boldsymbol{H}_{o_i}\,\mathbf{P} \qquad (128)$$

It becomes clear that this is a matrix differential equation with time-varying coefficients, and thus a general closed form solution to it cannot be derived. However, by employing the following lemma it is possible to derive an analytical expression for an upper bound on the covariance.

Lemma 3.3 *The maximum expected uncertainty for the position of a group of mobile robots performing Cooperative Localization is bounded by the solution of the following constant coefficient Riccati equation*

$$\dot{\bar{\mathbf{P}}} = \mathbf{Q} - \bar{\mathbf{P}}\,\mathbf{H}_o^T\mathbf{R}_u^{-1}\mathbf{H}_o\,\bar{\mathbf{P}} \qquad (129)$$

where $\mathbf{Q} = E\{\mathbf{Diag}(G_{c_i}^TQ_{c_i}G_{c_i})\} = \mathbf{Diag}(q_iI_{2\times 2})$ *is the average rate of noise influx due to the odometric measurements, and* $\mathbf{H}_o^T\mathbf{R}_u^{-1}\mathbf{H}_o = \sum_i\boldsymbol{H}_{o_i}^T\boldsymbol{R}_{u_i}^{-1}\boldsymbol{H}_{o_i}$.

Proof The proof of this lemma is straightforward, and follows from the fact that $\boldsymbol{R}_{u_i} \succeq \boldsymbol{R}_{o_i}$. Specifically, we have that

$$\begin{aligned}\boldsymbol{R}_{o_i} &\preceq \boldsymbol{R}_{u_i},\ i = 1..N \Rightarrow\\ \sum_i\boldsymbol{H}_{o_i}^T\boldsymbol{R}_{o_i}^{-1}\boldsymbol{H}_{o_i} &\succeq \sum_i\boldsymbol{H}_{o_i}^T\boldsymbol{R}_{u_i}^{-1}\boldsymbol{H}_{o_i} \Rightarrow\\ \mathbf{Diag}(G_{c_i}^TQ_{c_i}G_{c_i}) - \mathbf{P}\,\mathbf{H}_o^T\mathbf{R}_o{}^{-1}\mathbf{H}_o\,\mathbf{P} &\preceq \mathbf{Diag}(G_{c_i}^TQ_{c_i}G_{c_i}) - \mathbf{P}\,\mathbf{H}_o^T\mathbf{R}_u^{-1}\mathbf{H}_o\mathbf{P} \Rightarrow\\ E\{\dot{\mathbf{P}}\} &\preceq E\{\mathbf{Diag}(G_{c_i}^TQ_{c_i}G_{c_i})\} - E\{\mathbf{P}\mathbf{H}_o^T\mathbf{R}_u^{-1}\mathbf{H}_o\mathbf{P}\}\\ &\preceq E\{\mathbf{Diag}(G_{c_i}^TQ_{c_i}G_{c_i})\} - E\{\mathbf{P}\}\mathbf{H}_o^T\mathbf{R}_u^{-1}\mathbf{H}_oE\{\mathbf{P}\}\\ \Rightarrow E\{\dot{\mathbf{P}}\} &\preceq \mathbf{Q} - \bar{\mathbf{P}}\mathbf{H}_o^T\mathbf{R}_u^{-1}\mathbf{H}_o\bar{\mathbf{P}} = \dot{\bar{\mathbf{P}}}\end{aligned}$$

where we have employed Jensen's inequality and the fact that the function $f(\mathbf{P}) = \mathbf{P}\mathbf{H}_o^T\mathbf{R}_u^{-1}\mathbf{H}_o\mathbf{P}$ is matrix convex in $\mathbf{P}$ [14]. By setting the right hand side argument of the expression in the last line equal to $\dot{\bar{\mathbf{P}}}$, a Riccati equation in $\bar{\mathbf{P}}$ (Eq. (129)) is formed. Since $E\{\dot{\mathbf{P}}\} \preceq \dot{\bar{\mathbf{P}}}$, by selecting the initial conditions for $\bar{\mathbf{P}}$ equal to those for $\mathbf{P}$, it is clear that the solution to (Eq. (129)) is an upper bound for the expected positioning uncertainty of the robots.

■

We point out that in Eq. (129) we have replaced the matrix representing the information of the exteroceptive measurements with its lower bound, $\mathbf{H}_o^T\mathbf{R}_u^{-1}\mathbf{H}_o$. This lower bound corresponds to the case of a time invariant system, in which the exteroceptive measurements provide less (or equal) information than the information provided in the real system of the robot team. Additionally, the covariance of the system noise in the time invariant system is equal to the average covariance of the system noise in the real system. Therefore, the fact that the covariance of the estimates in the time invariant system is an upper bound on the covariance of the position estimates for the robots makes sense intuitively.

In order to solve the constant coefficient Riccati equation in Eq. (129), we first define the normalized covariance matrix:

$$\mathbf{P}_n(t) = \mathbf{Q}^{-1/2}\bar{\mathbf{P}}(t)\mathbf{Q}^{-1/2}$$

Substitution in Eq. (129) yields the Riccati equation for the normalized covariance:

$$\dot{\mathbf{P}}_n(t) = I_{2N\times 2N} - \mathbf{P}_n(t)\mathbf{Q}^{1/2}\mathbf{H}_o^T\mathbf{R}_u^{-1}\mathbf{H}_o\mathbf{Q}^{1/2}\mathbf{P}_n(t) \tag{130}$$

We define $\mathbf{C} = \mathbf{Q}^{1/2}\mathbf{H}_o^T\mathbf{R}_u^{-1}\mathbf{H}_o\mathbf{Q}^{1/2}$, and by substituting we have:

$$\dot{\mathbf{P}}_n(t) = I_{2N\times 2N} - \mathbf{P}_n(t)\mathbf{C}\mathbf{P}_n(t) \tag{131}$$

To solve this Riccati equation, we substitute $\mathbf{P}_n(t) = \mathbf{A}(t)\mathbf{B}^{-1}(t)$, and we form the Hamiltonian matrix of the system, $\mathcal{H}_2$. The derivation is analogous to the one-dimensional case which we have already presented, and yields the following solution for $\mathbf{A}(t)$, $\mathbf{B}(t)$:

$$\begin{bmatrix} \mathbf{B}(t) \\ \mathbf{A}(t) \end{bmatrix} = e^{\mathcal{H}_2 t}\begin{bmatrix} \mathbf{B}(0) \\ \mathbf{A}(0) \end{bmatrix} \tag{132}$$

where the Hamiltonian is given by

$$\mathcal{H}_2 = \begin{bmatrix} \mathbf{0}_{2N\times 2N} & \mathbf{C} \\ I_{2N\times 2N} & \mathbf{0}_{2N\times 2N} \end{bmatrix} \tag{133}$$

Note the similarity of this system of equations with the system described by Eq.s (93) and (94). If we denote the Singular Value Decomposition of $\mathbf{C}$ by $\mathbf{C} = \mathbf{U}\mathbf{\Lambda}\mathbf{U}^T$ it is evident that the solution for $\mathbf{A}(t)$ $\mathbf{B}(t)$ is given by

$$\begin{bmatrix} \mathbf{B}(t) \\ \mathbf{A}(t) \end{bmatrix} = e^{\mathcal{H}_2 t}\begin{bmatrix} I_{2N\times 2N} \\ \mathbf{P}_n(0) \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\mathbf{U}\,\mathrm{diag}\left(e^{\boldsymbol{\lambda}_i t} + e^{-\boldsymbol{\lambda}_i t}\right)\mathbf{U}^T + \frac{1}{2}\mathbf{U}\,\mathrm{diag}\left(\boldsymbol{\lambda}_i(e^{\boldsymbol{\lambda}_i t} - e^{-\boldsymbol{\lambda}_i t})\right)\mathbf{U}^T\mathbf{P}_n(0) \\ \frac{1}{2}\mathbf{U}\,\mathrm{diag}\left(\frac{e^{\boldsymbol{\lambda}_i t} - e^{-\boldsymbol{\lambda}_i t}}{\boldsymbol{\lambda}_i}\right)\mathbf{U}^T + \frac{1}{2}\mathbf{U}\,\mathrm{diag}\left(e^{\boldsymbol{\lambda}_i t} + e^{-\boldsymbol{\lambda}_i t}\right)\mathbf{U}^T\mathbf{P}_n(0) \end{bmatrix}$$

where the i-th eigenvalue of the symmetric matrix $\mathbf{C}$ has been denoted as $\boldsymbol{\lambda}_i^2$. In order to determine the covariance at steady state, we follow a course analogous to that of the 1D case. From the above formula we derive the following expression for the normalized covariance:

$$\begin{aligned} \mathbf{P}_n(t) &= \left(\frac{1}{2}\mathbf{U}\,\mathrm{diag}\left(\frac{e^{\boldsymbol{\lambda}_i t} - e^{-\boldsymbol{\lambda}_i t}}{\boldsymbol{\lambda}_i}\right)\mathbf{U}^T + \frac{1}{2}\mathbf{U}\,\mathrm{diag}\left(e^{\boldsymbol{\lambda}_i t} + e^{-\boldsymbol{\lambda}_i t}\right)\mathbf{U}^T\mathbf{P}_n(0)\right) \times \\ &\times \left(\frac{1}{2}\mathbf{U}\,\mathrm{diag}\left(e^{\boldsymbol{\lambda}_i t} + e^{-\boldsymbol{\lambda}_i t}\right)\mathbf{U}^T + \frac{1}{2}\mathbf{U}\,\mathrm{diag}\left(\boldsymbol{\lambda}_i(e^{\boldsymbol{\lambda}_i t} - e^{-\boldsymbol{\lambda}_i t})\right)\mathbf{U}^T\mathbf{P}_n(0)\right)^{-1} \\ &= \mathbf{U}\left(\mathrm{diag}\left(\frac{e^{\boldsymbol{\lambda}_i t} - e^{-\boldsymbol{\lambda}_i t}}{\boldsymbol{\lambda}_i}\right) + \mathrm{diag}\left(e^{\boldsymbol{\lambda}_i t} + e^{-\boldsymbol{\lambda}_i t}\right)\mathbf{U}^T\mathbf{P}_n(0)\mathbf{U}\right) \times \\ &\times \left(\mathrm{diag}\left(e^{\boldsymbol{\lambda}_i t} + e^{-\boldsymbol{\lambda}_i t}\right) + \mathrm{diag}\left(\boldsymbol{\lambda}_i(e^{\boldsymbol{\lambda}_i t} - e^{-\boldsymbol{\lambda}_i t})\right)\mathbf{U}^T\mathbf{P}_n(0)\mathbf{U}\right)^{-1}\mathbf{U}^T \\ &= \mathbf{U}(\mathbf{K} + \mathbf{L}P_0)(\mathbf{L} + \mathbf{\Lambda}\mathbf{K}P_0)^{-1}\mathbf{U}^T \end{aligned} \tag{134}$$

Where we have denoted

$$\mathbf{K} = \mathrm{diag}\left(\frac{e^{\boldsymbol{\lambda}_i t} - e^{-\boldsymbol{\lambda}_i t}}{\boldsymbol{\lambda}_i}\right)$$

$$\mathbf{L} = \mathrm{diag}\left(e^{\boldsymbol{\lambda}_i t} + e^{-\boldsymbol{\lambda}_i t}\right)$$

and

$$P_0 = \mathbf{U}^T \mathbf{P}_n(0)\mathbf{U}$$

Note that in the definition of $\mathbf{K}$, a problem arises when none of the robots receives position measurements. In this case, as shown in Appendix E.2, there exist two eigenvalues of $\mathbf{C}$ equal to zero. However, analogously to the 1D case, the quantity under consideration can be shown to exist, and equals

$$\mathbf{K} = \begin{bmatrix} \mathrm{diag}_{2N-2}\left(\frac{e^{\boldsymbol{\lambda}_i t} - e^{-\boldsymbol{\lambda}_i t}}{\boldsymbol{\lambda}_i}\right) & \mathbf{0}_{(2N-2)\times 2)} \\ \mathbf{0}_{2\times(2N-2)} & 2t\, I_{2\times 2} \end{bmatrix} \tag{135}$$

In order to preserve the clarity of the formulas, we will use the initial, less strict notation in the following derivations, bearing in mind that its true meaning is given by the last expression.

In the following, we determine bounds for the steady state uncertainty of the robots. This is accomplished by examining the behavior of the solution for the covariance after sufficient time. Notice the similarity of Eqs. (134) and (97). Applying analogous derivations, it is straightforward to show that $\mathbf{P}_n(t)$ can be written as:

$$\mathbf{P}_n(t) = \mathbf{U}\left(\mathbf{K}\mathbf{L}^{-1} + 4\mathbf{L}^{-1}P_0\left(I_{2N\times 2N} + \boldsymbol{\Lambda}\mathbf{K}\mathbf{L}^{-1}P_0\right)^{-1}\mathbf{L}^{-1}\right)\mathbf{U}^T \tag{136}$$

The behavior of the steady state covariance of the robots' position estimates depends on the availability of absolute positioning information. When absolute position measurements are available, all the eigenvalues of matrix $\mathbf{C}$ are positive, and the system of robots is observable from a Control Theoretic point of view. On the other hand, when the robots perform only relative position measurements, $\mathbf{C}$ has two eigenvalues equal to zero, and the system is unobservable (the proofs for the rank of $\mathbf{C}$ can be found in Appendix E.2). The two cases are examined separately in the following.

3.3.4 Observable System

If at least one of the robots receives absolute position measurements, all the eigenvalues of $\mathbf{C}$ are positive, and thus:

$$\begin{aligned} \lim_{t\to\infty} \mathbf{K}\mathbf{L}^{-1} &= \lim_{t\to\infty} \mathrm{diag}\left(\frac{e^{\boldsymbol{\lambda}_i t} - e^{-\boldsymbol{\lambda}_i t}}{\boldsymbol{\lambda}_i}\right)\mathrm{diag}\left(e^{\boldsymbol{\lambda}_i t} + e^{-\boldsymbol{\lambda}_i t}\right)^{-1} \\ &= \mathrm{diag}\left(\frac{1}{\boldsymbol{\lambda}_i}\right) \end{aligned}$$

Also

$$\begin{aligned} \lim_{t\to\infty} 4\mathbf{L}^{-1}P_0\left(I_{2N\times 2N} + \boldsymbol{\Lambda}\mathbf{K}\mathbf{L}^{-1}P_0\right)^{-1}\mathbf{L}^{-1} &= \mathbf{0}_{N\times N}P_0\left(I_{2N\times 2N} + \boldsymbol{\Lambda}^{1/2}P_0\right)^{-1}\mathbf{0}_{N\times N} \\ &= \mathbf{0}_{N\times N} \end{aligned}$$

Using the above two results, it is

$$\begin{aligned} \lim_{t\to\infty} \mathbf{P}_n(t) &= \lim_{t\to\infty} \mathbf{U}\left(\mathbf{K}\mathbf{L}^{-1} + P_0\left(I_{2N\times 2N} + \boldsymbol{\Lambda}\mathbf{K}\mathbf{L}^{-1}P_0\right)^{-1}\mathbf{L}^{-1}\right)\mathbf{U}^T \\ &= \mathbf{U}\,\mathrm{diag}\left(\frac{1}{\boldsymbol{\lambda}_i}\right)\mathbf{U}^T \end{aligned}$$

and therefore the upper bound for the position estimates' covariance at steady state is

$$\begin{aligned} \bar{\mathbf{P}}_{ss}(t) &= \mathbf{Q}^{1/2}\mathbf{U}\,\mathrm{diag}\left(\frac{1}{\boldsymbol{\lambda}_i}\right)\mathbf{U}^T\mathbf{Q}^{1/2} \\ &= \mathbf{Q}^{1/2}\sqrt{\mathbf{C}^{-1}}\mathbf{Q}^{1/2} \end{aligned} \tag{137}$$

Notice that the steady state uncertainty when at least one robot receives absolute position measurements is independent of the initial uncertainty. This result should be compared with the result for the case of an non-observable system, which we derive in the following.

3.3.5 Non-Observable System

When none of the robots has access to absolute position measurements, the two smallest eigenvalues of $\mathbf{C}$ are equal to zero, i.e., $\boldsymbol{\lambda}_{2N-1} = \boldsymbol{\lambda}_{2N} = 0$. In this case,

$$
\begin{aligned}
\mathbf{KL}^{-1} &= \operatorname{diag}\left(\frac{e^{\boldsymbol{\lambda}_i t} - e^{-\boldsymbol{\lambda}_i t}}{\boldsymbol{\lambda}_i}\right) \operatorname{diag}\left(e^{\boldsymbol{\lambda}_i t} + e^{-\boldsymbol{\lambda}_i t}\right)^{-1} \\
&= \begin{bmatrix} \operatorname{diag}_{2N-2}\left(\frac{e^{\boldsymbol{\lambda}_i t} - e^{-\boldsymbol{\lambda}_i t}}{\boldsymbol{\lambda}_i (e^{\boldsymbol{\lambda}_i t} + e^{-\boldsymbol{\lambda}_i t})}\right) & \mathbf{0}_{(2N-2)\times 2} \\ \mathbf{0}_{2\times(2N-2)} & \mathbf{0}_{2\times 2} \end{bmatrix} + \begin{bmatrix} \mathbf{0}_{(2N-2)\times(2N-2)} & \mathbf{0}_{(2N-2)\times 2} \\ \mathbf{0}_{2\times(2N-2)} & tI_{2\times 2} \end{bmatrix}
\end{aligned}
$$

But we note that

$$
\lim_{t\to\infty} \begin{bmatrix} \operatorname{diag}_{2N-2}\left(\frac{e^{\boldsymbol{\lambda}_i t} - e^{-\boldsymbol{\lambda}_i t}}{\boldsymbol{\lambda}_i (e^{\boldsymbol{\lambda}_i t} + e^{-\boldsymbol{\lambda}_i t})}\right) & \mathbf{0}_{(2N-2)\times 2} \\ \mathbf{0}_{2\times(2N-2)} & 0 \end{bmatrix} = \begin{bmatrix} \operatorname{diag}_{2N-2}\left(\frac{1}{\boldsymbol{\lambda}_i}\right) & \mathbf{0}_{(2N-2)\times 2} \\ \mathbf{0}_{2\times(2N-2)} & \mathbf{0}_{2\times 2} \end{bmatrix}
$$

and thus at steady state the term $\mathbf{KL}^{-1}$ becomes

$$
(\mathbf{KL}^{-1})_{ss} = \begin{bmatrix} \operatorname{diag}_{2N-2}\left(\frac{1}{\boldsymbol{\lambda}_i}\right) & \mathbf{0}_{(2N-2)\times 2} \\ \mathbf{0}_{2\times(2N-2)} & tI_{2\times 2} \end{bmatrix} \tag{138}
$$

Also,

$$
\begin{aligned}
&\lim_{t\to\infty} 4\mathbf{L}^{-1} P_0 \left(I_{2N\times 2N} + \boldsymbol{\Lambda}\mathbf{KL}^{-1} P_0\right)^{-1} \mathbf{L}^{-1} = \\
&\lim_{t\to\infty} 4 \begin{bmatrix} \operatorname{diag}_{2N-2}\left(\frac{1}{e^{\boldsymbol{\lambda}_i t} + e^{-\boldsymbol{\lambda}_i t}}\right) & \mathbf{0}_{(2N-2)\times 2} \\ \mathbf{0}_{2\times(2N-2)} & 2I_{2\times 2} \end{bmatrix} P_0 \left(I_{2N\times 2N} + \boldsymbol{\Lambda}\mathbf{KL}^{-1} P_0\right)^{-1} \begin{bmatrix} \operatorname{diag}_{2N-2}\left(\frac{1}{e^{\boldsymbol{\lambda}_i t} + e^{-\boldsymbol{\lambda}_i t}}\right) & \mathbf{0}_{(2N-2)\times 2} \\ \mathbf{0}_{2\times(2N-2)} & 2I_{2\times 2} \end{bmatrix} \\
&\quad = \begin{bmatrix} \mathbf{0}_{(2N-2)\times(2N-2)} & \mathbf{0}_{(2N-2)\times 2} \\ \mathbf{0}_{2\times(2N-2)} & I_{2\times 2} \end{bmatrix} P_0 \left(I_{2N\times 2N} + \boldsymbol{\Lambda}^{1/2} P_0\right)^{-1} \begin{bmatrix} \mathbf{0}_{(2N-2)\times(2N-2)} & \mathbf{0}_{(2N-2)\times 2} \\ \mathbf{0}_{2\times(2N-2)} & I_{2\times 2} \end{bmatrix}
\end{aligned}
$$

We denote

$$
P_0 \left(I_{2N\times 2N} + \boldsymbol{\Lambda}^{1/2} P_0\right)^{-1} = \begin{bmatrix} m_{11} & m_{12} & \dots & m_{1(2N)} \\ \vdots & \vdots & \vdots & \vdots \\ m_{(2N-1)1} & m_{(2N-1)2} & \dots & m_{(2N-1)2N} \\ m_{(2N)1} & m_{(2N)2} & \dots & m_{(2N)(2N)} \end{bmatrix}
$$

and thus

$$
\begin{aligned}
&\lim_{t\to\infty} 4\mathbf{L}^{-1} P_0 \left(I_{2N\times 2N} + \boldsymbol{\Lambda}\mathbf{KL}^{-1} P_0\right)^{-1} \mathbf{L}^{-1} = \\
&= \begin{bmatrix} \mathbf{0}_{(2N-2)\times(2N-2)} & \mathbf{0}_{(2N-2)\times 2} \\ \mathbf{0}_{2\times(2N-2)} & I_{2\times 2} \end{bmatrix} \begin{bmatrix} m_{11} & m_{12} & \dots & m_{2N} \\ \vdots & \vdots & \vdots & \vdots \\ m_{(2N-1)1} & m_{(2N-1)2} & \dots & m_{(2N-1)2N} \\ m_{(2N)1} & m_{(2N)2} & \dots & m_{(2N)(2N)} \end{bmatrix} \begin{bmatrix} \mathbf{0}_{)\times(2N-2)} & \mathbf{0}_{(2N-2)\times 2} \\ \mathbf{0}_{2\times(2N-2)} & I_{2\times 2} \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ m_{(2N-1)1} & m_{(2N-1)2} & \dots & m_{(2N-1)2N} \\ m_{(2N)1} & m_{(2N)2} & \dots & m_{(2N)(2N)} \end{bmatrix} \begin{bmatrix} \mathbf{0}_{(2N-2)\times(2N-2)} & \mathbf{0}_{(2N-2)\times 2} \\ \mathbf{0}_{2\times(2N-2)} & I_{2\times 2} \end{bmatrix} \\
&= \begin{bmatrix} \mathbf{0}_{(2N-2)\times(2N-2)} & \mathbf{0}_{(2N-2)\times 2} \\ \mathbf{0}_{2\times(2N-2)} & M_N \end{bmatrix}
\end{aligned} \tag{139}
$$

where

$$
M_N = \begin{bmatrix} m_{(2N-1)(2N-1)} & m_{(2N-1)(2N)} \\ m_{(2N)(2N-1)} & m_{(2N)(2N)} \end{bmatrix}
$$

The derivation of a closed form expression for M_N is analogous to the one-dimensional case presented in Section 3.2.2. We recall that $P_0 = \mathbf{U}^T \mathbf{P}_n(0)\mathbf{U}$, thus

$$\begin{aligned}
P_0 \left(I_{2N\times 2N} + \mathbf{\Lambda}^{1/2} P_0\right)^{-1} &= \mathbf{U}^T \mathbf{P}_n(0)\mathbf{U}\left(I_{2N\times 2N} + \mathbf{\Lambda}^{1/2}\mathbf{U}^T\mathbf{P}_n(0)\mathbf{U}\right)^{-1} \\
&= \mathbf{U}^T \mathbf{P}_n(0)\left(\mathbf{U}^T\right)^{-1}\left(I_{2N\times 2N} + \mathbf{\Lambda}^{1/2}\mathbf{U}^T\mathbf{P}_n(0)\mathbf{U}\right)^{-1}(\mathbf{U})^{-1}\mathbf{U} \\
&= \mathbf{U}^T \mathbf{P}_n(0)\left(\mathbf{U}\mathbf{U}^T + \mathbf{U}\mathbf{\Lambda}^{1/2}\mathbf{U}^T\mathbf{P}_n(0)\mathbf{U}\mathbf{U}^T\right)^{-1}\mathbf{U} \\
&= \mathbf{U}^T \mathbf{P}_n(0)\left(I_{2N\times 2N} + \sqrt{\mathbf{C}}\mathbf{P}_n(0)\right)^{-1}\mathbf{U} \\
&= \begin{bmatrix} \Psi^T \\ V^T \end{bmatrix} \mathbf{P}_n(0)\left(I_{2N\times 2N} + \sqrt{\mathbf{C}}\mathbf{P}_n(0)\right)^{-1}\begin{bmatrix} \Psi & V \end{bmatrix} \\
&= \begin{bmatrix} \Psi^T \mathbf{P}_n(0)\left(I_{2N\times 2N} + \sqrt{\mathbf{C}}\mathbf{P}_n(0)\right)^{-1}\Psi & \Psi^T \mathbf{P}_n(0)\left(I_{2N\times 2N} + \sqrt{\mathbf{C}}\mathbf{P}_n(0)\right)^{-1} V \\ V^T \mathbf{P}_n(0)\left(I_{2N\times 2N} + \sqrt{\mathbf{C}}\mathbf{P}_n(0)\right)^{-1}\Psi & V^T \mathbf{P}_n(0)\left(I_{2N\times 2N} + \sqrt{\mathbf{C}}\mathbf{P}_n(0)\right)^{-1} V \end{bmatrix}
\end{aligned}$$

where we have denoted by Ψ the matrix of dimensions $2N \times (2N-2)$, consisting of the $2N-2$ singular vectors of $\mathbf{C}$ corresponding to the nonzero singular values, and by V the $2N \times 2$ matrix consisting of the 2 singular vectors corresponding to the two zero singular values, i.e., $\mathbf{U} = [\Psi \;\; V]$.

M_N can now be written as

$$M_N = V^T \mathbf{P}_n(0)\left(I_{2N\times 2N} + \sqrt{\mathbf{C}}\mathbf{P}_n(0)\right)^{-1} V \tag{140}$$

Using Eq. (136) steady state normalized covariance is thus obtained:

$$\mathbf{P}_n(t) = \mathbf{U}\left(\begin{bmatrix} \mathrm{diag}_{2N-2}\left(\frac{1}{\boldsymbol{\lambda}_i}\right) & \mathbf{0}_{(2N-2)\times 2} \\ \mathbf{0}_{2\times(2N-2)} & tI_{2\times 2} \end{bmatrix} + \begin{bmatrix} \mathbf{0}_{(2N-2)\times(2N-2)} & \mathbf{0}_{(2N-2)\times 2} \\ \mathbf{0}_{2\times(2N-2)} & M_N \end{bmatrix}\right)\mathbf{U}^T$$

Thus the upper bound for the steady state uncertainty of the position estimates is

$$\begin{aligned}
\bar{\mathbf{P}}(t) &= \mathbf{Q}^{1/2}\mathbf{U}\begin{bmatrix} \mathrm{diag}_{2N-2}\left(\frac{1}{\boldsymbol{\lambda}_i}\right) & \mathbf{0}_{(2N-2)\times 2} \\ \mathbf{0}_{2\times(2N-2)} & \mathbf{0}_{2\times 2} \end{bmatrix}\mathbf{U}^T\mathbf{Q}^{1/2} + \\
&+ \mathbf{Q}^{1/2}\mathbf{U}\begin{bmatrix} \mathbf{0}_{(2N-2)\times(2N-2)} & \mathbf{0}_{(2N-2)\times 2} \\ \mathbf{0}_{2\times(2N-2)} & M_N \end{bmatrix}\mathbf{U}^T\mathbf{Q}^{1/2} \\
&+ \mathbf{Q}^{1/2}\mathbf{U}\begin{bmatrix} \mathbf{0}_{(2N-2)\times(2N-2)} & \mathbf{0}_{(2N-2)\times 2} \\ \mathbf{0}_{2\times(2N-2)} & tI_{2\times 2} \end{bmatrix}\mathbf{U}^T\mathbf{Q}^{1/2} \\
&= P_1 + P_2 + P_3(t)
\end{aligned} \tag{141}$$

In the above relation, the term P_1 is a constant term, that is independent of the initial uncertainty of the robots. The term P_2 can be written as

$$\begin{aligned}
P_2(t) &= \mathbf{Q}^{1/2}\mathbf{U}\begin{bmatrix} \mathbf{0}_{(2N-2)\times(2N-2)} & \mathbf{0}_{(2N-2)\times 2} \\ \mathbf{0}_{2\times(2N-2)} & M_N \end{bmatrix}\mathbf{U}^T\mathbf{Q}^{1/2} \\
&= \mathbf{Q}^{1/2}\begin{bmatrix} \Psi & V \end{bmatrix}\begin{bmatrix} \mathbf{0}_{(2N-2)\times(2N-2)} & \mathbf{0}_{(2N-2)\times 2} \\ \mathbf{0}_{2\times(2N-2)} & M_N \end{bmatrix}\begin{bmatrix} \Psi^T \\ V^T \end{bmatrix}\mathbf{Q}^{1/2} \\
&= \mathbf{Q}^{1/2} V V^T \mathbf{P}_n(0)\left(I_{2N\times 2N} + \sqrt{\mathbf{C}}\mathbf{P}_n(0)\right)^{-1} V V^T \mathbf{Q}^{1/2}
\end{aligned} \tag{142}$$

In Appendix E.2 we show that

$$VV^T = \mathbf{U}_{2N-1}\mathbf{U}_{2N-1}^T + \mathbf{U}_{2N}\mathbf{U}_{2N}^T = \mathbf{Q}^{-1/2}\begin{bmatrix} q_T & 0 & q_T & \cdots \\ 0 & q_T & 0 & \cdots \\ q_T & 0 & q_T & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}\mathbf{Q}^{-1/2} \tag{143}$$

where $\mathbf{U}_{2N-1}$ and $\mathbf{U}_{2N}$ are the eigenvectors associated with the zero eigenvalues of $\mathbf{C}$, and therefore

$$P_2(t) = q_T^2 \begin{bmatrix} 1 & 0 & 1 & \cdots \\ 0 & 1 & 0 & \cdots \\ 1 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \mathbf{Q}^{-1/2}\mathbf{P}_n(0)\left(I_{2N\times 2N} + \sqrt{\mathbf{C}}\mathbf{P}_n(0)\right)^{-1}\mathbf{Q}^{-1/2} \begin{bmatrix} 1 & 0 & 1 & \cdots \\ 0 & 1 & 0 & \cdots \\ 1 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \tag{144}$$

where $\frac{1}{q_T} = \sum_{i=1}^{N} \frac{1}{q_i}$.
From the above expression we can see that the value of P_2 is independent of the specific choice of the singular vectors. In order to further simplify the expression for P_2, and to reveal the special structure of this matrix, we choose for simplicity the following singular vectors for the zero singular values:

$$\mathbf{U}_{2N-1} = \sqrt{q_T}\mathbf{Q}^{-1/2} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ \vdots \end{bmatrix}$$

and

$$\mathbf{U}_{2N} = \sqrt{q_T}\mathbf{Q}^{-1/2} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ \vdots \end{bmatrix}$$

Using these we obtain

$$\begin{aligned} M_N &= V^T\mathbf{P}_n(0)\left(I_{2N\times 2N} + \sqrt{\mathbf{C}}\mathbf{P}_n(0)\right)^{-1} V \\ &= q_T \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ \vdots & \vdots \end{bmatrix}^T \mathbf{Q}^{-1/2}\mathbf{P}_n(0)\left(I_{2N\times 2N} + \sqrt{\mathbf{C}}\mathbf{P}_n(0)\right)^{-1}\mathbf{Q}^{-1/2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ \vdots & \vdots \end{bmatrix} \\ &= q_T \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ \vdots & \vdots \end{bmatrix}^T \mathbf{Q}^{-1}\mathbf{P}(0)\mathbf{Q}^{-1/2}\left(I_{2N\times 2N} + \sqrt{\mathbf{C}}\mathbf{Q}^{-1/2}\mathbf{P}(0)\mathbf{Q}^{-1/2}\right)^{-1}\mathbf{Q}^{-1/2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ \vdots & \vdots \end{bmatrix} \end{aligned} \tag{145}$$

We introduce the notation

$$\mathbf{W} = q_T\mathbf{Q}^{-1}\mathbf{P}(0)\mathbf{Q}^{-1/2}\left(I_{2N\times 2N} + \sqrt{\mathbf{C}}\mathbf{Q}^{-1/2}\mathbf{P}(0)\mathbf{Q}^{-1/2}\right)^{-1}\mathbf{Q}^{-1/2}$$

and we set

$$M_N = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \tag{146}$$

Due to the special structure of V, α is the sum of all the elements of $\mathbf{W}$ whose both indices are odd, δ is the sum of all the elements with two even indices, and γ is the sum of all the elements with an odd row index and an even column index. Due to symmetry, $\beta = \gamma$. Using Eq. (141) we obtain

$$
\begin{aligned}
P_2 &= \mathbf{Q}^{1/2}\mathbf{U} \begin{bmatrix} \mathbf{0}_{(2N-2)\times(2N-2)} & \mathbf{0}_{(2N-2)\times 2} \\ \mathbf{0}_{2\times(2N-2)} & M_N \end{bmatrix} \mathbf{U}^T \mathbf{Q}^{1/2} \\
&= \begin{bmatrix} \mathbf{Q}^{1/2}\Psi & \mathbf{Q}^{1/2}V \end{bmatrix} \begin{bmatrix} \mathbf{0}_{(2N-2)\times(2N-2)} & \mathbf{0}_{(2N-2)\times 2} \\ \mathbf{0}_{2\times(2N-2)} & M_N \end{bmatrix} \begin{bmatrix} (\mathbf{Q}^{1/2}\Psi)^T \\ (\mathbf{Q}^{1/2}V)^T \end{bmatrix} \\
&= q_T \begin{bmatrix} & 1 & 0 \\ & 0 & 1 \\ \frac{1}{\sqrt{q_T}}\mathbf{Q}^{1/2}\Psi & 1 & 0 \\ & 0 & 1 \\ & \vdots & \vdots \end{bmatrix} \begin{bmatrix} \mathbf{0}_{(2N-2)\times(2N-2)} & \mathbf{0}_{(2N-2)\times 2} \\ \mathbf{0}_{2\times(2N-2)} & \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \end{bmatrix} \begin{bmatrix} & \frac{1}{\sqrt{q_T}}(\mathbf{Q}^{1/2}\Psi)^T & & & & \\ 1 & 0 & 1 & 0 & 1 & \cdots \\ 0 & 1 & 0 & 1 & 0 & \cdots \end{bmatrix} \\
&= q_T \begin{bmatrix} & \alpha & \beta \\ & \gamma & \delta \\ \mathbf{0}_{2N\times(2N-2)} & \alpha & \beta \\ & \gamma & \delta \\ & \vdots & \vdots \end{bmatrix} \begin{bmatrix} & \frac{1}{\sqrt{q_T}}(\mathbf{Q}^{1/2}\Psi)^T & & & & \\ 1 & 0 & 1 & 0 & 1 & \cdots \\ 0 & 1 & 0 & 1 & 0 & \cdots \end{bmatrix} \\
&= q_T \begin{bmatrix} \alpha & \beta & \alpha & \beta & \cdots \\ \gamma & \delta & \gamma & \delta & \cdots \\ \alpha & \beta & \alpha & \beta & \cdots \\ \gamma & \delta & \gamma & \delta & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} = q_T \mathbf{1}_{N\times N} \otimes M_N
\end{aligned}
\tag{147}
$$

Using analogous derivations, it is straightforward to show that

$$
P_3(t) = t\, q_T \begin{bmatrix} 1 & 0 & 1 & 0 & \cdots \\ 0 & 1 & 0 & 1 & \cdots \\ 1 & 0 & 1 & 0 & \cdots \\ 0 & 1 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} = t\, q_T \mathbf{1}_{N\times N} \otimes I_{2\times 2} \tag{148}
$$

Using equations (147) and (148), the final expression for the maximum expected uncertainty at steady state becomes:

$$
\bar{\mathbf{P}}_{ss}(t) = \mathbf{Q}^{1/2}\mathbf{U} \begin{bmatrix} \mathrm{diag}_{2N-2}\left(\frac{1}{\boldsymbol{\lambda}_i}\right) & \mathbf{0}_{(2N-2)\times 2} \\ \mathbf{0}_{2\times(2N-2)} & \mathbf{0}_{2\times 2} \end{bmatrix} \mathbf{U}^T \mathbf{Q}^{1/2} + q_T \mathbf{1}_{N\times N} \otimes M_N + t\, q_T \mathbf{1}_{N\times N} \otimes I_{2\times 2} \tag{149}
$$

Thus the following lemma has been proven:

Lemma 3.4 *The maximum expected steady state uncertainty of a group of mobile robots performing cooperative localization is given by:*

$$
\bar{\mathbf{P}}_{ss}(t) = \mathbf{Q}^{1/2}\mathbf{U} \begin{bmatrix} \mathrm{diag}_{2N-2}\left(\frac{1}{\boldsymbol{\lambda}_i}\right) & \mathbf{0}_{(2N-2)\times 2} \\ \mathbf{0}_{2\times(2N-2)} & \mathbf{0}_{2\times 2} \end{bmatrix} \mathbf{U}^T \mathbf{Q}^{1/2} + q_T \mathbf{1}_{N\times N} \otimes M_N + t\, q_T \mathbf{1}_{N\times N} \otimes I_{2\times 2} \tag{150}
$$

where $\frac{1}{q_T} = \sum_{i=1}^{N} \frac{1}{q_i}$,

$$
M_N = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \tag{151}
$$

and the parameters α, β, γ, δ *are defined as follows: Let*

$$
\mathbf{W} = [w_{ij}] = q_T \mathbf{Q}^{-1}\mathbf{P}(0)\mathbf{Q}^{-1/2}\left(I_{2N\times 2N} + \sqrt{\mathbf{C}}\mathbf{Q}^{-1/2}\mathbf{P}(0)\mathbf{Q}^{-1/2}\right)^{-1}\mathbf{Q}^{-1/2}
$$

Then $\alpha = \sum_{i,j\ \mathrm{odd}} w_{ij}$ *is the sum of all elements of* $\mathbf{W} = [w_{ij}]$ *whose indices are both odd,* $\delta = \sum_{i,j\ \mathrm{even}} w_{ij}$ *is the sum of all elements with two even indices, and* $\gamma = \sum_{i\ \mathrm{odd,j\ even}} w_{ij}$ *is th sum of all elements with an odd row index and an even column index. Due to symmetry,* $\beta = \gamma$.

The first term of the above equation is a constant term, whose value depends on the topology of the RPMG and the characteristics of the sensors of the robots. The second term is a constant term that depends on the initial uncertainty, as well as the characteristics of the robots and the RPMG topology. Finally, the last term contributes with a constant rate of uncertainty increase that is proportional to q_T. At this point we should note that the rate of uncertainty increase is *independent* of the initial uncertainty P(0), the accuracy of the relative position measurements and the topology of the RPMG. From the definition of q_T, it becomes clear that will it be smaller than the smallest of the q_i's (notice that the definition of q_T is analogous to the expression for the resistance of resistors in parallel). This implies that it suffices to equip only *one* robot in the team with proprioceptive sensors of high accuracy, in order to achieve a desired rate of uncertainty increase. All the robots of the group will experience an improvement in the rate at which their uncertainty increases, and this improvement is more significant for robots with sensors of poor quality. We further discuss the significance of Eq. (150) in the last section, where the results of our simulations are presented.

3.4 RPMG Reconfigurations

The preceding analysis assumes that the topology of the graph describing the relative position measurements between robots does not change. However, this may be difficult to implement in a realistic scenario. For example, due to the robots' motion or due to obstacles in the environment, some robots may not be able to measure their relative positions. Additionally, the robots should allocate computational and communication resources to mission-specific tasks, and this may force them to reduce the number of measurements they process for localization purposes. Consequently, it is of considerable interest to study the effects of changes in the topology of the RPMG on the positioning performance of the team.

In this section we show that the same property that holds for the covariance of the position estimates in the 1D case (Lemma 3.2) holds also for the *upper bound* of the covariance in the case of robots performing cooperative localization in 2D. The derivations are analogous, with only minor modifications, to account for the fact that in the 2D case, the expressions provide an upper bound on the expected covariance, rather than an exact solution.

The following scenario is examined: At the initial stage of the deployment of a robotic team (Phase 1), the RPMG has a topology A, e.g., the complete graph shown in Fig. 8(a), and retains this topology until time instant t_1, when it assumes a different topology B, e.g., the ring graph shown in Fig. 8(b). We refer to the time interval during which the RPMG has topology B as Phase 2. Finally, at a second time instant t_2, the RPMG assumes the initial topology, A, once again (Phase 3). We assume that the time intervals $(0, t_1)$, (t_1, t_2) are of enough duration in order for the transient phenomena in the time evolution of uncertainty to subside. For this scenario, the following lemma applies:

Lemma 3.5 *The maximum expected steady state covariance of the robots during Phase 3 is* identical *to the maximum expected uncertainty the robot team would have if no RPMG reconfigurations had taken place. This result holds also for the case that during Phase 2 the robots perform DR (i.e., the RPMG is an* empty graph*).*

Proof We start the proof by noting that at time t_1 the normalized covariance will be

$$\mathbf{P}_{n_A}(t_1) = \mathbf{U}_A \begin{bmatrix} \mathrm{diag}_{2N-2}\left(\frac{1}{\boldsymbol{\lambda}_{Ai}}\right) & \mathbf{0}_{(2N-2)\times 2} \\ \mathbf{0}_{2\times(2N-2)} & t_1 I_{2\times 2} + M_A \end{bmatrix} \mathbf{U}_A^T \tag{152}$$

where

$$M_A = V^T \mathbf{P}_n(0) \left(I_{2N\times 2N} + \sqrt{\mathbf{C}_A}\mathbf{P}_n(0)\right)^{-1} V$$

and the subscript A has been appended to quantities that depend on the topology A.

At time t_1 the topology of the RPMG changes, and in order to compute the steady state covariance during the Phase 2, the covariance of the position estimates at time t_1 is required. If during Phase 2 the robots perform Dead Reckoning, then at time t_2 their normalized covariance will be

$$\mathbf{P}_{n_B}(t_2) = \mathbf{U}_A \begin{bmatrix} \mathrm{diag}_{2N-2}\left(\frac{1}{\boldsymbol{\lambda}_{Ai}}\right) + (t_2 - t_1) I_{(2N-2)\times(2N-2)} & \mathbf{0}_{(2N-2)\times 2} \\ \mathbf{0}_{2\times(2N-2)} & t_2 I_{2\times 2} + M_A \end{bmatrix} \mathbf{U}_A^T \tag{153}$$

while if the RPMG topology during phase 2 is a connected one, at time t_2 we have

$$\mathbf{P}_{n_B}(t) = \mathbf{U}_B \begin{bmatrix} \mathrm{diag}_{2N-2}\left(\frac{1}{\boldsymbol{\lambda}_{Bi}}\right) & \mathbf{0}_{(2N-2)\times 2} \\ \mathbf{0}_{2\times(2N-2)} & (t-t_1)I_{2\times 2} + M_B \end{bmatrix} \mathbf{U}_B^T \tag{154}$$

where

$$M_B \quad = \quad V^T \mathbf{P}_n(t_1)\left(I_{2N\times 2N} + \sqrt{\mathbf{C}_B}\mathbf{P}_n(t_1)\right)^{-1} V \tag{155}$$

and the subscript B denotes quantities associated with RPMG topology B. At this point we note that for computing the value of M_B, the *exact* value of the covariance at time t_1, would have to be employed. However, $P_n(t_1)$ is provides (after pre- and post multiplication with $\mathbf{Q}^{1/2}$) an upper bound on this covariance value. This is not a problem, since in Appendix I it is shown that M_B is a matrix increasing function of $\mathbf{P}_n(t_1)$, i.e.,

$$\mathbf{P}'_n \succeq \mathbf{P}_n \Rightarrow M'_B \succeq M_B \tag{156}$$

Therefore, by employing the upper bound of the covariance at t_1, the resulting expression remains an upper bound of the covariance during Phase 2. Thus, we are able to use the preceding expression in the derivations, since we only seek an upper bound of the steady state covariance during Phase 3. This is the only difference in the proof of Lemma 3.5, compared to Lemma 3.2. By comparison of the expressions in Eqs. (152)-(155) with those presented in Section 3.2.3, it becomes clear that by following derivations analogous to those for the 1D case the proof of Lemma 3.5 is straightforward.

■

This result is of great practical significance, and shows that if the robots of a team are forced to perform a small number of measurements (or even resort to mere DR) during a stage of their mission, due to communication or sensor failure, or in order allocate CPU and bandwidth resources to different tasks, then upon reverting to the initial RPMG topology, the team's positioning performance will have sustained *no degradation*. Additionally, it shows that it is beneficial to choose a dense topology for the RPMG during the initial phase of the deployment of a robot team. This may be possible for example, if during this initial phase the robots do not perform any other time-critical tasks, and are able to allocate a large proportion of the team's resources for localization purposes. As the robots start performing other tasks the RPMG topology may have to be reduced to a sparser one, in order to save computational and communication resources. However if, at a subsequent time instant, the RPMG resumes the initial, dense topology, the above lemma guarantees that the maximum expected uncertainty will be *identical* to the one that would arise if the dense RPMG topology were retained throughout the run of the robots. We remind that Lemma 3.5 holds under the assumption that the RPMG remains in each topology for sufficient time in order for the transient phenomena to die out.

3.5 Simulation Results

A series of experiments in simulation were conducted, with the aim of validating the preceding theoretical analysis. Robotic teams of different sizes and several topologies of the RPMG are considered, and the covariance values predicted by our theoretical analysis are compared to the experimental results. For the simulations the same two-layer approach to the estimation of the robot's pose is employed, that was used in the derivation of the theoretical bounds. For our experiments, the robots are restricted to move in an area of radius $r = 20$m, thus the maximum allowable distance between any two robots is $\rho_o = 40$m. The velocity of all robots is assumed to be constant, equal to $V_i = 0.25$m/sec. Note however, that our analysis does not require all the robots to move at the same speed. The orientation of the robots, while they move, changes randomly using samples drawn from a uniform distribution.

The parameters of the noise that corrupts the proprioceptive measurements of the simulated robots are identical to those measured on a iRobot PackBot robot ($\sigma_V = 0.0125$m/sec, $\sigma_\omega = 0.0384$rad/sec). The absolute orientation of each robot was measured by a simulated compass with $\sigma_\phi = 0.0524$rad. The robot tracker sensor returned range and bearing measurements corrupted by zero-mean white Gaussian noise with $\sigma_\rho = 0.01$m and $\sigma_\theta = 0.0349$rad. The above values are compatible with noise parameters observed in laboratory experiments [15]. All measurements were available at 1Hz.

In order to demonstrate the validity of the derived formulas for the steady state localization uncertainty of the robots, in Fig. 5 we plot the true value vs. the theoretical bound for the covariance along the x-axis of two robots performing cooperative localization. For this specific experiment the parameters for the proprioceptive sensors of the robots were chosen so that one of the robots has 5 times less accurate measurements (i.e, for this robot $\sigma_V = 0.0625$m/sec, $\sigma_\omega = 0.192$rad/sec). As evident, the true covariance consistently remains below the maximum expected value predicted by the theoretical analysis. This behavior for the localization uncertainty is a typical example of the results of our simulation experiments. In order to preserve the clarity of the figures in the following, a homogeneous robot team (i.e. a team whose robots are equipped with sensors of equal accuracy) is considered for the rest of the simulations. Note however, that homogeneity is not a prerequisite of our approach, as Fig. 5 demonstrates.

In Fig. 6 the theoretical upper bound for the expected localization uncertainty is compared with the true covariance provided by the simulations. Robotic teams consisting of 3, 5, 7 and 9 robots are considered, and in each plot, the theoretical bound as well as the true covariance for a fully connected RPMG and a ring RPMG are presented. In each plot, the true covariance is the average covariance over 20 runs of the simulation experiments. It becomes evident that the theoretical bound for the *rate* of uncertainty increase is quite tight, especially as the size of the team increases. We may also note that for the two radically different RPMGs considered (i.e a fully connected vs. a ring graph) the rate of uncertainty increase is identical, thus validating what was predicted theoretically. From comparison of the four plots in Fig. 6 we observe that for small teams, the difference in the localization uncertainty for two RPMG topologies is almost negligible. This implies that the performance improvement from employing a fully connected graph for the relative position measurements (and thus using up computational resources and communication bandwidth) are very small for small groups of robots, and the use of a sparser graph, (allowing for the allocation of computational and communication resources to other tasks) is favored.

In Fig. 7 the localization uncertainty evolution is presented for a team of 9 robots with changing RPMG topology. Initially, and up to $t = 200$sec the robots do not record any relative position measurements, and propagate their position estimates using Dead Reckoning (DR). At $t = 200$sec the robots start receiving relative measurements, and the topology of the RPMG is a fully connected one (Fig. 8(a)). The significant improvement in the rate of uncertainty increase that is achieved by using relative positioning information is demonstrated in this transition. At $t = 600$sec the RPMG assumes a ring topology (Fig. 8(b)). We note that the uncertainty undergoes a transient phase, during which it increases at a higher rate, and then, as soon as steady state is reached, the rate of increase is identical with the rate associated with the fully connected graph. This validates the result of Eq. (150), and shows that the dominant factor in determining the localization uncertainty for a team of robots is the quality of their proprioceptive sensors. At $t = 600$sec a supposed failure of the communication network occurs, and in the time interval between 600sec and 800sec only two robots are able to measure their relative position, (Fig. 8(c)). This case can be viewed as a degenerate case, where the 7 robots localize based solely on Dead Reckoning, while the other two robots form the team. We can observe that the rate of increase of the covariance is larger when the team consists of only two robots, instead of nine. At $t = 800$sec the RPMG assumes a non-canonical topology, i.e., random graph (Fig. 8(d)). This case is perhaps the most important for real applications, since robots will usually measure the distances of their neighbors, and due to the robots' motion, the topology of the RPMG can change randomly. In this case, the uncertainty increases at a rate identical to that of cases I and II of the graph's topology, as predicted by our theoretical analysis. It is also apparent, that the uncertainty for each robot converges to a set of lines with the same slope (rate of uncertainty increase), but different constant offset. This is due to the effect of the different degree of connectivity in the RPMG of each robot. Connection-rich robots have access to a higher rate of positioning information flow, and thus attain lower positioning uncertainty.

At $t = 1000$sec only one of the robots starts receiving GPS measurements while the RPMG retains the topology of (Fig. 8(d)) The GPS measurements are corrupted by noise with a standard deviation of $\sigma_{GPS} = 0.05$m in each axis. It is evident that the availability of absolute position measurements to *any* robot drastically reduces the localization uncertainty for *all* the robots in the group. Furthermore, the system becomes observable and the uncertainty is bounded for all robots in the group. As in the previous case, the constant value at which the uncertainty for each robot converges to depends on its degree of connectivity.

Acknowledgments

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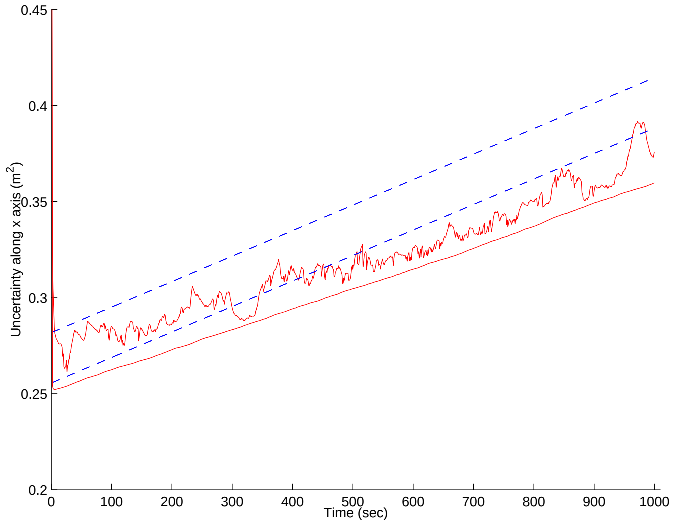
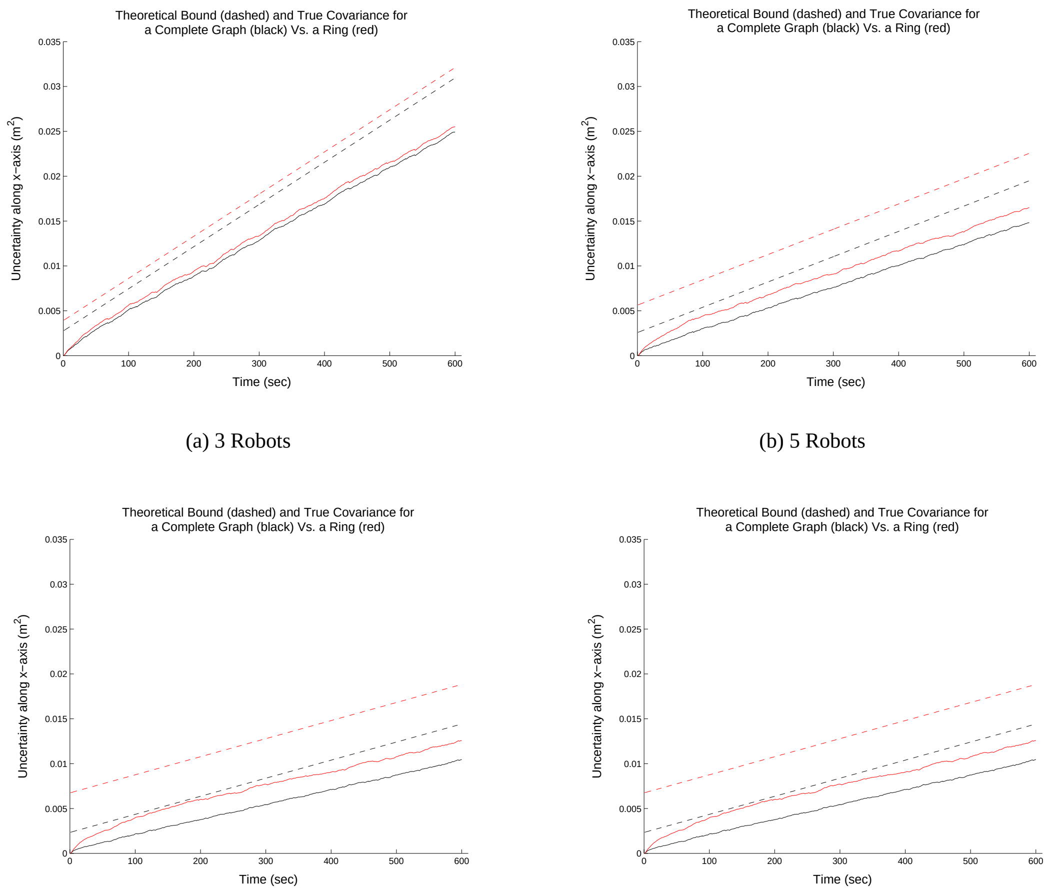


Figure 5: True covariance Vs. theoretical bound for a heterogeneous team of 2 robots.



(a) 3 Robots

(b) 5 Robots

(c) 7 Robots

(d) 9 Robots

Figure 6: True covariance vs. theoretical bound for homogeneous teams of robots. The plots correspond to fully connected and ring RPMGs.

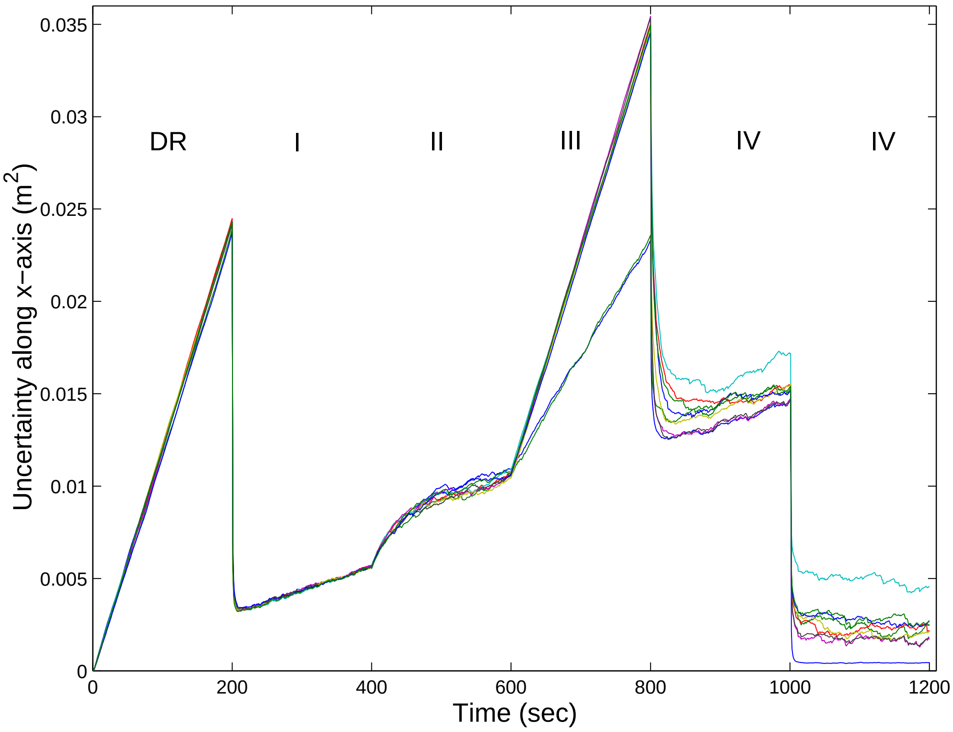
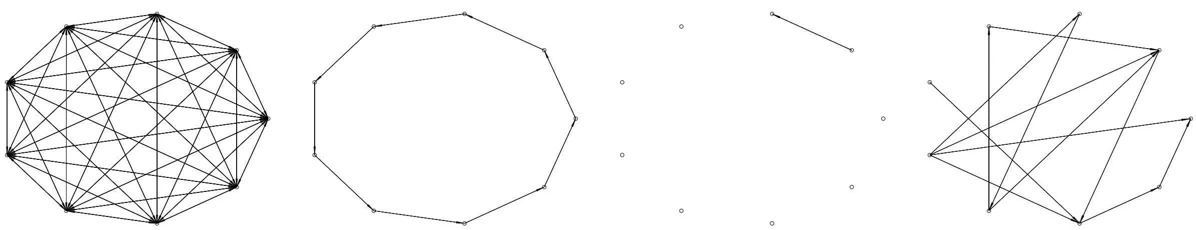


Figure 7: Uncertainty evolution for a RPMG with changing topology.



(a) Graph I (b) Graph II (c) Graph III (d) Graph IV

Figure 8: The four different measurement graph topologies considered in the simulations. Each arrow represents a relative position measurement, with the robot (node) where the arrow starts being the observing robot.

A Upper Bound Riccati Recursion

In this appendix we prove that if $\mathbf{R}_u \succeq \mathbf{R}_o(k)$ and $\mathbf{Q}_u \succeq \mathbf{Q}(k)$ for all $k \geq 0$, then the solutions to the following two Riccati recursions

$$\mathbf{P}_{k+1} = \mathbf{P}_k - \mathbf{P}_k \mathbf{H}_o^T \left(\mathbf{H}_o \mathbf{P}_k \mathbf{H}_o^T + \mathbf{R}_o(k+1)\right)^{-1} \mathbf{H}_o \mathbf{P}_k + \mathbf{Q}(k+1) \tag{157}$$

and

$$\mathbf{P}_{k+1}^u = \mathbf{P}_k^u - \mathbf{P}_k^u \mathbf{H}_o^T \left(\mathbf{H}_o \mathbf{P}_k^u \mathbf{H}_o^T + \mathbf{R}_u\right)^{-1} \mathbf{H}_o \mathbf{P}_k^u + \mathbf{Q}_u \tag{158}$$

with the *same* initial condition, $\mathbf{P}_0$, satisfy $\mathbf{P}_k^u \succeq \mathbf{P}_k$ for all $k \geq 0$. The proof is carried out by induction, and requires the following two intermediate results:

- **Monotonicity with respect to the measurement covariance matrix**

If $\mathbf{R}_1 \succeq \mathbf{R}_2$, then for any $\mathbf{P} \succeq 0$

$$\mathbf{P} - \mathbf{P}\mathbf{H}^T \left(\mathbf{H}\mathbf{P}\mathbf{H}^T + \mathbf{R}_1\right)^{-1} \mathbf{H}\mathbf{P} + \mathbf{Q}_o \succeq \mathbf{P} - \mathbf{P}\mathbf{H}^T \left(\mathbf{H}\mathbf{P}\mathbf{H}^T + \mathbf{R}_2\right)^{-1} \mathbf{H}\mathbf{P} + \mathbf{Q}_o \tag{159}$$

This statement is proven by taking into account the properties of linear matrix inequalities:

$$\begin{aligned}
\mathbf{R}_1 &\succeq \mathbf{R}_2 \Rightarrow \\
\mathbf{H}\mathbf{P}\mathbf{H}^T + \mathbf{R}_1 &\succeq \mathbf{H}\mathbf{P}\mathbf{H}^T + \mathbf{R}_2 \Rightarrow \\
\left(\mathbf{H}\mathbf{P}\mathbf{H}^T + \mathbf{R}_1\right)^{-1} &\preceq \left(\mathbf{H}\mathbf{P}\mathbf{H}^T + \mathbf{R}_2\right)^{-1} \Rightarrow \\
\mathbf{P}\mathbf{H}^T \left(\mathbf{H}\mathbf{P}\mathbf{H}^T + \mathbf{R}_1\right)^{-1} \mathbf{H}\mathbf{P} &\preceq \mathbf{P}\mathbf{H}^T \left(\mathbf{H}\mathbf{P}\mathbf{H}^T + \mathbf{R}_2\right)^{-1} \mathbf{H}\mathbf{P} \Rightarrow \\
-\mathbf{P}\mathbf{H}^T \left(\mathbf{H}\mathbf{P}\mathbf{H}^T + \mathbf{R}_1\right)^{-1} \mathbf{H}\mathbf{P} &\succeq -\mathbf{P}\mathbf{H}^T \left(\mathbf{H}\mathbf{P}\mathbf{H}^T + \mathbf{R}_2\right)^{-1} \mathbf{H}\mathbf{P} \Rightarrow \\
\mathbf{P} - \mathbf{P}\mathbf{H}^T \left(\mathbf{H}\mathbf{P}\mathbf{H}^T + \mathbf{R}_1\right)^{-1} \mathbf{H}\mathbf{P} + \mathbf{Q}_o &\succeq \mathbf{P} - \mathbf{P}\mathbf{H}^T \left(\mathbf{H}\mathbf{P}\mathbf{H}^T + \mathbf{R}_2\right)^{-1} \mathbf{H}\mathbf{P} + \mathbf{Q}_o
\end{aligned}$$

- **Monotonicity with respect to the state covariance matrix**

The solution to the Riccati recursion at time $k+1$ is monotonic with to the solution at time k, i.e., if $\mathbf{P}_k^{(1)}$ and $\mathbf{P}_k^{(2)}$ are two different solutions to the same Riccati recursion at time k, with $\mathbf{P}_k^{(1)} \succeq \mathbf{P}_k^{(2)}$ then $\mathbf{P}_{k+1}^{(1)} \succeq \mathbf{P}_{k+1}^{(2)}$. In order to prove the result in the general case, in which $\mathbf{P}_k^{(1)}$ and $\mathbf{P}_k^{(2)}$ are positive semidefinite, we use the following expression that relates the one-step ahead solutions to two Riccati recursions with identical $\mathbf{H}$, $\mathbf{Q}$ and $\mathbf{R}$ matrices, but different initial values $\mathbf{P}_k^{(1)}$ and $\mathbf{P}_k^{(2)}$ ([11]). It is

$$\mathbf{P}_{k+1}^{(2)} - \mathbf{P}_{k+1}^{(1)} = F_{p,k} \left(\left(\mathbf{P}_k^{(2)} - \mathbf{P}_k^{(1)}\right) - \left(\mathbf{P}_k^{(2)} - \mathbf{P}_k^{(1)}\right) \mathbf{H}^T \left(\mathbf{H}\mathbf{P}_k^{(2)}\mathbf{H}^T + \mathbf{R}\right) \mathbf{H} \left(\mathbf{P}_k^{(2)} - \mathbf{P}_k^{(1)}\right) \right) F_{p,k}^T \tag{160}$$

where $F_{p,k}$ is a matrix whose exact structure is not important for the purposes of this proof. Since we have assumed $\mathbf{P}_k^{(1)} \succeq \mathbf{P}_k^{(2)}$ we can write $\mathbf{P}_k^{(2)} - \mathbf{P}_k^{(1)} \preceq 0$. Additionally, the matrix

$$\left(\mathbf{P}_k^{(2)} - \mathbf{P}_k^{(1)}\right) \mathbf{H}^T \left(\mathbf{H}\mathbf{P}_k^{(2)}\mathbf{H}^T + \mathbf{R}\right) \mathbf{H} \left(\mathbf{P}_k^{(2)} - \mathbf{P}_k^{(1)}\right)$$

is positive semidefinite, and therefore we have

$$\begin{aligned}
-\left(\mathbf{P}_k^{(2)} - \mathbf{P}_k^{(1)}\right) \mathbf{H}^T \left(\mathbf{H}\mathbf{P}_k^{(2)}\mathbf{H}^T + \mathbf{R}\right) \mathbf{H} \left(\mathbf{P}_k^{(2)} - \mathbf{P}_k^{(1)}\right) &\preceq 0 \Rightarrow \\
\left(\mathbf{P}_k^{(2)} - \mathbf{P}_k^{(1)}\right) - \left(\mathbf{P}_k^{(2)} - \mathbf{P}_k^{(1)}\right) \mathbf{H}^T \left(\mathbf{H}\mathbf{P}_k^{(2)}\mathbf{H}^T + \mathbf{R}\right) \mathbf{H} \left(\mathbf{P}_k^{(2)} - \mathbf{P}_k^{(1)}\right) &\preceq 0 \Rightarrow \\
F_{p,k} \left(\left(\mathbf{P}_k^{(2)} - \mathbf{P}_k^{(1)}\right) - \left(\mathbf{P}_k^{(2)} - \mathbf{P}_k^{(1)}\right) \mathbf{H}^T \left(\mathbf{H}\mathbf{P}_k^{(2)}\mathbf{H}^T + \mathbf{R}\right) \mathbf{H} \left(\mathbf{P}_k^{(2)} - \mathbf{P}_k^{(1)}\right) \right) F_{p,k}^T &\preceq 0 \Rightarrow \\
\mathbf{P}_{k+1}^{(2)} - \mathbf{P}_{k+1}^{(1)} &\preceq 0
\end{aligned}$$

The last line implies that $\mathbf{P}_{k+1}^{(1)} \succeq \mathbf{P}_{k+1}^{(2)}$, which is the desired result.

We can now employ induction to prove the main statement of this appendix. Assuming that at some time instant i, $\mathbf{P}_i^u \succeq \mathbf{P}_i$, we can write

$$\begin{aligned}
\mathbf{P}_{i+1}^u &= \mathbf{P}_i^u - \mathbf{P}_i^u \mathbf{H}_o^T \left(\mathbf{H}_o \mathbf{P}_i^u \mathbf{H}_o^T + \mathbf{R}_u\right)^{-1} \mathbf{H}_o \mathbf{P}_i^u + \mathbf{Q}_u \\
&\succeq \mathbf{P}_i - \mathbf{P}_i \mathbf{H}_o^T \left(\mathbf{H}_o \mathbf{P}_i \mathbf{H}_o^T + \mathbf{R}_u\right)^{-1} \mathbf{H}_o \mathbf{P}_i + \mathbf{Q}_u \\
&\succeq \mathbf{P}_i - \mathbf{P}_i \mathbf{H}_o^T \left(\mathbf{H}_o \mathbf{P}_i \mathbf{H}_o^T + \mathbf{R}_u\right)^{-1} \mathbf{H}_o \mathbf{P}_i + \mathbf{Q}(k+1) \\
&\succeq \mathbf{P}_i - \mathbf{P}_i \mathbf{H}_o^T \left(\mathbf{H}_o \mathbf{P}_i \mathbf{H}_o^T + \mathbf{R}_o(k+1)\right)^{-1} \mathbf{H}_o \mathbf{P}_i + \mathbf{Q}(k+1) = \mathbf{P}_{i+1}
\end{aligned}$$

where the monotonicity of the Riccati recursion with respect to the covariance matrix, the property $\mathbf{Q}_u \succeq \mathbf{Q}(k+1)$ and the monotonicity of the Riccati recursion with respect to the measurement covariance matrix have been used in the last three lines. Thus $\mathbf{P}_i^u \succeq \mathbf{P}_i \Rightarrow \mathbf{P}_{i+1}^u \succeq \mathbf{P}_{i+1}$. For $i = 0$ the condition $\mathbf{P}_i^u \succeq \mathbf{P}_i$ holds with equality, and therefore for any $i > 0$, the solution to the Riccati recursion in Eq. (157) is an upper bound to the solution of the recursion in Eq. (158).

B Riccati Recursion for the Upper Bound on the Average Covariance

In this appendix we prove that if $\bar{\mathbf{R}}$ and $\bar{\mathbf{Q}}$ are matrices such that $\bar{\mathbf{R}} = E\{\mathbf{R}_o(k)\}$ and $\bar{\mathbf{Q}} = \{\mathbf{Q}(k)\}$ for all $k \geq 0$, then the solutions to the following two Riccati recursions

$$\mathbf{P}_{k+1} = \mathbf{P}_k - \mathbf{P}_k \mathbf{H}_o^T \left(\mathbf{H}_o \mathbf{P}_k \mathbf{H}_o^T + \mathbf{R}_o(k+1)\right)^{-1} \mathbf{H}_o \mathbf{P}_k + \mathbf{Q}(k+1) \tag{161}$$

and

$$\bar{\mathbf{P}}_{k+1} = \bar{\mathbf{P}}_k - \bar{\mathbf{P}}_k \mathbf{H}_o^T \left(\mathbf{H}_o \bar{\mathbf{P}}_k \mathbf{H}_o^T + \bar{\mathbf{R}}\right)^{-1} \mathbf{H}_o \bar{\mathbf{P}}_k + \bar{\mathbf{Q}} \tag{162}$$

with the *same* initial condition, $\mathbf{P}_0$, satisfy $\bar{\mathbf{P}}_k \succeq E\{\mathbf{P}_k\}$ for all $k \geq 0$. We first prove a useful intermediate result:

- **Concavity of the Riccati recursion**

We note that the Riccati recursion

$$P_{k+1} = P_k - P_k H^T \left(H P_k H^T + R_{k+1}\right)^{-1} H P_k + Q_{k+1} \tag{163}$$

can equivalently be written as

$$\begin{aligned}
P_{k+1} = &\begin{bmatrix} I & \mathbf{0} \end{bmatrix} \begin{bmatrix} P_k & \mathbf{0} \\ \mathbf{0} & R_{k+1} \end{bmatrix} \begin{bmatrix} I \\ \mathbf{0} \end{bmatrix} \\
&- \begin{bmatrix} I & \mathbf{0} \end{bmatrix} \begin{bmatrix} P_k & \mathbf{0} \\ \mathbf{0} & R_{k+1} \end{bmatrix} \begin{bmatrix} H^T \\ \mathbf{0} \end{bmatrix} \left(\begin{bmatrix} H & I \end{bmatrix} \begin{bmatrix} P_k & \mathbf{0} \\ \mathbf{0} & R_{k+1} \end{bmatrix} \begin{bmatrix} H^T \\ \mathbf{I} \end{bmatrix} \right)^{-1} \begin{bmatrix} H & \mathbf{0} \end{bmatrix} \begin{bmatrix} P_k & \mathbf{0} \\ \mathbf{0} & R_{k+1} \end{bmatrix} \begin{bmatrix} I \\ \mathbf{0} \end{bmatrix} \\
&+ Q_{k+1}
\end{aligned}$$

our goal is to show that the above expression is concave with respect to the matrix

$$\begin{bmatrix} P_k & \mathbf{0} \\ \mathbf{0} & R_{k+1} \end{bmatrix}$$

A sufficient condition for this is that the function

$$f(X) = AXB \left(CXC^T\right)^{-1} B^T X A^T \tag{164}$$

is convex with respect to the positive semidefinite matrix X, when A,B,C are arbitrary matrices of compatible dimensions. This is equivalent to proving the convexity of the function of the scalar variable t

$$f_t(t) = A(X_o + tZ_o)B\left(C(X_o + tZ_o)C^T\right)^{-1} B^T(X_o + tZ_o)A^T \tag{165}$$

with domain those values of t for which $X_o + tZ_o \succeq 0, X_o \succeq 0$ is convex [14]. $f_t(t)$ is convex if and only if the scalar function

$$f_t(t) = z^T A(X_o + tZ_o)B\left(C(X_o + tZ_o)C^T\right)^{-1} B^T(X_o + tZ_o)A^T z \tag{166}$$

is convex for any vector z of appropriate dimensions [14]. Moreover, it is well known that a function is convex if and only if its epigraph is a convex set, and therefore we obtain the following convexity condition for $f(X)$:

$$f(X) \text{ convex } \Leftrightarrow \{s,t|z^T A(X_o + tZ_o)B\left(C(X_o + tZ_o)C^T\right)^{-1} B^T(X_o + tZ_o)A^T z \leq s\} \text{ is convex}$$

However, from the properties of Schur complements it is well known that if $A_o \succ 0$ then

$$\begin{bmatrix} A_o & B_o \\ B_o^T & C_o \end{bmatrix} \succeq 0 \Leftrightarrow C_o - B_o^T C_o^{-1} B \succeq 0$$

In our problem, the matrix $C(X_o + tZ_o)C^T$ is clearly positive definite, and thus we can write

$$z^T A(X_o{+}tZ_o)B\left(C(X_o + tZ_o)C^T\right)^{-1} B^T(X_o{+}tZ_o)A^T z \leq s \Leftrightarrow \begin{bmatrix} C(X_o + tZ_o)C^T & B^T(X_o + tZ_o)A^T z \\ z^T A(X_o + tZ_o)B & s \end{bmatrix} \succeq 0$$

However, the defining matrix inequality of the epigraph is equivalent to

$$\begin{bmatrix} CX_oC^T & B^T X_o A^T z \\ z^T AX_oB & 0 \end{bmatrix} + t\begin{bmatrix} CZ_oC^T & B^T Z_o A^T z \\ z^T AZ_oB & 0 \end{bmatrix} + s\begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix} \succeq 0$$

which defines a convex set in (s,t) [14].

Thus, by the preceding analysis $f(X)$ is a convex function, and consequently P_{k+1} is a concave function of the matrix

$$\begin{bmatrix} P_k & \mathbf{0} \\ \mathbf{0} & R_{k+1} \end{bmatrix}$$

■

We now employ this result to prove the main result of this appendix. The proof is carried out by induction. Assuming that at time step k the inequality $\bar{\mathbf{P}}_k \succeq E\{\mathbf{P}_k\}$ holds, we will show that it also holds for the time step $k+1$. We have

$$\begin{aligned}
\mathbf{P}_{k+1} &= \mathbf{P}_k - \mathbf{P}_k\mathbf{H}_o^T\left(\mathbf{H}_o\mathbf{P}_k\mathbf{H}_o^T + \mathbf{R}_o(k+1)\right)^{-1}\mathbf{H}_o\mathbf{P}_k + \mathbf{Q}(k+1) \Rightarrow \\
E\{\mathbf{P}_{k+1}\} &= E\{\mathbf{P}_k - \mathbf{P}_k\mathbf{H}_o^T\left(\mathbf{H}_o\mathbf{P}_k\mathbf{H}_o^T + \mathbf{R}_o(k+1)\right)^{-1}\mathbf{H}_o\mathbf{P}_k + \mathbf{Q}(k+1)\} \\
&= E\{\mathbf{P}_k - \mathbf{P}_k\mathbf{H}_o^T\left(\mathbf{H}_o\mathbf{P}_k\mathbf{H}_o^T + \mathbf{R}_o(k+1)\right)^{-1}\mathbf{H}_o\mathbf{P}_k\} + E\{\mathbf{Q}(k+1)\} \\
&\preceq E\{\mathbf{P}_k\} - E\{\mathbf{P}_k\}\mathbf{H}_o^T\left(\mathbf{H}_oE\{\mathbf{P}_k\}\mathbf{H}_o^T + E\{\mathbf{R}_o(k+1)\}\right)^{-1}\mathbf{H}_oE\{\mathbf{P}_k\} + E\{\mathbf{Q}(k+1)\}
\end{aligned}$$

where in the last line the concavity of Jensen's inequality was applied [14], in order to exploit tht concavity of the Riccati. By assumption, $\bar{\mathbf{P}}_k \succeq E\{\mathbf{P}_k\}$ and employing the property of the monotonicity of the Riccati with respect to the covariance matrix (cf. Appendix A), we can write

$$\begin{aligned}
E\{\mathbf{P}_{k+1}\} &\preceq \bar{\mathbf{P}}_k - \bar{\mathbf{P}}_k\mathbf{H}_o^T\left(\mathbf{H}_o\bar{\mathbf{P}}_k\mathbf{H}_o^T + E\{\mathbf{R}_o(k+1)\}\right)^{-1}\mathbf{H}_o\bar{\mathbf{P}}_k + E\{\mathbf{Q}(k+1)\} \\
&= \bar{\mathbf{P}}_k - \bar{\mathbf{P}}_k\mathbf{H}_o^T\left(\mathbf{H}_o\bar{\mathbf{P}}_k\mathbf{H}_o^T + \bar{\mathbf{R}}\}\right)^{-1}\mathbf{H}_o\bar{\mathbf{P}}_k + \bar{\mathbf{Q}} \\
&= \bar{\mathbf{P}}_{k+1}
\end{aligned}$$

Thus, $\bar{\mathbf{P}}_k \succeq E\{\mathbf{P}_k\} \Rightarrow \bar{\mathbf{P}}_{k+1} \succeq E\{\mathbf{P}_{k+1}\}$. For $k = 0$ the condition $\bar{\mathbf{P}}_k \succeq E\{\mathbf{P}_k\}$ holds with equality, and the proof is complete.

C Continuous Time Riccati Equation

For a linear continuous time system, where the state measurements are available continuously, the state model equations are

$$\begin{aligned} \dot{x}(t) &= F(t)x(t) + B(t)u(t) + G(t)w(t) \quad (167) \\ z(t) &= H(t)x(t) + n(t) \quad (168) \end{aligned}$$

where $u(t)$ is the input to the system, $w(t)$ is the dynamic driving noise process having covariance $Q(t)$, $n(t)$ is the measurement noise process, with covariance $R(t)$, $F(t)$ is the matrix describing the dynamic behavior of the states, $B(t)$ is the matrix describing the affect of the inputs on the states, and $H(t)$ is the measurement matrix.

The continuous time Riccati equation, describing the evolution of the state covariance is

$$\dot{P} = FP + PF^T + GQG^T - PH^TR^{-1}HP \quad (169)$$

where the time indices have been dropped for simplicity.

D Appendix: Taylor Series Expansion of the Hyperbolic Sine and Cosine Functions

The Taylor series expansion of the exponential function is given by:

$$e^{at} = \Sigma_{k=0}^{\infty} \frac{a^k t^k}{k!} = 1 + \frac{at}{1!} + \frac{a^2t^2}{2!} + \frac{a^3t^3}{3!} + \frac{a^4t^4}{4!} + \cdots$$

The above relation, when substituting $-t$ instead of t yields:

$$e^{-at} = \Sigma_{k=0}^{\infty} \frac{a^k (-t)^k}{k!} = 1 - \frac{at}{1!} + \frac{a^2t^2}{2!} - \frac{a^3t^3}{3!} + \frac{a^4t^4}{4!} - \cdots$$

Thus, by subtracting and adding the previous two relations, we get:

$$\frac{e^{at} + e^{-at}}{2} = 1 + \frac{1}{2!}a^2t^2 + \frac{1}{4!}a^4t^4 + \cdots$$

and

$$\frac{e^{at} - e^{-at}}{2} = \frac{1}{1!}at + \frac{1}{3!}a^3t^3 + \frac{1}{5!}a^5t^5 + \cdots$$

The last two functions are the hyperbolic cosine and sine respectively.

E Rank and Nullspace of the Measurement Matrices

In this appendix we present some results concerning the rank of the measurement matrices in CL, as well as the rank and eigenvectors of the matrix:

$$\mathbf{C}_s = \mathbf{Q}_s^{1/2}\mathbf{H}_o^T\mathbf{R}_s^{-1}\mathbf{H}_o\mathbf{Q}_s^{1/2}$$

Where the matrices $\mathbf{Q}_s^{1/2}$ and $\mathbf{R}_s$ can be substituted for either by the upper bounds, or by the average values of the corresponding covariance matrices.

We first note that,in the case in which the robots receive only relative position measurements, $\mathbf{H}_o$ consists of block rows of the form

$$\left[\begin{array}{ccccccc} \mathbf{0}_{2\times2} & .. & -I_{2\times2} & .. & I_{2\times2} & .. & \mathbf{0}_{2\times2} \end{array} \right] = \left[\begin{array}{ccccccc} 0 & .. & -1 & .. & 1 & .. & 0 \end{array} \right] \otimes I_{2\times2}$$

while if the some of the robots additionally receive absolute position measurements, $\mathbf{H}_o$ also has some block rows of the form

$$\left[\begin{array}{ccccc} \mathbf{0}_{2\times2} & .. & I_{2\times2} & .. & \mathbf{0}_{2\times2} \end{array} \right] = \left[\begin{array}{ccccc} 0 & .. & 1 & .. & 0 \end{array} \right] \otimes I_{2\times2}$$

We therefore conclude, that in any case, the matrix $\mathbf{H}_o$ can be expressed as

$$\mathbf{H}_o = H \otimes I_{2\times 2} \tag{170}$$

where H is an appropriate matrix, consisting of rows having one of the two following structures:

$$H_{ij} = \begin{bmatrix} 0 & .. & -1 & .. & 1 & .. & 0 \end{bmatrix}$$

or

$$H_\ell = \begin{bmatrix} 0 & .. & 1 & .. & 0 \end{bmatrix}$$

It becomes clear that the matrix H will be the measurement matrix associated with a 1D CL system model, in which the robot team has the same RPMG as the team of robots performing localization in 2D (cf. Section 3.2).

Employing the properties of the Kronecker product, from Eq. (170) we conclude that

$$\mathrm{rank}(\mathbf{H}_o) = \mathrm{rank}(H)\,\mathrm{rank}(I_{2\times 2}) = 2 \cdot \mathrm{rank}(H) \tag{171}$$

and therefore we can determine the rank of $\mathbf{H}_o$ by first studying the properties of the 1D-measurement matrix H. For this reason, we start by presenting the results for the, simpler, one-dimensional CL case (cf. Section 3.2).

E.1 Cooperative Localization in 1D

For the one-dimensional case, when no absolute position measurements are available, the measurement matrix H is defined by

$$H = \begin{bmatrix} H_{ij} \\ \vdots \\ H_{kl} \\ \vdots \\ H_{mn} \end{bmatrix} \tag{172}$$

where each row of H corresponds to one relative position measurement, or equivalently, to one edge of the RPMG. Each of the rows contains a “-1”, at the column that corresponds to the robot i registering the relative position measurement, and a “1” at the column that corresponds to the robot being observed. This matrix is identical to the *incidence matrix* defined for any directed graph. In [16] it is shown that the incidence matrix of a directed graph is of rank $N-1$, whenever the graph is connected, and therefore the rank of H is $N-1$, where we have imposed the constraint that the measurement graph is connected[4].

Having determined the rank of H, we are now able to study the rank and eigenvectors of the matrix

$$C = Q^{1/2} H^T R^{-1} H Q^{1/2}$$

where Q and R are diagonal and positive definite. In order to determine the rank of this matrix, we use the following lemma from linear algebra [16]:

Lemma E.1 *The rank of the product of two matrices A, B is given by*

$$\mathrm{rank}(AB) = \mathrm{rank}(B) - \dim\left(N(A) \bigcap R(B)\right) \tag{173}$$

where $\dim\ X \bigcap Y$ denotes the dimension of the subspace formed by the intersection of the subspaces X and Y, $N(A)$ is the nullspace of matrix A, and $R(B)$ is the range of B.

Note that the matrix product $H^T R^{-1} H$ can be written as $H^T R^{-1/2} R^{-1/2} H = (R^{-1/2} H)^T R^{-1/2} H$. We now apply the above lemma to the matrix product $M = R^{-1/2} H$. Since $R^{-1/2}$ is an invertible matrix, its nullspace is of dimension 0, and we have $\mathrm{rank}(M) = \mathrm{rank}(R^{-1/2} H) = \mathrm{rank}(H) = N-1$. Moreover, it is evident that the

[4] This is not a restraining assumption. The case in which the RPMG is not connected is a degenerate one. In this case, the robots that are not connected by an edge to any robot of the team, do not actually belong to the team, and therefore, we can study this case by a considering each connected subgraph independently.

nullspace of M will be the same with the nullspace of H. In order to find the rank of $H^T R^{-1} H = M^T M$ we employ the above lemma once again:

$$\mathrm{rank}(H^T R^{-1} H) = \mathrm{rank}(M^T M) = \mathrm{rank}(M) - \dim\left(N(M) \bigcap R(M)\right)$$

Since the nullspace and the range of any matrix are disjoint sets, $\mathrm{rank}(H^T R^{-1} H) = N - 1$. By consecutive application of the above lemma to the matrix products $(H^T R^{-1} H) Q^{1/2}$ and $Q^{1/2}(H^T R^{-1} H Q^{1/2})$ it is easy to show that $\mathrm{rank}(C) = N - 1$.

A direct consequence of this result is that C has one eigenvalue equal to zero, and that its nullspace is of dimension 1. Note that since the sum of all elements of the rows of H is zero, we obtain

$$H \mathbf{1}_{N\times 1} = \mathbf{0}_{N\times 1}$$

hence the basis of the nullspace of H is the vector $x_N = \mathbf{1}_{N\times 1}$. As a result, we deduce that the basis vector for the nullspace of C is given by

$$U_N = \frac{1}{||Q^{-1/2}\mathbf{1}_{N\times 1}||} Q^{-1/2}\mathbf{1}_{N\times 1}$$

since

$$CU_N = \frac{1}{||Q^{-1/2}\mathbf{1}_{N\times 1}||} Q^{1/2} H^T R^{-1} H Q^{1/2} Q^{-1/2}\mathbf{1}_{N\times 1} = \frac{1}{||Q^{-1/2}\mathbf{1}_{N\times 1}||} Q^{1/2} H^T R^{-1} \left(H\mathbf{1}_{N\times 1}\right) = \mathbf{0}_{N\times 1}$$

Simpe calculations show that

$$U_N = \frac{1}{||Q^{-1/2}\mathbf{1}_{N\times 1}||} Q^{-1/2}\mathbf{1}_{N\times 1} = \frac{1}{\left(\sum_{i=1}^{N}\frac{1}{q_i}\right)^{1/2}} Q^{-1/2}\mathbf{1}_{N\times 1} = \sqrt{q_{total}}\, Q^{-1/2}\mathbf{1}_{N\times 1}$$

where

$$\frac{1}{q_{total}} = \sum_{i=1}^{N}\frac{1}{q_i}$$

Finally, by applying simple vector-matrix multiplication, we obtain the following result, which is useful in the derivations in Section 3.2:

$$Q^{1/2} U_N U_N^T Q^{1/2} = q_{total}\mathbf{1}_{N\times N} \tag{174}$$

If in addition to the relative position measurements, some of the robots receive absolute positioning information, then the measurement matrix has a number of rows (at least one) of the form $H_{i_A} = [0\ ..\ 1\ ..\ 0]$, with the "1"s being at the columns corresponding to the robots receiving absolute positioning information. In this case C can be written as

$$C = Q^{1/2}\left(H^T R^{-1} H + \sum_k \frac{1}{\sigma_A^2} H_{k_A}^T H_{k_A}\right) Q^{1/2} = C + Q^{1/2}\sum_k \frac{1}{\sigma_{A_k}^2} H_{k_A}^T H_{k_A} Q^{1/2} = C + C_A \tag{175}$$

where the sum is over all robots receiving absolute position measurements, $\sigma_{A_k}^2$ are the variances of these measurements, and C is the matrix of the previous case, in which only relative position information were available.

We now prove that C is positive definite, by showing that $x^T C x = 0 \Leftrightarrow x = 0$. Assume that there exists a vector x such that

$$x^T C x = 0 \Rightarrow x^T C x + x^T C_A x = 0$$

Clearly, both terms in the last expression are always nonnegative, since the involved matrices are positive semidefinite. Thus $x^T C x = 0$ implies $x^T C x = x^T C_A x = 0$. The term $x^T C x$ assumes the zero value only when $x = aU_N$, where $a \in \mathbb{R}$ and U_N is the basis vector of the nullspace of C. But

$$a^2 U_N^T\left(Q^{1/2}\sum_k \frac{1}{\sigma_{A_k}^2} H_{k_A}^T H_{k_A} Q^{1/2}\right) U_N = a^2 q_{total}\sum_k \frac{1}{\sigma_{A_k}^2}$$

and therefore this quantity is equal to zero only when $a = 0$. Thus $x^T C x = 0 \Rightarrow x = 0$, which implies that when at least one robot has access to absolute position information, C is positive definite.

E.2 Cooperative Localization in 2D

We can now employ the results of the preceding 1D analysis to the 2D case. Using the result of Eq. (171), we immediately see that when the robots of the a team performing CL in 2D only record relative position measurements, then $\mathrm{rank}(\mathbf{H}_o) = 2N - 2$, while if at least one of the robots has access to absolute position measurements, we have $\mathrm{rank}(\mathbf{H}_o) = 2N$.

Regarding the rank and eigenvectors of $\mathbf{C}_s$, it is straightforward to see that

$$\mathrm{rank}(\mathbf{H}_o) = 2N \Rightarrow \mathrm{rank}(\mathbf{C}_s) = 2N$$

since in this case $\mathbf{C}_s$ is the product of full-rank matrices. Similarly, we can use Lemma E.1 in the same way as in the 1D case, to show that $\mathrm{rank}(\mathbf{C}_s) = 2N - 2$. As a result, the nullspace of $\mathbf{C}_s$ is of dimension 2, and is spanned by 2 orthogonal basis vectors. We can find two such vectors by observing that

$$\begin{aligned}
\mathbf{C}_s \left(\mathbf{Q}_s^{-1/2}\mathbf{1}_{N\times 1} \otimes I_{2\times 2}\right) &= \mathbf{Q}_s^{1/2}\mathbf{H}_o^T\mathbf{R}_s^{-1}\mathbf{H}_o\mathbf{Q}_s^{1/2}\left(\mathbf{Q}_s^{-1/2}\mathbf{1}_{N\times 1} \otimes I_{2\times 2}\right) \\
&= \mathbf{Q}_s^{1/2}\mathbf{H}_o^T\mathbf{R}_s^{-1}\mathbf{H}_o\left(\mathbf{1}_{N\times 1} \otimes I_{2\times 2}\right) \\
&= \mathbf{Q}_s^{1/2}\mathbf{H}_o^T\mathbf{R}_s^{-1}\left(H \otimes I_{2\times 2}\right)\left(\mathbf{1}_{N\times 1} \otimes I_{2\times 2}\right)
\end{aligned}$$

But employing the properties of the Kronecker product yields

$$\left(H \otimes I_{2\times 2}\right)\left(\mathbf{1}_{N\times 1} \otimes I_{2\times 2}\right) = \left(H\mathbf{1}_{N\times 1}\right) \otimes I_{2\times 2} = \mathbf{0}_{2N\times 2}$$

and therefore

$$\mathbf{C}_s \left(\mathbf{Q}_s^{-1/2}\mathbf{1}_{N\times 1} \otimes I_{2\times 2}\right) = \mathbf{0}_{2N\times 2}$$

The columns of the matrix $\mathbf{Q}_s^{-1/2}\mathbf{1}_{N\times 1} \otimes I_{2\times 2}$ are

$$c_1 = \mathbf{Q}_s^{-1/2}\begin{bmatrix}1\\0\\1\\0\\\vdots\end{bmatrix}$$

and

$$c_1 = \mathbf{Q}_s^{-1/2}\begin{bmatrix}0\\1\\0\\1\\\vdots\end{bmatrix}$$

which are orthogonal (this is easily verified by computing the dot product $c_1^T c_2$). Therefore, a basis for the nullspace of $\mathbf{C}_s$ is given by the vectors

$$\mathbf{U}_{2N-1} = \frac{c_1}{||c_1||} = \sqrt{q_{s_T}}\mathbf{Q}_s^{-1/2}\begin{bmatrix}1\\0\\1\\0\\\vdots\end{bmatrix} \tag{176}$$

and

$$\mathbf{U}_{2N} = \frac{c_2}{||c_2||} = \sqrt{q_{s_T}}\mathbf{Q}_s^{-1/2}\begin{bmatrix}0\\1\\0\\1\\\vdots\end{bmatrix} \tag{177}$$

where, under the assumption that $\mathbf{Q}_s$ is diagonal with diagonal elements q_{s_i} (which holds for all the cases of interest in this work), q_{s_T} is defined by the relation

$$\frac{1}{q_{s_T}} = \sum_{i=1}^{N} \frac{1}{q_{s_i}}$$

F On the Use of the Zero Eigenvalue

In Eq. (95) the inverse of the diagonal matrix of the eigenvalues of C appears, which, in the case of an non-observable system, does not exist, since the smallest eigenvalue is zero. Although this is wrong from a strict mathematical point of view, we argue here that this notation can be justified. In order to compute the submatrix element (2,1) of $e^{\mathcal{H}t}$ we have:

$$\begin{aligned} e^{\mathcal{H}t}(2,1) &= tI_{N\times N} + C\frac{t^3}{3!} + C^2\frac{t^5}{5!} + \cdots \\ &= tI_{N\times N} + U\Lambda U^T\frac{t^3}{3!} + U\Lambda^2 U^T\frac{t^5}{5!} + \cdots \\ &= U(tI_{N\times N} + \Lambda\frac{t^3}{3!} + \Lambda^2\frac{t^5}{5!} + \cdots)U^T \end{aligned} \tag{178}$$

The derivative with respect to time of the above expression is:

$$\begin{aligned} \left(e^{\mathcal{H}t}(2,1)\right) &= I_{N\times N} + C\frac{t^2}{2!} + C^2\frac{t^4}{4!} + \cdots \\ &= tI_{N\times N} + U\Lambda U^T\frac{t^2}{2!} + U\Lambda^2 U^T\frac{t^4}{4!} + \cdots \\ &= U(tI_{N\times N} + \Lambda\frac{t^2}{2!} + \Lambda^2\frac{t^4}{4!} + \cdots)U^T \\ &= \frac{1}{2}U\,\mathrm{diag}(e^{\lambda_i t} + e^{-\lambda_i t})U^T \\ &= \frac{1}{2}U\begin{bmatrix} \mathrm{diag}_{N-1}(e^{\lambda_i t} + e^{-\lambda_i t}) & \mathbf{0}_{1\times(N-1)} \\ \mathbf{0}_{(N-1)\times 1} & 2 \end{bmatrix}U^T \end{aligned}$$

In the last line, we have simply written out the diagonal matrix, in order to underline the fact that the last element is a constant. Integration of the above relation yields:

$$e^{\mathcal{H}t}(2,1) = \frac{1}{2}U\begin{bmatrix} \mathrm{diag}_{N-1}\left(\frac{e^{\lambda_i t} - e^{-\lambda_i t}}{\lambda_i}\right) & \mathbf{0}_{1\times(N-1)} \\ \mathbf{0}_{(N-1)\times 1} & 2t \end{bmatrix}U^T + c$$

where c is a constant matrix term resulting from the integration operation. In order to evaluate this constant term, we note from Eq. (178) that $e^{\mathcal{H}0}(2,1) = 0$, thus substitution in the above relation yields $c = 0$, and therefore

$$e^{\mathcal{H}t}(2,1) = \frac{1}{2}U\begin{bmatrix} \mathrm{diag}_{N-1}\left(\frac{e^{\lambda_i t} - e^{-\lambda_i t}}{\lambda_i}\right) & \mathbf{0}_{1\times(N-1)} \\ \mathbf{0}_{(N-1)\times 1} & 2t \end{bmatrix}U^T \tag{179}$$

This last relation is mathematically correct, since the term of the form $0/0$ that appears in Eq. (178) does not appear here. However, this expression in quite cumbersome, and its use will make the resulting formulas unappealing and difficult to understand. Since the notation in Eq. (178) is much simpler, we will use it, bearing at all times in mind that the true meaning of it is given by Eq. (179).

In this section the matrix that appears in the analysis of robots moving in 1D has been treated. However, it is clear that the 2D case can be treated in the same manner, and that the matrix

$$\frac{1}{2}\mathbf{U}\,\mathrm{diag}\left(\frac{e^{\boldsymbol{\lambda}_i t} - e^{-\boldsymbol{\lambda}_i t}}{\boldsymbol{\lambda}_i}\right)\mathbf{U}^T$$

that appears in the derivations, should be formally interpreted as

$$\frac{1}{2}\mathbf{U}\left[\begin{array}{cc} \mathrm{diag}_{2N-2}\left(\frac{e^{\boldsymbol{\lambda}_i t}-e^{-\boldsymbol{\lambda}_i t}}{\boldsymbol{\lambda}_i}\right) & \mathbf{0}_{(2N-2)\times 2)} \\ \mathbf{0}^T_{(2N-2)\times 2} & 2tI_{2\times 2} \end{array}\right]\mathbf{U}^T \tag{180}$$

G Relationship between continuous and discrete time position propagation model

The discrete-time motion equations for a robot moving in 2D are

$$\begin{aligned} x(k+1) &= x(k) + V(k)\delta t\cos(\phi(k)) \\ y(k+1) &= y(k) + V(k)\delta t\sin(\phi(k)) \end{aligned}$$

where $1/\delta t$ is the frequency at which odometry measurements are being processed. By linearizing these equations the error propagation equations in discrete time are readily derived:

$$\begin{aligned} \left[\begin{array}{c} \widetilde{x}(k+1) \\ \widetilde{y}(k+1) \end{array}\right] &= \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right]\left[\begin{array}{c} \widetilde{x}(k) \\ \widetilde{y}(k) \end{array}\right] + \left[\begin{array}{cc} \delta t\cos(\hat{\phi}(k)) & -V_m(k)\delta t\sin(\hat{\phi}(k)) \\ \delta t\sin(\hat{\phi}(k)) & V_m(k)\delta t\cos(\hat{\phi}(k)) \end{array}\right]\left[\begin{array}{c} w_{V_d}(k) \\ \widetilde{\phi}(k) \end{array}\right] \\ \Leftrightarrow \widetilde{X}(k+1) &= \Phi(k)\widetilde{X}(k) + G_d(k)W_d(k) \end{aligned}$$

The covariance matrix of the system noise is $G_d(k)Q_d(k)G_d^T(k)$ where Q_d is the covariance matrix of the discrete-time velocity and orientation measurements,

$$Q_d = E\{W_d(k)W_d^T(k)\} = \left[\begin{array}{cc} \sigma^2_{V_d} & 0 \\ 0 & \sigma^2_{\phi_d} \end{array}\right]$$

and therefore

$$G_d(k)Q_d(k)G_d^T(k) = \delta t^2\left[\begin{array}{cc} \sigma^2_{V_d}\cos^2(\hat{\phi}(k)) + \sigma^2_{\phi_d}V_m^2(k)\sin^2(\hat{\phi}(k)) & \left(\sigma^2_{V_d} - \sigma^2_{\phi_d}V_m^2(k)\right)\sin(\hat{\phi}(k))\cos(\hat{\phi}(k)) \\ \left(\sigma^2_{V_d} - \sigma^2_{\phi_d}V_m^2(k)\right)\sin(\hat{\phi}(k))\cos(\hat{\phi}(k)) & \sigma^2_{V_d}\sin^2(\hat{\phi}(k)) + \sigma^2_{\phi_d}V_m^2(k)\cos^2(\hat{\phi}(k)) \end{array}\right] \tag{181}$$

The matrix $G_d(k)Q_d(k)G_d^T(k)$ represents the influx of uncertainty to the system over one sampling period, and therefore in order to create an equivalent continuous time system, the following must hold:

$$\begin{aligned} G_d(k)Q_d(k)G_d^T(k) &= \int_{t_k}^{t_{k+1}}\int_{t_k}^{t_{k+1}} G_c(\tau)E\{W_c(t)W_c^T(\tau)\}G_c^T(\tau)d\tau dt \\ &= \int_{t_k}^{t_{k+1}} G_c(\tau)\left[\begin{array}{cc} \sigma^2_{V_c} & 0 \\ 0 & \sigma^2_{\phi_c} \end{array}\right]G_c^T(\tau)d\tau \\ &= \delta t\left[\begin{array}{cc} \sigma^2_{V_c}\cos^2(\hat{\phi}(t)) + \sigma^2_{\phi_c}V_m^2(t)\sin^2(\hat{\phi}(t)) & \left(\sigma^2_{V_c} - \sigma^2_{\phi_c}V_m^2(t)\right)\sin(\hat{\phi}(t))\cos(\hat{\phi}(t)) \\ \left(\sigma^2_{V_c} - \sigma^2_{\phi_c}V_m^2(t)\right)\sin(\hat{\phi}(t))\cos(\hat{\phi}(t)) & \sigma^2_{V_c}\sin^2(\hat{\phi}(t)) + \sigma^2_{\phi_c}V_m^2(t)\cos^2(\hat{\phi}(t)) \end{array}\right] \end{aligned}$$

where $\sigma^2_{V_c}$ and $\sigma^2_{\phi_c}$ are the variances of the velocity measurements and orientation estimates of the equivalent continuous time system, respectively. By comparison of the last expression with the expression in Eq. (181), the expressions for defining the variance of the noise in the equivalent continuous time system follow:

$$\sigma_{V_c} = \sqrt{\delta t}\sigma_{V_d}, \text{ and } \sigma_{\phi_c} = \sqrt{\delta t}\sigma_{\phi_d} \tag{182}$$

H Matrix Inversion Lemma

If A is $n\times n$, B is $n\times m$, C is $m\times m$ and D is $m\times n$ then:

$$(A^{-1} + BC^{-1}D)^{-1} = A - AB(DAB+C)^{-1}DA \tag{183}$$

I Matrix Monotonicity of M_N

In this appendix we show that the matrix

$$M = V^T \mathbf{X} \left(I_{2N\times 2N} + h(\mathbf{C}_s)\mathbf{X}\right)^{-1} V \tag{184}$$

is matrix increasing in the argument $\mathbf{X}$, i.e.,

$$\mathbf{X}' \succeq \mathbf{X} \Rightarrow M' \succeq M \tag{185}$$

We note that if $\mathbf{X}$ is invertible (which is the case of interest), then

$$M = V^T \left(\mathbf{X}^{-1} + h(\mathbf{C}_s)\right)^{-1} V \tag{186}$$

And from the last relation it follows that

$$\begin{aligned}
\mathbf{X}' &\succeq \mathbf{X} \Rightarrow \\
\mathbf{X}'^{-1} &\preceq \mathbf{X}^{-1} \Rightarrow \\
\mathbf{X}'^{-1} + h(\mathbf{C}_s) &\preceq \mathbf{X}^{-1} + h(\mathbf{C}_s) \Rightarrow \\
\left(\mathbf{X}'^{-1} + h(\mathbf{C}_s)\right)^{-1} &\succeq \left(\mathbf{X}^{-1} + h(\mathbf{C}_s)\right)^{-1} \Rightarrow \\
V^T \left(\mathbf{X}'^{-1} + h(\mathbf{C}_s)\right)^{-1} V &\succeq V^T \left(\mathbf{X}^{-1} + h(\mathbf{C}_s)\right)^{-1} V \Rightarrow \\
M' &\succeq M
\end{aligned}$$

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