Performance Bounds for Cooperative Simultaneous Localization and Mapping (C-SLAM)

Anastasios I. Mourikis and Stergios I. Roumeliotis
{mourikis|stergios}@cs.umn.edu

Dept. of Computer Science & Engineering
University of Minnesota
Minneapolis, MN 55455

Center for
Distributed Robotics

Technical Report
Number -2004-0004
September 2004

Dept. of Computer Science & Engineering
University of Minnesota
4-192 EE/CS Building
200 Union St. S.E.
Minneapolis, MN 55455
Tel: (612) 625-2217
Fax: (612) 625-0572
URL: http://www.cs.umn.edu/˜mourikis

Performance Bounds for Cooperative Simultaneous Localization and Mapping (C-SLAM)

Anastasios I. Mourikis and Stergios I. Roumeliotis
{mourikis|stergios}@cs.umn.edu

Dept. of Computer Science & Engineering
University of Minnesota
Minneapolis, MN 55455

Robotics & Autonomous Systems Laboratory, TR-2004-0004
September 2004

Abstract

In this Technical Report we study the time evolution of the position estimates' covariance in Cooperative Simultaneous Localization and Mapping (C-SLAM), and obtain *analytical upper bounds* for the positioning uncertainty. The derived bounds provide descriptions of the asymptotic positioning performance of a team of robots in a mapping task, as a function of the characteristics of the proprioceptive and exteroceptive sensors of the robots, and of the graph of relative position measurements recorded by the robots. A study of the properties of the Riccati recursion which describes the propagation of uncertainty through time, yields (i) the *guaranteed accuracy* for a robot team in a given C-SLAM application, as well as (ii) the maximum *expected* steady state uncertainty of the robots and landmarks, when the spatial distribution of features in the environment can be modeled by a known distribution.

1 Introduction

In order for a multirobot team to coordinate while navigating autonomously within an area, all robots must be able to determine their positions with respect to a common frame of reference. In an ideal scenario, each robot would have direct access to measurements of its absolute position, such as those provided by a GPS receiver, or those inferred by detecting previously mapped features. However, reliance on GPS is not feasible in a number of situations, since GPS signals are not available everywhere (e.g., indoors), or, triangulation techniques based on them may provide erroneous results due to multiple reflections (e.g., in the vicinity of tall structures and buildings). Moreover, compiling a detailed map of the environment is a tedious and time consuming process, while numerous applications require robots to operate in unknown surroundings, whose structure cannot be determined in advance.

In situations where absolute position information is not available, the robots of a team can improve their localization accuracy by recording robot-to-robot relative position measurements, and processing them in order to update their position estimates [1, 2, 3]. This method results in a substantial improvement in estimation accuracy compared to simple Dead-Reckoning localization schemes. However performing Cooperative Localization (CL) solely based on relative position measurements has the limitation that the uncertainty of the robots' position estimates continuously increases, and the attained accuracy may not be sufficient for certain applications. An alternative approach is for the robots to localize while concurrently building a map of the environment, in which case the uncertainty in their position estimates remains bounded [4]. This introduces the problem of Cooperative Simultaneous Localization And Mapping (C-SLAM) that has recently attracted the interest of many researchers.

In this Technical Report we study the time evolution of the position estimates' covariance in C-SLAM and obtain *analytical upper bounds* for the positioning uncertainty. A study of the properties of the Riccati recursion which describes the propagation of uncertainty through time, yields (i) the *guaranteed accuracy* for a robot team in a given C-SLAM application, as well as (ii) the maximum *expected* steady state uncertainty of the robots and landmarks, when the spatial distribution of features in the environment can be modeled by a known distribution. In the next section the problem formulation is presented, and in Section 3 the Riccati recursion is formulated, and the aforementioned bounds for its steady state solution are derived.

2 Problem Formulation

Consider a group of M mobile robots, denoted as r_1, r_2, ..., r_M, moving on a planar surface, in an environment that contains N landmarks, denoted as L_1, L_2, ..., L_N. The robots use proprioceptive measurements (e.g., from odometric or inertial sensors) to propagate their state (position) estimates, and are equipped with exteroceptive sensors (e.g., laser range finders) that enable them to measure the relative position of other robots and landmarks. All the measurements are fused using an Extended Kalman Filter (EKF) in order to produce estimates of the position of the robots and the landmarks. In our formulation, it is assumed that an upper bound for the variance of the errors in the robots' orientation estimates can be determined a priori. This allows us to decouple the task of position estimation from that of orientation estimation and facilitates the derivation of an analytical upper bound on the positioning uncertainty.

The robots' orientation uncertainty is bounded when, for example, absolute orientation measurements from a compass or sun sensor are available, or when the perpendicularity of the walls in an indoor environment is used to infer orientation. In cases where neither approach is possible, our analysis still holds under the condition that a conservative upper bound for the orientation uncertainty of each robot is determined by alternative means, e.g., by estimating the maximum orientation error accumulated, over a certain period of time, due to the integration of noise in the odometric measurements [5]. It should be noted that the requirement for bounded orientation error covariance is not too restrictive: In the EKF framework, the nonlinear state propagation and measurement equations are linearized around the estimates of the robots' orientation. If the errors in these estimates are allowed to increase unbounded, the linearization will unavoidably become erroneous and the estimates will diverge. Furthermore, large errors in the estimates for the robots' orientation in SLAM result in erroneous data association, that may have detrimental effects on the filter stability. Thus, in the vast majority of practical situations, provisions are made in order to constrain the robots' orientation uncertainty within given limits.

In this work, C-SLAM is considered within the *Stochastic Mapping* framework [6], [7]. We assume that the mobile robots move randomly in a planar environment, while recording measurements of the relative positions (i.e., range and bearing) of other robots in the team, and of static point landmarks that exist in the environment. A means of describing the exteroceptive measurements that are recorded at each time step is the associated *Relative Position Measurement Graph* (RPMG), i.e., the graph whose vertices represent the robots and landmarks, while its directed edges correspond to the *robot-to-robot* and *robot-to-landmark* measurements. We impose the constraint that the RPMG is a *connected* graph, i.e., that there exists a path between any two of its nodes. This constraint arises naturally and is not a restrictive one, since if an RPMG is not connected, then it can always be decomposed into smaller, connected sub-graphs. Each of these sub-graphs corresponds to an isolated group of robots and/or landmarks, whose position estimation problem can be studied independently.

In our formulation, the metric employed for describing the accuracy of position estimation in C-SLAM is the covariance matrix of the position estimates. It is well known that the time evolution of the covariance matrix in the EKF framework is described by the propagation and update equations (cf. Eqs. (9) and (24)). Combining these equations yields the Riccati recursion (cf. Eq. (34)), whose solution is the covariance of the error in the state estimate at each time step, right after the propagation phase of the EKF. In the case of C-SLAM, the matrix coefficients in this recursion are time varying and a general closed form expression for the time evolution of the covariance matrix does not exist. We thus resort to deriving *upper bounds* for the covariance, by exploiting the convexity and monotonicity properties of the Riccati recursion (cf. Lemmas 3.1 and 3.2). These properties allow for the formulation of *constant coefficient* Riccati recursions, whose solutions provide upper bounds for the positioning uncertainty in C-SLAM.

2.1 Position propagation

The discrete-time kinematic equations for the i-th robot are

$$x_{r_i}(k+1) = x_{r_i}(k) + V_i(k)\delta t \cos(\phi_i(k)) \quad (1)$$

$$y_{r_i}(k+1) = y_{r_i}(k) + V_i(k)\delta t \sin(\phi_i(k)) \quad (2)$$

where $V_i(k)$ denotes the robot's translational velocity at time k and δt is the sampling period. In the Kalman filter framework, the estimates of the robot's position are propagated using the measurements of the robot's velocity, $V_{m_i}(k)$, and the estimates of the robot's orientation, $\hat{\phi}_i(k)$:

$$
\begin{aligned}
\hat{x}_{r_{i_{k+1|k}}} &= \hat{x}_{r_{i_{k|k}}} + V_{m_i}(k)\delta t \cos(\hat{\phi}_i(k)) \\
\hat{y}_{r_{i_{k+1|k}}} &= \hat{y}_{r_{i_{k|k}}} + V_{m_i}(k)\delta t \sin(\hat{\phi}_i(k))
\end{aligned}
$$

Clearly, these equations are time varying and nonlinear due to the dependence on the robot's orientation. By linearizing Eqs. (1) and (2), the error propagation equation for the robot's position is readily derived:

$$
\begin{bmatrix} \widetilde{x}_{r_{i_{k+1|k}}} \\ \widetilde{y}_{r_{i_{k+1|k}}} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \widetilde{x}_{r_{i_{k|k}}} \\ \widetilde{y}_{r_{i_{k|k}}} \end{bmatrix} + \begin{bmatrix} \delta t \cos(\hat{\phi}_i(k)) & -V_{m_i}(k)\delta t \sin(\hat{\phi}_i(k)) \\ \delta t \sin(\hat{\phi}_i(k)) & V_{m_i}(k)\delta t \cos(\hat{\phi}_i(k)) \end{bmatrix} \begin{bmatrix} w_{V_i}(k) \\ \widetilde{\phi}_i(k) \end{bmatrix}
$$
$$
\Leftrightarrow \widetilde{X}_{r_{i_{k+1|k}}} = I_{2\times 2} \; \widetilde{X}_{r_{i_{k|k}}} + G_{r_i}(k) \; W_i(k) \tag{3}
$$

where[1] $w_{V_i}(k)$ is a zero-mean white Gaussian noise sequence of variance $\sigma^2_{V_i}$, affecting the velocity measurements and $\widetilde{\phi}_i(k)$ is the error in the robot's orientation estimate at time k. This is modeled as a zero-mean white Gaussian noise sequence of variance $\sigma^2_{\phi_i}$.

From Eq. (3), we deduce that the covariance matrix of the system noise affecting the i-th robot is:

$$
\begin{aligned}
Q_{r_i}(k) &= E\{G_{r_i}(k)W_i(k)W_i^T(k)G_{r_i}^T(k)\} \\
&= G_{r_i}(k)E\{W_i(k)W_i^T(k)\}G_{r_i}^T(k) \\
&= \begin{bmatrix} \delta t \cos(\hat{\phi}_i(k)) & -V_{m_i}(k)\delta t \sin(\hat{\phi}_i(k)) \\ \delta t \sin(\hat{\phi}_i(k)) & V_{m_i}(k)\delta t \cos(\hat{\phi}_i(k)) \end{bmatrix} \begin{bmatrix} \sigma^2_{V_i} & 0 \\ 0 & \sigma^2_{\phi_i} \end{bmatrix} \begin{bmatrix} \delta t \cos(\hat{\phi}_i(k)) & -V_{m_i}(k)\delta t \sin(\hat{\phi}_i(k)) \\ \delta t \sin(\hat{\phi}_i(k)) & V_{m_i}(k)\delta t \cos(\hat{\phi}_i(k)) \end{bmatrix}^T \\
&= \begin{bmatrix} \cos(\hat{\phi}_i(k)) & -\sin(\hat{\phi}_i(k)) \\ \sin(\hat{\phi}_i(k)) & \cos(\hat{\phi}_i(k)) \end{bmatrix} \begin{bmatrix} \delta t^2 \sigma^2_{V_i} & 0 \\ 0 & \delta t^2 V^2_{m_i}(k)\sigma^2_{\phi_i} \end{bmatrix} \begin{bmatrix} \cos(\hat{\phi}_i(k)) & -\delta t \sin(\hat{\phi}_i(k)) \\ \sin(\hat{\phi}_i(k)) & \delta t \cos(\hat{\phi}_i(k)) \end{bmatrix}^T \\
&= C(\hat{\phi}_i(k)) \begin{bmatrix} \delta t^2 \sigma^2_{V_i} & 0 \\ 0 & \delta t^2 V^2_m(k)\sigma^2_{\phi_i} \end{bmatrix} C^T(\hat{\phi}_i(k))
\end{aligned} \tag{4}
$$

where $C(\hat{\phi}_i)$ denotes the rotation matrix associated with $\hat{\phi}_i$.

The landmarks are modeled as static points in 2D space, and therefore the state propagation equations are

$$
X_{L_i}(k+1) = X_{L_i}(k), \quad \text{for} \;\; i = 1 \dots N
$$

Hence, the estimates for the landmark positions are propagated using the relations

$$
\hat{X}_{L_i k+1|k} = \hat{X}_{L_i k|k}, \quad \text{for} \;\; i = 1 \dots N
$$

while the errors are propagated by

$$
\widetilde{X}_{L_i k+1|k} = \widetilde{X}_{L_i k|k}, \quad \text{for} \;\; i = 1 \dots N
$$

Using these results we can now write the error propagation equations for the entire system, comprising of M robots and N landmarks:

$$
\widetilde{X}_{k+1|k} = I_{\xi\times\xi}\widetilde{X}_{k|k} + \begin{bmatrix} G_{r_1}(k) & \mathbf{0}_{2\times 2} & \cdots & \mathbf{0}_{2\times 2} \\ \mathbf{0}_{2\times 2} & G_{r_2}(k) & \cdots & \mathbf{0}_{2\times 2} \\ & & \ddots & \\ \mathbf{0}_{2\times 2} & & & G_{r_M}(k) \\ & \mathbf{0}_{2N\times 2M} & & \end{bmatrix} \begin{bmatrix} w_{V_1}(k) \\ \widetilde{\phi}_1(k) \\ w_{V_2}(k) \\ \widetilde{\phi}_2(k) \\ \vdots \\ w_{V_M}(k) \\ \widetilde{\phi}_M(k) \end{bmatrix}
$$
$$
\Leftrightarrow \widetilde{X}_{k+1|k} = \mathbf{\Phi}(k)\widetilde{X}_{k|k} + \mathbf{G}_t(k) \; \mathbf{W}(k) \tag{5}
$$

[1] Throughout this paper, $\mathbf{0}_{m\times n}$ denotes the $m \times n$ matrix of zeros, $\mathbf{1}_{m\times n}$ denotes the $m \times n$ matrix of ones, and $I_{n\times n}$ denotes the $n \times n$ identity matrix.

where $\xi = 2M + 2N$ is the size of the state vector of the entire system, defined as the stacked vector comprising of the positions of the robots and landmarks, i.e.,

$$X = \begin{bmatrix} X_{r_1} \\ \vdots \\ X_{r_M} \\ X_{L_1} \\ \vdots \\ X_{L_N} \end{bmatrix}$$

The covariance matrix of the system noise is given by

$$\begin{aligned}
\mathbf{Q}(k) &= E\{\mathbf{G}_t(k)\mathbf{W}(k)\mathbf{W}^T(k)\mathbf{G}_t^T(k)\} \\
&= \begin{bmatrix} E\{G_1(k)W_1(k)W_1^T(k)G_1^T(k)\} & \cdots & \mathbf{0}_{2\times 2} & \\ \vdots & \ddots & \vdots & \mathbf{0}_{2M\times 2N} \\ \mathbf{0}_{2\times 2} & \cdots & E\{G_M(k)W_M(k)W_M^T(k)G_M^T(k)\} & \\ & \mathbf{0}_{2N\times 2M} & & \mathbf{0}_{2N\times 2N} \end{bmatrix} \\
&= \begin{bmatrix} Q_{r_1}(k) & \cdots & \mathbf{0}_{2\times 2} & \\ \vdots & \ddots & \vdots & \mathbf{0}_{2M\times 2N} \\ \mathbf{0}_{2\times 2} & \cdots & Q_{r_M}(k) & \\ & \mathbf{0}_{2N\times 2M} & & \mathbf{0}_{2N\times 2N} \end{bmatrix} \\
&= \mathbf{G}_o\mathbf{Q}_r(k)\mathbf{G}_o^T
\end{aligned} \tag{6}$$

where

$$\mathbf{G}_o = \begin{bmatrix} I_{2M\times 2M} \\ \mathbf{0}_{2N\times 2M} \end{bmatrix} = G_o \otimes I_{2\times 2}, \quad \text{with} \quad G_o = \begin{bmatrix} I_{M\times M} \\ \mathbf{0}_{N\times M} \end{bmatrix} \tag{7}$$

and

$$\mathbf{Q}_r(k) = \mathbf{Diag}\left(Q_{r_i}(k)\right) \tag{8}$$

i.e., $\mathbf{Q}_r(k)$ is a block diagonal matrix with elements $Q_{r_i}(k)$, $i = 1 \ldots M$. Thus the equation for propagating the covariance matrix of the state error is written as

$$\mathbf{P}_{k+1|k} = \mathbf{P}_{k|k} + \mathbf{G}_o\mathbf{Q}_r(k)\mathbf{G}_o^T \tag{9}$$

where $\mathbf{P}_{k+1|k} = E\{\widetilde{X}_{k+1|k}\widetilde{X}_{k+1|k}^T\}$ and $\mathbf{P}_{k|k} = E\{\widetilde{X}_{k|k}\widetilde{X}_{k|k}^T\}$ are the covariance of the error in the estimate of $X(k+1)$ and $X(k)$ respectively, after measurements up to time k have been processed.

2.2 Measurement Model

At every time step, the robots perform robot-to-robot and robot-to-landmark relative position measurements. The relative position measurement between robots r_i and r_m is given by:

$$z_{r_i r_m} = C^T(\phi_i)\left(X_{r_m} - X_{r_i}\right) + n_{z_{r_i r_m}} \tag{10}$$

where r_i (r_m) is the observing (observed) robot, and $n_{z_{r_i r_m}}$ is the noise affecting this measurement. Similarly, the measurement of the relative position between r_i and L_n is given by:

$$z_{r_i L_n} = C^T(\phi_i)\left(X_{L_n} - X_{r_i}\right) + n_{z_{r_i L_n}} \tag{11}$$

The similarity of the preceding two measurement equations allows us to treat both types of measurements in a uniform manner. We denote by T_{ij} the target of the j-th measurement performed by robot i, i.e.,

$$T_{ij} \in \{r_1, r_2, \cdots, r_M, L_1, L_2, \cdots, L_N\} \setminus \{r_i\}$$

Thus, the general form of the relative position measurement equation is:

$$z_{ij} = C^T(\phi_i)\left(X_{T_{ij}} - X_{r_i}\right) + n_{z_{ij}} \tag{12}$$

Assuming that the i-th robot performs M_i relative position measurements, the index j assumes integer values in the range $[1, M_i]$ to describe these measurements. By linearizing the last expression, the measurement error equation is obtained:

$$\begin{aligned}
\widetilde{z}_{ij}(k+1) &= z_{ij}(k+1) - \hat{z}_{ij}(k+1) \\
&= C^T(\hat{\phi}_i(k+1))\left(\widetilde{X}_{T_{ij}\ k+1|k} - \widetilde{X}_{r_i\ k+1|k}\right) - C^T(\hat{\phi}_i(k+1))J\left(\hat{X}_{T_{ij}\ k+1|k} - \hat{X}_{r_i\ k+1|k}\right)\widetilde{\phi}_i(k+1) + n_{z_{ij}}(k+1) \\
&= C^T(\hat{\phi}_i(k+1))\begin{bmatrix} 0_{2\times2} & \cdots & \underbrace{-I_{2\times2}}_{r_i} & \cdots & \underbrace{I_{2\times2}}_{T_{ij}} & \cdots & 0_{2\times2}\end{bmatrix}\begin{bmatrix} \vdots \\ \widetilde{X}_{r_i} \\ \vdots \\ \widetilde{X}_{T_{ij}} \\ \vdots \end{bmatrix}_{k+1|k} \\
&+ \begin{bmatrix} I_{2\times2} & -C^T(\hat{\phi}_i(k+1))J\widehat{\Delta p}_{ij\ k+1|k}\end{bmatrix}\begin{bmatrix} n_{z_{ij}}(k+1) \\ \widetilde{\phi}_i(k+1)\end{bmatrix} \\
&= H_{ij}(k+1)\widetilde{X}_{k+1|k} + \Gamma_{ij}(k+1)n_{ij}(k+1)
\end{aligned} \tag{13}$$

where

$$J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad \widehat{\Delta p}_{ij_{k+1|k}} = \hat{X}_{T_{ij}\ k+1|k} - \hat{X}_{r_i\ k+1|k}$$

and we note that the measurement matrix for this relative position measurement can be written as

$$H_{ij}(k+1) = C^T(\hat{\phi}_i(k+1))\begin{bmatrix} 0_{2\times2} & \cdots & \underbrace{-I_{2\times2}}_{r_i} & \cdots & \underbrace{I_{2\times2}}_{T_{ij}} & \cdots & 0_{2\times2}\end{bmatrix} = C^T(\hat{\phi}_i(k+1))H_{o_{ij}} \tag{14}$$

At each time instant robot i records M_i relative position measurements, described by the measurement matrix $\boldsymbol{H}_i(k+1)$, i.e., a matrix whose block rows are $H_{ij}(k+1)$, $j = 1 \ldots M_i$, i.e.:

$$\boldsymbol{H}_i(k+1) = \begin{bmatrix} C^T(\hat{\phi}_i(k+1))H_{o_{i1}} \\ C^T(\hat{\phi}_i(k+1))H_{o_{i2}} \\ \vdots \\ C^T(\hat{\phi}_i(k+1))H_{o_{iM_i}} \end{bmatrix} = \boldsymbol{\Xi}^T_{\hat{\phi}_i}(k+1)\boldsymbol{H}_{o_i} \tag{15}$$

in the last expression $\boldsymbol{H}_{o_i}$ is a constant matrix whose block rows are $H_{o_{ij}}$, $j = 1 \ldots M_i$, and $\boldsymbol{\Xi}_{\hat{\phi}_i}(k+1) = I_{M_i \times M_i} \otimes C(\hat{\phi}_i(k+1))$, with $\otimes$ denoting the Kronecker matrix product. The covariance for the error of the j-th measurement of robot i is given by

$$\begin{aligned}
{}^iR_{jj}(k+1) &= \Gamma_{ij}(k+1)E\{n_{ij}(k+1)n^T_{ij}(k+1)\}\Gamma^T_{ij}(k+1) \\
&= R_{z_{ij}}(k+1) + R_{\tilde{\phi}_{ij}}(k+1)
\end{aligned} \tag{16}$$

This expression encapsulates all sources of noise and uncertainty that contribute to the measurement error $\widetilde{z}_{ij}(k+1)$. More specifically, $R_{z_{ij}}(k+1)$ is the covariance of the noise $n_{ij}(k+1)$ in the recorded relative position measurement $z_{ij}(k+1)$ and $R_{\tilde{\phi}_{ij}}(k+1)$ is the additional covariance term due to the error $\tilde{\phi}_i(k+1)$ in the orientation estimate of the measuring robot. This is given by:

$$\begin{aligned}
R_{\tilde{\phi}_{ij}}(k+1) &= C^T(\hat{\phi}_i(k+1))J\widehat{\Delta p}_{ij_{k+1|k}}E\{\tilde{\phi}_i^{\,2}\}\widehat{\Delta p}^T_{ij_{k+1|k}}J^TC(\hat{\phi}_i(k+1)) \\
&= \sigma^2_{\phi_i}C^T(\hat{\phi}_i(k+1))J\widehat{\Delta p}_{ij_{k+1|k}}\widehat{\Delta p}^T_{ij_{k+1|k}}J^TC(\hat{\phi}_i(k+1))
\end{aligned} \tag{17}$$

From this expression we conclude that the uncertainty $\sigma^2_{\phi_i}$ in the orientation estimate $\hat{\phi}_i(k+1)$ of the robot is amplified by the distance between the robot and corresponding landmark.

Each relative position measurement is comprised of the distance ρ_{ij} and bearing θ_{ij} to the target, expressed in the measuring robot's local coordinate frame, i.e.,

$$z_{ij}(k+1) \quad = \quad \left[\begin{array}{c} \rho_{ij}(k+1)\cos\theta_{ij}(k+1) \\ \rho_{ij}(k+1)\sin\theta_{ij}(k+1) \end{array} \right] + n_{z_{ij}}(k+1)$$

By linearizing, the noise in this measurement can be expressed as:

$$n_{z_{ij}}(k+1) \simeq \left[\begin{array}{cc} \cos\hat{\theta}_{ij} & -\hat{\rho}_{ij}\sin\hat{\theta}_{ij} \\ \sin\hat{\theta}_{ij} & \hat{\rho}_{ij}\cos\hat{\theta}_{ij} \end{array} \right] \left[\begin{array}{c} n_{\rho_{ij}}(k+1) \\ n_{\theta_{ij}}(k+1) \end{array} \right]$$

where $n_{\rho_{ij}}$ is the error in the range measurement, $n_{\theta_{ij}}$ is the error in the bearing measurement, assumed to be independent white zero-mean Gaussian sequences, and

$$\begin{aligned} \hat{\rho}^2_{ij} &= \widehat{\Delta p}^T_{ij_{k+1|k}} \widehat{\Delta p}_{ij_{k+1|k}} \\ \hat{\theta}_{ij} &= \mathrm{Atan2}(\widehat{\Delta y}_{ij_{k+1|k}}, \widehat{\Delta x}_{ij_{k+1|k}}) - \hat{\phi}_i(k+1) \end{aligned}$$

are the estimates of the range and bearing to the landmark, expressed with respect to the robot's coordinate frame. At this point we note that

$$\begin{aligned} C(\hat{\phi}_i(k+1))n_{z_{ij}}(k+1) &= \left[\begin{array}{cc} \cos\hat{\phi}_i(k+1) & -\sin\hat{\phi}_i(k+1) \\ \sin\hat{\phi}_i(k+1) & \cos\hat{\phi}_i(k+1) \end{array} \right] \left[\begin{array}{cc} \cos\hat{\theta}_{ij} & -\hat{\rho}_{ij}\sin\hat{\theta}_{ij} \\ \sin\hat{\theta}_{ij} & \hat{\rho}_{ij}\cos\hat{\theta}_{ij} \end{array} \right] \left[\begin{array}{c} n_{\rho_{ij}}(k+1) \\ n_{\theta_{ij}}(k+1) \end{array} \right] \\ &= \left[\begin{array}{cc} \cos(\hat{\phi}_i(k+1)+\hat{\theta}_{ij}) & -\hat{\rho}_{ij}\sin(\hat{\phi}_i(k+1)+\hat{\theta}_{ij}) \\ \sin(\hat{\phi}_i(k+1)+\hat{\theta}_{ij}) & \hat{\rho}_{ij}\cos(\hat{\phi}_i(k+1)+\hat{\theta}_{ij}) \end{array} \right] \left[\begin{array}{c} n_{\rho_{ij}}(k+1) \\ n_{\theta_{ij}}(k+1) \end{array} \right] \\ &= \left[\begin{array}{cc} \frac{1}{\hat{\rho}_{ij}}\widehat{\Delta p}_{ij} & J\widehat{\Delta p}_{ij} \end{array} \right] \left[\begin{array}{c} n_{\rho_{ij}}(k+1) \\ n_{\theta_{ij}}(k+1) \end{array} \right] \end{aligned}$$

and therefore the quantity $R_{z_{ij}}(k+1)$ can be written as:

$$\begin{aligned} R_{z_{ij}}(k+1) &= E\{n_{z_{ij}}(k+1)n^T_{z_{ij}}(k+1)\} \\ &= C^T(\hat{\phi}_i(k+1)) \left[\begin{array}{cc} \frac{1}{\hat{\rho}_{ij}}\widehat{\Delta p}_{ij} & J\widehat{\Delta p}_{ij} \end{array} \right] E\left\{ \left[\begin{array}{c} n_{\rho_{ij}} \\ n_{\theta_{ij}} \end{array} \right] \left[\begin{array}{c} n_{\rho_{ij}} \\ n_{\theta_{ij}} \end{array} \right]^T \right\} \left[\begin{array}{cc} \frac{1}{\hat{\rho}_{ij}}\widehat{\Delta p}_{ij} & J\widehat{\Delta p}_{ij} \end{array} \right]^T C(\hat{\phi}_i(k+1)) \\ &= C^T(\hat{\phi}_i(k+1)) \left[\begin{array}{cc} \frac{1}{\hat{\rho}_{ij}}\widehat{\Delta p}_{ij} & J\widehat{\Delta p}_{ij} \end{array} \right] \left[\begin{array}{cc} \sigma^2_{\rho_i} & 0 \\ 0 & \sigma^2_{\theta_i} \end{array} \right] \left[\begin{array}{cc} \frac{1}{\hat{\rho}_{ij}}\widehat{\Delta p}_{ij} & J\widehat{\Delta p}_{ij} \end{array} \right]^T C(\hat{\phi}_i(k+1)) \\ &= C^T(\hat{\phi}_i(k+1)) \left(\frac{\sigma^2_{\rho_i}}{\hat{\rho}^2_{ij}} \widehat{\Delta p}_{ij}\widehat{\Delta p}^T_{ij} + \sigma^2_{\theta_i} J\widehat{\Delta p}_{ij}\widehat{\Delta p}^T_{ij}J^T \right) C(\hat{\phi}_i(k+1)) \\ &= C^T(\hat{\phi}_i(k+1)) \left(\frac{\sigma^2_{\rho_i}}{\hat{\rho}^2_{ij}} \left(\hat{\rho}^2_{ij} I_{2\times 2} - J\widehat{\Delta p}_{ij}\widehat{\Delta p}^T_{ij}J^T \right) + \sigma^2_{\theta_i} J\widehat{\Delta p}_{ij}\widehat{\Delta p}^T_{ij}J^T \right) C(\hat{\phi}_i(k+1)) \\ &= C^T(\hat{\phi}_i(k+1)) \left(\sigma^2_{\rho_i} I_{2\times 2} + \left(\sigma^2_{\theta_i} - \frac{\sigma^2_{\rho_i}}{\hat{\rho}^2_{ij}} \right) J\widehat{\Delta p}_{ij}\widehat{\Delta p}^T_{ij}J^T \right) C(\hat{\phi}_i(k+1)) \end{aligned} \tag{18}$$

where the variance of the noise in the distance and bearing measurements is given by

$$\sigma^2_{\rho_i} = E\{n^2_{\rho_i}\} \;\; , \;\; \sigma^2_{\theta_i} = E\{n^2_{\theta_i}\}$$

respectively. Due to the existence of the error component attributed to $\tilde{\phi}_i(k+1)$, the exteroceptive measurements that each robot performs at a given time instant are correlated. The matrix of correlation between the errors in the measurements $z_{ij}(k+1)$ and $z_{i\ell}(k+1)$ is

$$\begin{aligned} {}^iR_{j\ell}(k+1) &= \Gamma_{ij}(k)E\{n_{ij}(k+1)n^T_{i\ell}(k+1)\}\Gamma^T_{i\ell}(k) \\ &= \sigma^2_{\phi_i} C^T(\hat{\phi}_i(k+1)) J \widehat{\Delta p}_{ij_{k+1|k}} \widehat{\Delta p}^T_{i\ell_{k+1|k}} J^T C(\hat{\phi}_i(k+1)) \end{aligned} \tag{19}$$

The covariance matrix of all the measurements performed by robot i at the time instant $k+1$ can now be computed. This is a block matrix whose mn-th 2×2 submatrix element is ${}^{i}R_{mn}$, for $m, n = 1 \dots M_i$. Using the results of Eqs. (17), (18), and (19), this matrix can be written as

$$\boldsymbol{R}_i(k+1) = \boldsymbol{\Xi}^T_{\hat{\phi}_i}(k+1)\boldsymbol{R}_{o_i}(k+1)\boldsymbol{\Xi}_{\hat{\phi}_i}(k+1) \tag{20}$$

where

$$\begin{aligned}
\boldsymbol{R}_{o_i}(k+1) &= \begin{bmatrix} \sigma^2_{\rho_i} I_{2\times2} + \left(\sigma^2_{\phi_i} + \sigma^2_{\theta_i} - \frac{\sigma^2_{\rho_i}}{\hat{\rho}^2_{i1}}\right) J\widehat{\Delta p}_{i1}\widehat{\Delta p}^T_{i1}J^T & \dots & \sigma^2_{\phi_i} J\widehat{\Delta p}_{i1}\widehat{\Delta p}^T_{iM_i}J^T \\ \vdots & \ddots & \vdots \\ \sigma^2_{\phi_i} J\widehat{\Delta p}_{iM_i}\widehat{\Delta p}^T_{i1}J^T & \dots & \sigma^2_{\rho_i} I_{2\times2} + \left(\sigma^2_{\phi_i} + \sigma^2_{\theta_i} - \frac{\sigma^2_{\rho_i}}{\hat{\rho}^2_{iM_i}}\right) J\widehat{\Delta p}_{iM_i}\widehat{\Delta p}^T_{iM_i}J^T \end{bmatrix} \\
&= \sigma^2_{\rho_i} I_{2N\times2N} + D_i(k+1)\left(\sigma^2_{\theta_i} I_{N\times N} + \sigma^2_{\phi_i}\mathbf{1}_{N\times N} - \text{diag}\left(\frac{\sigma^2_{\rho_{ij}}}{\hat{\rho}^2_i}\right)\right) D^T_i(k+1) \\
&= \underbrace{\sigma^2_{\rho_i} I_{2N\times2N} - D_i(k+1)\,\text{diag}\left(\frac{\sigma^2_{\rho_{ij}}}{\hat{\rho}^2_i}\right) D^T_i(k+1)}_{R_1(k+1)} + \underbrace{\sigma^2_{\theta_i} D_i(k+1)D^T_i(k+1)}_{R_2(k+1)} + \underbrace{\sigma^2_{\phi_i} D_i(k+1)\mathbf{1}_{N\times N}D^T_i(k+1)}_{R_3(k+1)}
\end{aligned} \tag{21}$$

where

$$D_i(k+1) = \begin{bmatrix} J\widehat{\Delta p}_{i1_{k+1|k}} & \dots & 0_{2\times1} \\ \vdots & \ddots & \vdots \\ 0_{2\times1} & \dots & J\widehat{\Delta p}_{iM_{i_{k+1|k}}} \end{bmatrix} = \mathbf{Diag}\left(J\widehat{\Delta p}_{ij_{k+1|k}}\right)$$

is a $2M_1 \times M_i$ block diagonal matrix, depending on the estimated positions of the robots and landmarks. In Eq. (21) the covariance term $R_1(k+1)$ is the covariance of the error due to the noise in the range measurements, $R_2(k+1)$ is the covariance term due to the error in the bearing measurements, and $R_3(k+1)$ is the covariance term due to the error in the orientation estimates of the robot. The measurement matrix $\mathbf{H}(k+1)$ describing all the measurements that are performed by the robots at time step $k+1$ is a matrix with block rows $\boldsymbol{H}_i(k+1)$, $i = 1 \dots M$, i.e.,

$$\mathbf{H}(k+1) = \begin{bmatrix} \boldsymbol{\Xi}^T_{\hat{\phi}_1}(k+1)\boldsymbol{H}_{o_1} \\ \boldsymbol{\Xi}^T_{\hat{\phi}_2}(k+1)\boldsymbol{H}_{o_2} \\ \vdots \\ \boldsymbol{\Xi}^T_{\hat{\phi}_M}(k+1)\boldsymbol{H}_{o_M} \end{bmatrix} = \mathbf{Diag}\left(\boldsymbol{\Xi}^T_{\hat{\phi}_i}(k+1)\right)\begin{bmatrix} \boldsymbol{H}_{o_1} \\ \boldsymbol{H}_{o_2} \\ \vdots \\ \boldsymbol{H}_{o_M} \end{bmatrix} = \boldsymbol{\Xi}^T(k+1)\mathbf{H}_o \tag{22}$$

where $\boldsymbol{\Xi}(k+1) = \mathbf{Diag}\left(\boldsymbol{\Xi}_{\hat{\phi}_i}(k+1)\right)$ is a block diagonal matrix with block elements $\boldsymbol{\Xi}_{\hat{\phi}_i}(k+1)$, for $i = 1 \dots M$, and $\mathbf{H}_o$ is a matrix with block rows $\boldsymbol{H}_{o_i}$, $i = 1 \dots M$. Since the measurements performed by different robots are independent, the measurement coviarance matrix for the entire system is given by

$$\mathbf{R}(k+1) = \mathbf{Diag}\left(\boldsymbol{R}_i(k+1)\right) = \mathbf{Diag}\left(\boldsymbol{\Xi}^T_{\hat{\phi}_i}\boldsymbol{R}_{o_i}(k+1)\boldsymbol{\Xi}_{\hat{\phi}_i}\right) = \boldsymbol{\Xi}^T(k+1)\mathbf{R}_o(k+1)\boldsymbol{\Xi}(k+1) \tag{23}$$

where $\mathbf{R}_o$ is a block diagonal matrix with block elements $\boldsymbol{R}_{o_i}$, $i = 1 \dots M$.

We now write the covariance update equation, which is

$$\begin{aligned}
\mathbf{P}_{k+1|k+1} &= \mathbf{P}_{k+1|k} - \mathbf{P}_{k+1|k}\mathbf{H}^T(k+1)\left(\mathbf{H}(k+1)\mathbf{P}_{k+1|k}\mathbf{H}^T(k+1) + \mathbf{R}(k+1)\right)^{-1}\mathbf{H}(k+1)\mathbf{P}_{k+1|k} \\
&= \mathbf{P}_{k+1|k} \\
&\quad - \mathbf{P}_{k+1|k}\mathbf{H}^T_o\boldsymbol{\Xi}(k+1)\left(\boldsymbol{\Xi}^T(k+1)\mathbf{H}_o\mathbf{P}_{k+1|k}\mathbf{H}^T_o\boldsymbol{\Xi}(k+1) + \boldsymbol{\Xi}^T(k+1)\mathbf{R}_o(k+1)\boldsymbol{\Xi}(k+1)\right)^{-1}\boldsymbol{\Xi}^T(k+1)\mathbf{H}_o\mathbf{P}_{k+1|k} \\
&= \mathbf{P}_{k+1|k} - \mathbf{P}_{k+1|k}\mathbf{H}^T_o\left(\mathbf{H}_o\mathbf{P}_{k+1|k}\mathbf{H}^T_o + \mathbf{R}_o(k+1)\right)^{-1}\mathbf{H}_o\mathbf{P}_{k+1|k}
\end{aligned} \tag{24}$$

In order to derive the last expression, property $\boldsymbol{\Xi}^T(k+1) = \boldsymbol{\Xi}^{-1}(k+1)$ was employed. This property is a consequence of the definition of matrix $\boldsymbol{\Xi}(k+1)$, and the fact that the rotation matrices satisfy $C^T(\hat{\phi}_i) = C^{-1}(\hat{\phi}_i)$.

Note that in the formulation presented up to this point, the measurement and covariance matrices have been partitioned based on the subsets of measurements that are performed by each robot (cf. Eqs. (22) and (23)). A different partitioning, however, turns out to be more convenient in the study of the asymptotic properties of the covariance matrix of the position estimates. Specifically, we permute the block rows $H_{ij}(k+1)$ of $\mathbf{H}(k+1)$ (and equivalently, the block rows $H_{o_{ij}}$ of $\mathbf{H}_o$) so that all the robot-to-robot measurements are stacked together. The measurement matrix $\mathbf{H}'(k+1)$ that arises is related to $\mathbf{H}(k+1)$ by the transformation

$$\mathbf{H}'(k+1) = \mathcal{P}\mathbf{H}(k+1) \;\; \Leftrightarrow \;\; \mathbf{H}(k+1) = \mathcal{P}^T\mathbf{H}'(k+1) \tag{25}$$

where $\mathcal{P}$ is an appropriate permutation matrix. As a result of this permutation, the covariance matrix of the measurements is also transformed by a similarity transformation, yielding the new covariance matrix

$$\mathbf{R}'(k+1) = \mathcal{P}\mathbf{R}(k+1)\mathcal{P}^T \;\; \Leftrightarrow \;\; \mathbf{R}(k+1) = \mathcal{P}^T\mathbf{R}'(k+1)\mathcal{P} \tag{26}$$

Similarly, the transformations

$$\mathbf{H}'_o(k+1) = \mathcal{P}\mathbf{H}_o(k+1) \;\; \Leftrightarrow \;\; \mathbf{H}_o(k+1) = \mathcal{P}^T\mathbf{H}'_o(k+1) \tag{27}$$

and

$$\mathbf{R}'_o(k+1) = \mathcal{P}\mathbf{R}_o(k+1)\mathcal{P}^T \;\; \Leftrightarrow \;\; \mathbf{R}_o(k+1) = \mathcal{P}^T\mathbf{R}'_o(k+1)\mathcal{P} \tag{28}$$

are defined. The permutation of the rows of the measurement matrix is selected so as to yield a measurement matrix in which the robot-to-robot measurements correspond to the first block rows of $\mathbf{H}'(k+1)$. As a result, the matrix $\mathbf{H}'_o$ can be partitioned as

$$\mathbf{H}'_o = \begin{bmatrix} \mathbf{H}_R & \mathbf{0}_{2M_{RR}\times 2N} \\ \mathbf{H}_1 & \mathbf{H}_2 \end{bmatrix} \tag{29}$$

where M_{RR} is the total number of robot-to-robot measurements, $\mathbf{H}_R$ is a $2M_{RR} \times 2M$ matrix describing these measurements. Due to the structure of the measurement equations, each of the $2 \times 2M$ block rows of $\mathbf{H}_R$ has a special form. Specifically, the block row that corresponds to the relative position measurement between robots r_i and r_j is

$$H_{R_{ij}} = \begin{bmatrix} \mathbf{0}_{2\times 2} & \dots & \underbrace{-I_{2\times 2}}_{i-\text{th block}} & \dots & \underbrace{I_{2\times 2}}_{j-\text{th block}} & \dots & \mathbf{0}_{2\times 2} \end{bmatrix} \tag{30}$$

In Eq. (29) $\mathbf{H}_1$ is a $2M_{RL} \times 2M$ matrix, and $\mathbf{H}_2$ is a $2M_{RL} \times 2N$ matrix, where M_{RL} denotes the total number of robot-to-landmark measurements. Each $2 \times \xi$ block row of the submatrix $[\mathbf{H}_1 \;\; \mathbf{H}_2]$ describes one such measurement, and thus the block rows of $\mathbf{H}_1$ and $\mathbf{H}_2$ have special structure. If robot r_ℓ measures the relative position of landmark L_m, then the following block rows exist in $\mathbf{H}_1$ and $\mathbf{H}_2$ respectively:

$$H_{1_{\ell m}} = \begin{bmatrix} \mathbf{0}_{2\times 2} & \dots & \underbrace{-I_{2\times 2}}_{\ell-\text{th block}} & \dots & \mathbf{0}_{2\times 2} \end{bmatrix} \;\; \text{and} \;\; H_{2_{\ell m}} = \begin{bmatrix} \mathbf{0}_{2\times 2} & \dots & \underbrace{I_{2\times 2}}_{m-\text{th block}} & \dots & \mathbf{0}_{2\times 2} \end{bmatrix} \tag{31}$$

At this point we note that $\mathbf{H}'_o$ can be expressed as

$$\mathbf{H}'_o = H'_o \otimes I_{2\times 2} = \begin{bmatrix} H_R & 0_{M_{RR}\times N} \\ H_1 & H_2 \end{bmatrix} \otimes I_{2\times 2} \tag{32}$$

where the matrices H'_o, H_R, H_1 and H_2 are easily derived from $\mathbf{H}'_o$, $\mathbf{H}_R$, $\mathbf{H}_1$ and $\mathbf{H}_2$, respectively.

Substitution from Eqs. (27) and (28) in Eq. (24), and application of the property $\mathcal{P}^{-1} = \mathcal{P}^T$, which holds for any permutation matrix, an equivalent expression of the covariance update equation of the EKF:

$$\begin{aligned}
\mathbf{P}_{k+1|k+1} &= \mathbf{P}_{k+1|k} - \mathbf{P}_{k+1|k}\mathbf{H}_o^T\left(\mathbf{H}_o\mathbf{P}_{k+1|k}\mathbf{H}_o^T + \mathbf{R}_o(k+1)\right)^{-1}\mathbf{H}_o\mathbf{P}_{k+1|k} \\
&= \mathbf{P}_{k+1|k} - \mathbf{P}_{k+1|k}\mathbf{H}_o'^T\mathcal{P}\left(\mathcal{P}^T\mathbf{H}'_o\mathbf{P}_{k+1|k}\mathbf{H}_o'^T\mathcal{P} + \mathcal{P}^T\mathbf{R}'_o(k+1)\mathcal{P}\right)^{-1}\mathcal{P}^T\mathbf{H}'_o\mathbf{P}_{k+1|k} \\
&= \mathbf{P}_{k+1|k} - \mathbf{P}_{k+1|k}\mathbf{H}_o'^T\left(\mathbf{H}'_o\mathbf{P}_{k+1|k}\mathbf{H}_o'^T + \mathbf{R}'_o(k+1)\right)^{-1}\mathbf{H}'_o\mathbf{P}_{k+1|k}
\end{aligned} \tag{33}$$

3 SLAM Positioning Accuracy Characterization

3.1 The Riccati Recursion

The metric we employ in order to characterize the positioning performance of C-SLAM is the covariance matrix of the robots' and landmarks' position estimates. By combining Eqs. (9) and (33) we derive the discrete-time Riccati recursion, that describes the time evolution of the covariance matrix:

$$\mathbf{P}_{k+2|k+1} = \mathbf{P}_{k+1|k} - \mathbf{P}_{k+1|k}\mathbf{H}_o'^T\left(\mathbf{H}_o'\mathbf{P}_{k+1|k}\mathbf{H}_o'^T + \mathbf{R}_o'(k+1)\right)^{-1}\mathbf{H}_o'\mathbf{P}_{k+1|k} + \mathbf{G}_o\mathbf{Q}_r(k+1)\mathbf{G}_o^T$$

This recursion provides the value of the covariance matrix at each time step, right after the propagation phase of the EKF. To simplify the notation, we set $\mathbf{P}_k = \mathbf{P}_{k+1|k}$ and $\mathbf{P}_{k+1} = \mathbf{P}_{k+2|k+1}$, and therefore we can write

$$\mathbf{P}_{k+1} = \mathbf{P}_k - \mathbf{P}_k\mathbf{H}_o'^T\left(\mathbf{H}_o'\mathbf{P}_k\mathbf{H}_o'^T + \mathbf{R}_o'(k+1)\right)^{-1}\mathbf{H}_o'\mathbf{P}_k + \mathbf{G}_o\mathbf{Q}_r(k+1)\mathbf{G}_o^T \quad (34)$$

We note that the matrices $\mathbf{Q}_r(k+1)$ and $\mathbf{R}_o'(k+1)$ in this Riccati recursion are time varying, and this does not allow the derivation of any closed form expressions for the time evolution of $\mathbf{P}_k$, in the general case. We therefore have to resort to deriving *bounds* for the covariance of the C-SLAM position estimates. The following two lemmas are the basis of our analysis:

Lemma 3.1 *If $\mathbf{R}_u'$ and $\mathbf{Q}_u$ are matrices such that $\mathbf{R}_u' \succeq \mathbf{R}_o'(k)$ and $\mathbf{Q}_u \succeq \mathbf{Q}_r(k)$ for all $k \geq 0$, then the solution to the Riccati recursion*

$$\mathbf{P}_{k+1}^u = \mathbf{P}_k^u - \mathbf{P}_k^u\mathbf{H}_o'^T\left(\mathbf{H}_o'\mathbf{P}_k^u\mathbf{H}_o'^T + \mathbf{R}_u'\right)^{-1}\mathbf{H}_o'\mathbf{P}_k^u + \mathbf{G}_o\mathbf{Q}_u\mathbf{G}_o^T \quad (35)$$

with the initial condition $\mathbf{P}_0^u = \mathbf{P}_0$, satisfies $\mathbf{P}_k^u \succeq \mathbf{P}_k$ for all $k \geq 0$.

Lemma 3.2 *If $\bar{\mathbf{R}}'$ and $\bar{\mathbf{Q}}_r$ are matrices such that $\bar{\mathbf{R}}' = E\{\mathbf{R}_o'(k)\}$ and $\bar{\mathbf{Q}}_r = \{\mathbf{Q}_r(k)\}$ for all $k \geq 0$, then the solution to the Riccati recursion*

$$\bar{\mathbf{P}}_{k+1} = \bar{\mathbf{P}}_k - \bar{\mathbf{P}}_k\mathbf{H}_o'^T\left(\mathbf{H}_o'\bar{\mathbf{P}}_k\mathbf{H}_o'^T + \bar{\mathbf{R}}'\right)^{-1}\mathbf{H}_o'\bar{\mathbf{P}}_k + \mathbf{G}_o\bar{\mathbf{Q}}_r\mathbf{G}_o \quad (36)$$

with the initial condition $\bar{\mathbf{P}}_0 = \mathbf{P}_0$, satisfies $\bar{\mathbf{P}}_k \succeq E\{\mathbf{P}_k\}$ for all $k \geq 0$.

Essentially, Lemma 3.1 maintains that in order to derive an upper bound on the *worst-case* covariance matrix of the position estimates in C-SLAM, it suffices to derive *upper bounds* for the covariance matrices of the system and measurement noise, and to solve a *constant coefficient* Riccati recursion. Similarly, Lemma 3.2 states that an upper bound on the *expected* positioning uncertainty of C-SLAM is determined as the solution of a constant coefficient Riccati recursion, where the covariance matrices of the system and measurement noise have been replaced by their *average* values. The proofs for these lemmas are given in Appendices A and B respectively. In the remainder of this section, we derive appropriate upper bounds, as well as the average values of the matrices $\mathbf{Q}_r(k)$ and $\mathbf{R}_o'(k)$ respectively.

- **Derivation of upper bounds for $\mathbf{Q}_r(k)$ and $\mathbf{R}_o'(k)$**

In order to derive an upper bound for the covariance matrix $\mathbf{Q}_r(k)$ we note that (cf. Eqs. (4) and (8))

$$\mathbf{Q}_r(k) = \begin{bmatrix} Q_{r_1}(k) & \cdots & \mathbf{0}_{2\times 2} \\ \vdots & \ddots & \vdots \\ \mathbf{0}_{2\times 2} & \cdots & Q_{r_M}(k) \end{bmatrix} \quad (37)$$

where

$$Q_{r_i}(k) = C(\hat{\phi}_i(k))\begin{bmatrix} \delta t^2\sigma_{V_i}^2 & 0 \\ 0 & \delta t^2 V_{m_i}^2(k)\sigma_{\phi_i}^2 \end{bmatrix} C^T(\hat{\phi}_i(k))$$

From the properties of rotation matrices it is known that $C^{-1}(\hat{\phi}_i(k)) = C^T(\hat{\phi}_i(k))$, and thus $Q_{r_i}(k)$ is related by a similarity transformation to the matrix

$$\begin{bmatrix} \delta t^2\sigma_{V_i}^2 & 0 \\ 0 & \delta t^2 V_{m_i}^2(k)\sigma_{\phi_i}^2 \end{bmatrix}$$

which implies that the eigenvalues of $Q_{r_i}(k)$ are $\delta t^2 \sigma_{V_i}^2$ and $\delta t^2 V_{m_i}^2(k) \sigma_{\phi_i}^2$. We assume that the velocity of each robot is approximately constant, and equal to V_i, and denote

$$q_i = \max\left(\delta t^2 \sigma_{V_i}^2, \delta t^2 V_{m_i}^2(k) \sigma_{\phi_i}^2\right) \simeq \max\left(\delta t^2 \sigma_{V_i}^2, \delta t^2 V_i^2 \sigma_{\phi_i}^2\right) \tag{38}$$

This definition states that q_i is the largest eigenvalue of $Q_{r_i}(k)$, and therefore

$$Q_{r_i}(k) \preceq q_i I_{2\times 2} \Rightarrow \mathbf{Q}_r(k) \preceq \mathbf{Diag}(q_i I_{2\times 2}) = \mathbf{Q}_u \tag{39}$$

In order to derive an upper bound for $\mathbf{R}'_o(k)$ we first derive an upper bound for $\mathbf{R}_o(k)$, and employ the property

$$\mathbf{R}_u \succeq \mathbf{R}_o(k) \ \ \Rightarrow \ \ \mathcal{P}\mathbf{R}_u\mathcal{P}^T \succeq \mathcal{P}\mathbf{R}_o(k)\mathcal{P}^T \ = \ \mathbf{R}'_o(k)$$

The upper bound on $\mathbf{R}_o(k)$ is obtained by considering each if its block diagonal elements, $\boldsymbol{R}_{o_i}(k)$. Referring to Eq. (21), we examine the terms $R_1(k)$, $R_2(k)$ and $R_3(k)$ separately: the term expressing the effect of the noise in the range measurements is

$$R_1(k) = \sigma_{\rho_i}^2 I_{2N\times 2N} - D_i(k)\,\mathrm{diag}\left(\frac{\sigma_{\rho_i}^2}{\hat{\rho}_{ij}^2}\right) D_i^T(k) \preceq \sigma_{\rho_i}^2 I_{2N\times 2N} \tag{40}$$

The last matrix inequality follows from the fact that the term being subtracted from $\sigma_{\rho_i}^2 I_{2N\times 2N}$ is a positive semidefinite matrix. The covariance term due to the noise in the bearing measurement is

$$\begin{aligned} R_2(k) &= \sigma_{\theta_i}^2 D_i(k) D_i^T(k) \\ &= \sigma_{\theta_i}^2 \mathbf{Diag}\left(\hat{\rho}_{ij}^2 \begin{bmatrix} \sin^2(\hat{\theta}_{ij}) & \sin(\hat{\theta}_{ij})\cos(\hat{\theta}_{ij}) \\ \sin(\hat{\theta}_{ij})\cos(\hat{\theta}_{ij}) & \cos^2(\hat{\theta}_{ij}) \end{bmatrix}\right) \\ &\preceq \sigma_{\theta_i}^2 \mathbf{Diag}\left(\hat{\rho}_{ij}^2 I_{2\times 2}\right) \\ &\preceq \sigma_{\theta_i}^2 \rho_o^2 I_{2N\times 2N} \end{aligned} \tag{41}$$

where ρ_o is the maximum range at which a measurement can occur, determined either by the characteristics of the robots' sensors or by the properties of the area in which the robots move. Finally, the covariance term due to the error in the orientation of the measuring robot is $R_3(k) = \sigma_{\phi_i}^2 D_i(k) \mathbf{1}_{N\times N} D_i^T(k)$. Calculation of the eigenvalues of the matrices $\mathbf{1}_{N\times N}$ and $I_{N\times N}$ verifies that $\mathbf{1}_{N\times N} \preceq N I_{N\times N}$, and thus we can write $R_3(k) \preceq N\sigma_{\phi_i}^2 D_i(k) D_i^T(k)$. By derivations analogous to those employed to yield an upper bound for $R_2(k)$, we can show that

$$R_3(k) \preceq N\sigma_{\phi_i}^2 \rho_o^2 I_{2N\times 2N}$$

By combining this result with those of Eqs. (40), (41), we can write $\boldsymbol{R}_{o_i}(k) = R_1(k) + R_2(k) + R_3(k) \preceq \mathbf{R}_i^u$, where

$$\mathbf{R}_i^u = \left(\sigma_{\rho_i}^2 + N\sigma_{\phi_i}^2 \rho_o^2 + \sigma_{\theta_i}^2 \rho_o^2\right) I_{2N\times 2N} = r_i I_{2N\times 2N} \tag{42}$$

with

$$r_i = \sigma_{\rho_i}^2 + N\sigma_{\phi_i}^2 \rho_o^2 + \sigma_{\theta_i}^2 \rho_o^2 \tag{43}$$

Thus, we can write

$$\mathbf{R}_o(k) = \mathbf{Diag}(\boldsymbol{R}_{o_i}(k)) \preceq \mathbf{Diag}(r_i I_{M_i\times M_i}) = \mathbf{R}_u \tag{44}$$

Therefore an upper bound for $\mathbf{R}'_o(k)$ is given by

$$\mathbf{R}'_o(k) \preceq \mathcal{P}\mathbf{Diag}(r_i I_{M_i\times M_i})\mathcal{P}^T = \mathbf{R}'_u$$

- **Derivation of the Expected Values of $\mathbf{Q}_r(k)$ and $\mathbf{R}'_o(k)$**

  In order to derive the average value of $\mathbf{Q}_r(k)$ we note that

$$\begin{aligned}
Q_{r_i}(k) &= C(\hat{\phi}_i(k)) \begin{bmatrix} \delta t^2 \sigma_{V_i}^2 & 0 \\ 0 & \delta t^2 V_{m_i}^2(k)\sigma_{\phi_i}^2 \end{bmatrix} C^T(\hat{\phi}_i(k)) \\
&= \delta t^2 \begin{bmatrix} \sigma_{V_i}^2 \cos^2(\hat{\phi}_i) + V_{m_i}^2(k)\sigma_{\phi_i}^2 \sin^2(\hat{\phi}_i) & \left(\sigma_{V_i}^2 - V_{m_i}^2(k)\sigma_{\phi_i}^2\right)\sin(\hat{\phi}_i)\cos(\hat{\phi}_i) \\ \left(\sigma_{V_i}^2 - V_{m_i}^2(k)\sigma_{\phi_i}^2\right)\sin(\hat{\phi}_i)\cos(\hat{\phi}_i) & \sigma_{V_i}^2 \sin^2(\hat{\phi}_i) + V_{m_i}^2(k)\sigma_{\phi_i}^2 \cos^2(\hat{\phi}_i) \end{bmatrix}
\end{aligned}$$

and therefore, by averaging over all values of orientation, the expected value of $Q_{r_i}(k)$ is derived:

$$E\{Q_{r_i}(k)\} = \delta t^2 \frac{\sigma_V^2 + V_i^2 \sigma_{\phi_i}^2}{2} I_{2\times 2} = \bar{q}_i I_{2\times 2}$$

where

$$\bar{q}_i = \delta t^2 \frac{\sigma_V^2 + V_i^2 \sigma_{\phi_i}^2}{2}$$

Thus,

$$E\{\mathbf{Q}_r(k)\} = \mathbf{Diag}\left(E\{Q_{r_i}(k)\}\right) = \mathbf{Diag}(\bar{q}_i I_{2\times 2}) = \bar{\mathbf{Q}}_r \tag{45}$$

The average value of $\mathbf{R}'_o(k)$ is derived by first considering the matrix $\mathbf{R}_o(k)$, and employing the property

$$\begin{aligned}
E\{\mathbf{R}'_o(k)\} &= E\{\mathcal{P}\mathbf{R}_o(k)\mathcal{P}^T\} \\
&= \mathcal{P}E\{\mathbf{R}_o(k)\}\mathcal{P}^T \\
&= \mathcal{P}E\{\mathbf{Diag}(\boldsymbol{R}_{o_i}(k))\}\mathcal{P}^T \\
&= \mathcal{P}\mathbf{Diag}(E\{\boldsymbol{R}_{o_i}(k)\})\mathcal{P}^T
\end{aligned} \tag{46}$$

We therefore see that the average values of the matrices $\boldsymbol{R}_{o_i}(k)$, $i = 1 \ldots M$ need to be determined, in order to compute $E\{\mathbf{R}'_o(k)\}$. From Eq. (21) we note that evaluation of the average value of $\boldsymbol{R}_{o_i}(k)$ requires the computation of the expected values of the following terms:

$$T_1 = \frac{\widehat{\Delta p}_{ij}\widehat{\Delta p}_{ij}^T}{\hat{\rho}_{ij}^2}, \quad T_2 = \widehat{\Delta p}_{ij}\widehat{\Delta p}_{ij}^T, \quad \text{and} \quad T_3 = \widehat{\Delta p}_{ij}\widehat{\Delta p}_{i\ell}^T \tag{47}$$

for $j, \ell = 1 \ldots M_i$. The average value of T_1 is easily derived by employing the polar coordinate description of the vector $\widehat{\Delta p}_{ij}$ in terms of $\hat{\rho}_{ij}$ and $\hat{\theta}_{ij}$, which yields

$$\begin{aligned}
T_1 &= \frac{\widehat{\Delta p}_{ij}\widehat{\Delta p}_{ij}^T}{\hat{\rho}_{ij}^2} \\
&= \frac{1}{\hat{\rho}_{ij}^2} \begin{bmatrix} \hat{\rho}_{ij}^2 \cos^2(\hat{\theta}_{ij}) & \hat{\rho}_{ij}^2 \sin(\hat{\theta}_{ij})\cos(\hat{\theta}_{ij}) \\ \hat{\rho}_{ij}^2 \sin(\hat{\theta}_{ij})\cos(\hat{\theta}_{ij}) & \hat{\rho}_{ij}^2 \sin^2(\hat{\theta}_{ij}) \end{bmatrix} \\
&= \begin{bmatrix} \cos^2(\hat{\theta}_{ij}) & \sin(\hat{\theta}_{ij})\cos(\hat{\theta}_{ij}) \\ \sin(\hat{\theta}_{ij})\cos(\hat{\theta}_{ij}) & \sin^2(\hat{\theta}_{ij}) \end{bmatrix}
\end{aligned}$$

From the last expression we conclude that for any probability density function that guarantees a uniform distribution for the bearing angle of the measurements (i.e., any symmetric probability density function), the average value of the term T_1 is

$$E\{T_1\} = \frac{1}{2} I_{2\times 2}$$

In order to compute the expected value of the terms T_2 and T_3, we assume that the robots and landmarks are located in a square arena of side α, and that their positions are described by uniformly distributed random variables in the interval $[-\alpha/2, \alpha/2]$. We can thus write

$$\begin{aligned}
E\{T_2\} = E\{\widehat{\Delta p}_{ij}\widehat{\Delta p}_{ij}^T\} &= E\left\{\begin{bmatrix} \widehat{\Delta x}_{ij}^2 & \widehat{\Delta x}_{ij}\widehat{\Delta y}_{ij} \\ \widehat{\Delta y}_{ij}\widehat{\Delta x}_{ij} & \widehat{\Delta y}_{ij}^2 \end{bmatrix}\right\} \\
&= \begin{bmatrix} E\{x_j^2 - 2x_i x_j + x_i^2\} & E\{x_j y_j - x_j y_i - x_i y_j + x_i y_i\} \\ E\{y_j x_j - y_j x_i - y_i x_j + y_i x_i\} & E\{y_j^2 - 2y_j y_i + y_i^2\} \end{bmatrix} \\
&= \begin{bmatrix} 2E\{x_i^2\} & 0 \\ 0 & 2E\{y_i^2\} \end{bmatrix} \\
&= \begin{bmatrix} \frac{\alpha^2}{6} & 0 \\ 0 & \frac{\alpha^2}{6} \end{bmatrix} \\
&= \frac{\alpha}{12} I_{2\times 2}
\end{aligned}$$

and similarly,

$$\begin{aligned}
E\{T_3\} = E\{\widehat{\Delta p}_{ij}\widehat{\Delta p}_{i\ell}^T\} &= E\left\{\begin{bmatrix} \widehat{\Delta x}_{ij}\widehat{\Delta x}_{i\ell} & \widehat{\Delta x}_{ij}\widehat{\Delta y}_{i\ell} \\ \widehat{\Delta y}_{ij}\widehat{\Delta x}_{i\ell} & \widehat{\Delta y}_{ij}\widehat{\Delta y}_{i\ell} \end{bmatrix}\right\} \\
&= \begin{bmatrix} E\{x_j x_\ell - x_i x_\ell - x_j x_i + x_i^2\} & E\{x_j y_\ell - x_j y_i - x_i y_\ell + x_i y_i\} \\ E\{y_j x_\ell - y_j x_i - y_i x_\ell + y_i x_i\} & E\{y_j y_\ell - y_i y_\ell - y_j y_i + y_i^2\} \end{bmatrix} \\
&= \begin{bmatrix} E\{x_i^2\} & 0 \\ 0 & E\{y_i^2\} \end{bmatrix} \\
&= \begin{bmatrix} \frac{\alpha^2}{12} & 0 \\ 0 & \frac{\alpha^2}{12} \end{bmatrix} \\
&= \frac{\alpha}{12} I_{2\times 2}
\end{aligned}$$

These results enable us to obtain the average value of the matrices $\boldsymbol{R}_{o_i}(k)$, $i = 1 \ldots M$. Employing the linearity of the expectation operator yields

$$\begin{aligned}
\bar{\mathbf{R}}_i &= E\{\boldsymbol{R}_{o_i}(k)\} \\
&= \begin{bmatrix} \left(\frac{1}{2}\sigma_{\rho_i}^2 + \frac{1}{6}\sigma_{\phi_i}^2 + \frac{1}{6}\sigma_{\theta_i}^2\right) I_{2\times 2} & \cdots & \frac{1}{12}\sigma_{\phi_i}^2 I_{2\times 2} \\ \vdots & \ddots & \vdots \\ \frac{1}{12}\sigma_{\phi_i}^2 I_{2\times 2} & \cdots & \left(\frac{1}{2}\sigma_{\rho_i}^2 + \frac{1}{6}\sigma_{\phi_i}^2 + \frac{1}{6}\sigma_{\theta_i}^2\right) I_{2\times 2} \end{bmatrix} \\
&= \left(\frac{1}{2}\sigma_{\rho_i}^2 + \frac{1}{12}\sigma_{\phi_i}^2 + \frac{1}{6}\sigma_{\theta_i}^2\right) I_{2M_i\times 2M_i} + \frac{1}{12}\sigma_{\phi_i}^2 \left(\mathbf{1}_{M_i\times M_i} \otimes I_{2\times 2}\right)
\end{aligned}$$

The average value of $\mathbf{R}_o(k)$ is therefore

$$\begin{aligned}
\bar{\mathbf{R}} &= E\{\mathbf{R}_o(k)\} \\
&= \mathbf{Diag}(\bar{\mathbf{R}}_i)
\end{aligned} \tag{48}$$

while the average value of $\mathbf{R}_o'(k)$ is

$$\bar{\mathbf{R}}' = \mathcal{P}^T \bar{\mathbf{R}} \mathcal{P}$$

3.2 Steady State Covariance Bounds

Lemmas 3.1 and 3.2 allow the evaluation of upper bounds on the worst case uncertainty and on the average uncertainty of the position estimates in C-SLAM, at *any* time instant after the beginning of the exploration task. This can be trivially achieved, for example, by numerical evaluation of the solution to the recursions in Eqs. (35) and (36) respectively.

It is well known that in C-SLAM the covariance of the landmarks' position estimates decreases monotonically, and asymptotically assumes a steady state value. Thus, for many applications, it is important to characterize the *steady state* accuracy of the estimates. In this section, we determine an upper bound on the asymptotic value of the covariance matrix, by deriving the limit of $\mathbf{P}_k^u$ and $\bar{\mathbf{P}}_k$ after sufficient time, i.e., as $k \to \infty$.

We note at this point that the Riccati recursions of Eqs. (35) and (36) essentially describe the time evolution of the covariance of the position estimates in two hypothetical C-SLAM scenarios, where the system model is a Linear Time Invariant (LTI) one. Therefore, the problem of computing the upper bounds on the steady state positioning uncertainty in C-SLAM reduces to the problem of *determining the steady state covariance matrix for a LTI C-SLAM system model*. In the following, we consider a C-SLAM scenario with the following LTI system model:

$$X_o(k+1) = X_o(k) + \mathbf{G}_o w_o(k) \tag{49}$$

$$z_o(k) = \mathbf{H}'_o X_o(k) + n_o(k) \tag{50}$$

where the measurement covariance matrix is a constant matrix equal to

$$E\{n_o(k)n_o(k)^T\} = \mathbf{R}_s = \begin{bmatrix} \mathbf{R}_1 & \mathbf{R}_2 \\ \mathbf{R}_2^T & \mathbf{R}_3 \end{bmatrix} = \begin{bmatrix} R_1 & R_2 \\ R_2^T & R_3 \end{bmatrix} \otimes I_{2\times 2} = R_s \otimes I_{2\times 2} \tag{51}$$

while the system noise covariance matrix is the constant matrix

$$E\{w_o(k)w_o(k)^T\} = \mathbf{Q}_s = Q_s \otimes I_{2\times 2} \tag{52}$$

For this LTI system model the time evolution of the state covariance matrix is described by the following Riccati recursion:

$$\mathbf{P}_{k+1}^s = \mathbf{P}_k^s - \mathbf{P}_k^s \mathbf{H}_o'^T \left(\mathbf{H}'_o \mathbf{P}_k^s \mathbf{H}_o'^T + \mathbf{R}_s\right)^{-1} \mathbf{H}'_o \mathbf{P}_k^s + \mathbf{G}_o \mathbf{Q}_s \mathbf{G}_o^T \tag{53}$$

After deriving the steady state solution of this recursion, we employ the substitutions

$$\mathbf{R}_s \to \mathbf{R}_u, \qquad \mathbf{Q}_s \to \mathbf{Q}_u$$

and

$$\mathbf{R}_s \to \bar{\mathbf{R}}', \qquad \mathbf{Q}_s \to \bar{\mathbf{Q}}_r$$

in order to obtain the steady state solutions of the Riccati recursions of Lemmas (3.1) and (3.2) respectively.

Our analysis is based upon the following result, which is proven in [8] (Section 8.6, Lemmas 8.6.2 and 8.6.3):

Lemma 3.3 *Suppose $P_k^{(0)}$ is the solution to the discrete-time Riccati recursion*

$$P_{k+1} = FP_kF^T + GQG^T - (FP_kH^T + GS)(HP_kH^T + R)^{-1}(FP_kH^T + GS)^T, \tag{54}$$

with initial value $P_0 = 0$. Then the solution to the Riccati recursion with the same $\{F, G, H\}$ and $\{Q, R, S\}$ matrices, but with an arbitrary initial condition Π_0 is defined by the identity

$$P_{k+1} - P_{k+1}^{(0)} = \Phi_p^{(0)}(k+1,0)\left[I + \Pi_0 \mathcal{O}_k^{(0)}\right]^{-1} \Pi_0 \Phi_p^{(0)}(k+1,0)^T$$

where $\Phi_p^{(0)}(k+1,0)$ is given by

$$\Phi_p^{(0)}(k+1,0) = (F - K_pH)^{k+1}\left[I + PJ_{k+1}\right]$$

and

$$\mathcal{O}_k^{(0)} = J_{k+1}$$

In these expressions P is any solution to the Discrete Algebraic Riccati Equation (DARE)

$$P = FPF^T + GQG^T - (FPH^T + GS)(HPH^T + R)^{-1}(FPH^T + GS)^T,$$

$K_p = \left(FPH^T + GS\right)\left(R + HPH^T\right)^{-1}$ *and J_k denotes the solution to the* dual *Riccati recursion with zero initial condition, which, in the case $S = 0$, is written as*

$$J_{k+1} = FJ_kF^T + H^TR^{-1}H - F^TJ_kG(Q^{-1} + G^TJ_kG)^{-1}J_kF, \quad J_0 = 0$$

Introducing the substitutions

$$P_k \leftrightarrow \mathbf{P}_k^s, \quad G \leftrightarrow \mathbf{G}_o, \quad Q \leftrightarrow \mathbf{Q}_s, \quad H \leftrightarrow \mathbf{H}'_o, \quad R \leftrightarrow \mathbf{R}_s, \quad S \leftrightarrow \mathbf{0}_{2\times(2N+2)}$$

allows us to specialize Lemma 54 to our problem as follows:

Lemma 3.4 *Suppose* $\mathbf{P}_k^{s(0)}$ *is the solution to the Riccati recursion*

$$\mathbf{P}_{k+1}^s \quad = \quad \mathbf{P}_k^s - \mathbf{P}_k^s \mathbf{H}_o'^T \left(\mathbf{H}'_o \mathbf{P}_k^s \mathbf{H}_o'^T + \mathbf{R}_s\right)^{-1} \mathbf{H}'_o \mathbf{P}_k^s + \mathbf{G}_o \mathbf{Q}_s \mathbf{G}_o^T \tag{55}$$

with zero initial condition. Then the solution to this recursion when the initial covariance matrix is an arbitrary positive semidefinite matrix Π_0 *is defined by the relation*

$$\mathbf{P}_{k+1}^s - \mathbf{P}_{k+1}^{s(0)} = \Phi_p^{(0)}(k+1,0) \left[I_{\xi\times\xi} + \Pi_0 \mathbf{J}_{k+1}\right]^{-1} \Pi_0 \Phi_p^{(0)}(k+1,0)^T \tag{56}$$

where

$$\Phi_p^{(0)}(k+1,0) = \left(I_{\xi\times\xi} - \mathbf{P}\mathbf{H}_o'^T \left(\mathbf{R}_s + \mathbf{H}'_o \mathbf{P} \mathbf{H}_o'^T\right)^{-1} \mathbf{H}'_o\right)^{k+1} \left[I_{\xi\times\xi} + \mathbf{P}\mathbf{J}_{k+1}\right] \tag{57}$$

In these expressions $\mathbf{P}$ *is any solution to the Discrete Algebraic Riccati Equation (DARE)*

$$\mathbf{P} = \mathbf{P} - \mathbf{P}\mathbf{H}_o'^T \left(\mathbf{R}_s + \mathbf{H}'_o \mathbf{P} \mathbf{H}_o'^T\right)^{-1} \mathbf{H}'_o \mathbf{P} + \mathbf{G}_o \mathbf{Q}_s \mathbf{G}_o^T$$

and $\mathbf{J}_k$ *denotes the solution to the* dual *Riccati recursion with zero initial condition:*

$$\mathbf{J}_{k+1} = \mathbf{J}_k + \mathbf{H}_o'^T \mathbf{R}_s^{-1} \mathbf{H}'_o - \mathbf{J}_k \mathbf{G}_o \left(\mathbf{Q}_s + \mathbf{G}_o^T \mathbf{J}_k \mathbf{G}_o\right)^{-1} \mathbf{G}_o^T \mathbf{J}_k, \quad \mathbf{J}_0 = \mathbf{0}_{\xi\times\xi} \tag{58}$$

In order to derive the steady state value of $\mathbf{P}_k^s$, we will evaluate Eq. (56) in the limit as $k \to \infty$. To this end, we first evaluate $\lim_{k\to\infty} \mathbf{P}_k^{s(0)}$, i.e., the steady state covariance of the position estimates when the initial uncertainty is zero, and then we evaluate the limit value of the right hand side member of Eq. (56). In the derivations that follow, it will be convenient to manipulate the matrices $\mathbf{P}_k^s$ and $\mathbf{J}_k$ as partitioned matrices, i.e.,

$$\mathbf{P}_k^s = \begin{bmatrix} \mathbf{P}_{rr_k}^s & \mathbf{P}_{Lr_k}^{s\ T} \\ \mathbf{P}_{Lr_k}^s & \mathbf{P}_{LL_k}^s \end{bmatrix} \quad \text{and} \quad \mathbf{J}_k = \begin{bmatrix} \mathbf{J}_{rr_k} & \mathbf{J}_{Lr_k}^T \\ \mathbf{J}_{Lr_k} & \mathbf{J}_{LL_k} \end{bmatrix} \tag{59}$$

where the matrices $\mathbf{P}_{rr_k}^s$ and $\mathbf{J}_{rr_k}$ are $2M \times 2M$ matrices corresponding to robots' position estimates, $\mathbf{P}_{LL_k}^s$ and $\mathbf{J}_{LL_k}$ are $2N \times 2N$ matrices corresponfing to the landmarks' position estimates, while $\mathbf{P}_{Lr_k}^s$ and $\mathbf{J}_{Lr_k}$ are $2N \times 2M$ matrices, corresponding to the cross-correlations between the robots and landmarks.

3.2.1 Solution with Zero Initial Covariance

The derivation of the steady state value of $\mathbf{P}_k^{s(0)}$ can be greatly simplified by considering the physical interpretation of the quantities that appear in Eq. (53). This Riccati recursion describes the time evolution of the covariance of the position estimates for a LTI C-SLAM scenario, in which the initial covariance matrix, $\mathbf{P}_0^s$, is zero. This implies that our initial knowledge about the position of the robots and landmarks is perfect. The landmarks are static, and thus the estimates about their position will *not* degrade as time progresses. The matrix $\mathbf{P}_{LL_k}^s$ will *remain* equal to zero, for all time steps $k > 0$, and since $\mathbf{P}_k^{s(0)}$ is a positive semidefinite matrix, we conclude that the matrix $\mathbf{P}_{Lr_k}^s$ will also remain equal to zero. The physical interpretation of this is that the robots actually perform map-based localization with a perfectly known map, while simultaneously recording relative position measurements between them. The measurements of the landmarks' positions are equivalent to *absolute* measurements of the robots position, based on a perfect map. We note that by application of the matrix inversion lemma, the Riccati recursion can be expressed as

$$\begin{aligned} \mathbf{P}_{k+1}^s &= \mathbf{P}_k^s - \mathbf{P}_k^s \mathbf{H}_o'^T \left(\mathbf{H}'_o \mathbf{P}_k^s \mathbf{H}_o'^T + \mathbf{R}_s\right)^{-1} \mathbf{H}'_o \mathbf{P}_k^s + \mathbf{G}_o \mathbf{Q}_s \mathbf{G}_o^T \\ &= \left(I_{\xi\times\xi} - \mathbf{P}_k^s \mathbf{H}_o'^T \left(\mathbf{H}'_o \mathbf{P}_k^s \mathbf{H}_o'^T + \mathbf{R}_s\right)^{-1} \mathbf{H}'_o\right) \mathbf{P}_k^s + \mathbf{G}_o \mathbf{Q}_s \mathbf{G}_o^T \\ &= \left(I_{\xi\times\xi} + \mathbf{P}_k^s \mathbf{H}_o'^T \mathbf{R}_s^{-1} \mathbf{H}'_o\right)^{-1} \mathbf{P}_k^s + \mathbf{G}_o \mathbf{Q}_s \mathbf{G}_o^T \end{aligned}$$

Setting

$$\mathbf{P}_k^s = \begin{bmatrix} \mathbf{P}_{rr_k}^s & \mathbf{0}_{2M\times 2N} \\ \mathbf{0}_{2N\times 2M} & \mathbf{0}_{2N\times 2N} \end{bmatrix}$$

yields

$$\begin{aligned}
\begin{bmatrix} \mathbf{P}_{rr_{k+1}}^s & \mathbf{0}_{2M\times 2N} \\ \mathbf{0}_{2N\times 2M} & \mathbf{0}_{2N\times 2N} \end{bmatrix} &= \left(I_{\xi\times\xi} + \begin{bmatrix} \mathbf{P}_{rr_k}^s & \mathbf{0}_{2M\times 2N} \\ \mathbf{0}_{2N\times 2M} & \mathbf{0}_{2N\times 2N} \end{bmatrix} \begin{bmatrix} \mathbf{I}_{rr} & \mathbf{I}_{Lr}^T \\ \mathbf{I}_{Lr} & \mathbf{I}_{LL} \end{bmatrix}\right)^{-1} \begin{bmatrix} \mathbf{P}_{rr_k}^s & \mathbf{0}_{2M\times 2N} \\ \mathbf{0}_{2N\times 2M} & \mathbf{0}_{2N\times 2N} \end{bmatrix} + \mathbf{G}_o \mathbf{Q}_s \mathbf{G}_o^T \\
&= \begin{bmatrix} I_{2M\times 2M} + \mathbf{P}_{rr_k}^s \mathbf{I}_{rr} & \mathbf{P}_{rr_k}^s \mathbf{I}_{Lr}^T \\ \mathbf{0}_{2N\times 2M} & I_{2N\times 2N} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{P}_{rr_k}^s & \mathbf{0}_{2M\times 2N} \\ \mathbf{0}_{2N\times 2M} & \mathbf{0}_{2N\times 2N} \end{bmatrix} + \mathbf{G}_o \mathbf{Q}_s \mathbf{G}_o^T \\
&= \begin{bmatrix} \left(I_{2M\times 2M} + \mathbf{P}_{rr_k}^s \mathbf{I}_{rr}\right)^{-1} & -\left(I_{2M\times 2M} + \mathbf{P}_{rr_k}^s \mathbf{I}_{rr}\right)^{-1} \mathbf{P}_{rr_k}^s \mathbf{I}_{Lr}^T \\ \mathbf{0}_{2N\times 2M} & I_{2N\times 2N} \end{bmatrix} \begin{bmatrix} \mathbf{P}_{rr_k}^s & \mathbf{0}_{2M\times 2N} \\ \mathbf{0}_{2N\times 2M} & \mathbf{0}_{2N\times 2N} \end{bmatrix} \\
&\quad + \mathbf{G}_o \mathbf{Q}_s \mathbf{G}_o^T \\
&= \begin{bmatrix} \left(I_{2M\times 2M} + \mathbf{P}_{rr_k}^s \mathbf{I}_{rr}\right)^{-1} \mathbf{P}_{rr_k}^s & \mathbf{0}_{2M\times 2N} \\ \mathbf{0}_{2N\times 2M} & \mathbf{0}_{2N\times 2N} \end{bmatrix} + \mathbf{G}_o \mathbf{Q}_s \mathbf{G}_o^T \qquad (60)
\end{aligned}$$

where we have defined

$$\begin{aligned}
\mathbf{H}_o'^T \mathbf{R}_s^{-1} \mathbf{H}_o' &= \left(H_o'^T \otimes I_{2\times 2}\right) \left(R_s^{-1} \otimes I_{2\times 2}\right) \left(H_o' \otimes I_{2\times 2}\right) \\
&= \begin{bmatrix} H_R^T F_1 H_R + H_1^T F_2^T H_R + H_R^T F_2 H_1 + H_1^T F_4 H_1 & H_R^T F_2 H_2 + H_1^T F_4 H_2 \\ H_2^T F_2^T H_R + H_2^T F_4 H_1 & H_2^T F_4 H_2 \end{bmatrix} \otimes I_{2\times 2} \\
&= \begin{bmatrix} I_{rr} & I_{Lr}^T \\ I_{Lr} & I_{LL} \end{bmatrix} \otimes I_{2\times 2} \\
&= \begin{bmatrix} \mathbf{I}_{rr} & \mathbf{I}_{Lr}^T \\ \mathbf{I}_{Lr} & \mathbf{I}_{LL} \end{bmatrix}
\end{aligned}$$

with

$$\begin{aligned}
\mathbf{R}_s^{-1} &= R_s^{-1} \otimes I_{2\times 2} \\
&= \begin{bmatrix} R_1 & R_2 \\ R_2^T & R_4 \end{bmatrix}^{-1} \otimes I_{2\times 2} \\
&= \begin{bmatrix} \left(R_1 - R_2 R_4^{-1} R_2^T\right)^{-1} & -\left(R_1 - R_2 R_4^{-1} R_2^T\right)^{-1} R_2 R_4^{-1} \\ -R_4 R_2^T \left(R_1 - R_2 R_4^{-1} R_2^T\right)^{-1} & \left(R_4 - R_2^T R_1^{-1} R_2\right)^{-1} \end{bmatrix} \otimes I_{2\times 2} \\
&= \begin{bmatrix} F_1 & F_2 \\ F_2^T & F_4 \end{bmatrix} \otimes I_{2\times 2}
\end{aligned}$$

At this point we note that the quantities $\mathbf{I}_{rr}$ and I_{rr} can be expressed alternatively as:

$$\mathbf{I}_{rr} = \begin{bmatrix} I_{2M\times 2M} & \mathbf{0}_{2N\times 2M} \end{bmatrix} \mathbf{H}_o'^T \mathbf{R}_s^{-1} \mathbf{H}_o' \begin{bmatrix} I_{2M\times 2M} \\ \mathbf{0}_{2N\times 2M} \end{bmatrix} \text{ and } I_{rr} = \begin{bmatrix} I_{M\times M} & \mathbf{0}_{N\times M} \end{bmatrix} \mathbf{H}_o'^T \mathbf{R}_s^{-1} \mathbf{H}_o' \begin{bmatrix} I_{M\times M} \\ \mathbf{0}_{N\times M} \end{bmatrix} \quad (61)$$

These expressions are simpler, and will be useful in expressing the final result in a more intuitive form.

The Riccati recursion for $\mathbf{P}_{rr_k}^s$ is derived from Eq. (60):

$$\mathbf{P}_{rr_{k+1}}^s = \left(I_{2M\times 2M} + \mathbf{P}_{rr_k}^s \mathbf{I}_{rr}\right)^{-1} \mathbf{P}_{rr_k}^s + \mathbf{Q}_s, \quad \mathbf{P}_{rr_0}^s = \mathbf{0}_{2M\times 2M} \qquad (62)$$

At this point we note that all the matrices that appear in this recursion can be expressed as the Kronecker product of some matrix with the 2×2 identity matrix, while the initial value of the recursion is zero. Employing the result of Appendix C we conclude that at any time step $k > 0$, the solution to the recursion will be of the form $P_{rr_k}^s \otimes I_{2\times 2}$, where $P_{rr_k}^s$ is a $M \times M$ matrix. The time evolution of $P_{rr_k}^s$ is described by the Riccati recursion

$$P_{rr_{k+1}}^s = \left(I_{M\times M} + P_{rr_k}^s I_{rr}\right)^{-1} P_{rr_k}^s + Q_s, \quad P_{rr_0}^s = \mathbf{0}_{2M\times 2M}$$

and its steady state value can be found by solving the equation

$$\begin{aligned} P^s_{rr_\infty} &= \left(I_{M\times M} + P^s_{rr_k} I_{rr}\right)^{-1} P^s_{rr_\infty} + Q_s \\ &= \left(P^{s\ -1}_{rr_\infty} + I_{rr}\right)^{-1} + Q_s \end{aligned} \tag{63}$$

Pre- and post-multiplying the last expression by $Q_s^{-1/2}$ yields

$$\begin{aligned} Q_s^{-1/2} P^s_{rr_\infty} Q_s^{-1/2} &= Q_s^{-1/2} \left(P^{s\ -1}_{rr_\infty} + I_{rr}\right)^{-1} Q_s^{-1/2} + I_{M\times M} \Rightarrow \\ P_n &= \left(P_n^{-1} + C\right)^{-1} + I_{M\times M} \end{aligned} \tag{64}$$

where we have defined

$$P_n = Q_s^{-1/2} P^s_{rr_\infty} Q_s^{-1/2} \tag{65}$$

and

$$C = Q_s^{1/2} I_{rr} Q_s^{1/2}$$

At this point we employ the singular value decomposition of C, which we denote as

$$C = U \Lambda U^T = U \operatorname{diag}(\lambda_i) U^T$$

and Eq. (64) is written as

$$\begin{aligned} P_n &= \left(P_n^{-1} + U\Lambda U^T\right)^{-1} + I_{M\times M} \Rightarrow \\ U^T P_n U &= U^T \left(P_n^{-1} + U\Lambda U^T\right)^{-1} U + U^T U \Rightarrow \\ U^T P_n U &= \left(U^T P_n^{-1} U + U^T U \Lambda U^T U\right)^{-1} U + U^T U \end{aligned}$$

but $U^T U = I_{M\times M}$ and by defining

$$P_{nn} = U^T P_n U \tag{66}$$

we can write

$$P_{nn} = \left(P_{nn}^{-1} + \Lambda\right)^{-1} + I_{M\times M}$$

In order to find a solution for P_{nn} in the last equation, we assume that P_{nn} is diagonal, i.e.,

$$P_{nn} = \operatorname{diag}(P_{nn_i}) \tag{67}$$

In that case we can find the diagonal elements P_{nn_i},$i = 1 \ldots 2M$ by solving the M equations

$$P_{nn_i} = \left(\frac{1}{P_{nn_i}} + \lambda_i\right)^{-1} + 1, \qquad i = 1 \ldots 2M \tag{68}$$

This is a set of M equations with scalar unknowns, that can be trivially solved, yielding

$$P_{nn_i} = \frac{1}{2} \pm \sqrt{\frac{1}{4} + \frac{1}{\lambda_i}}$$

Since $P^s_{rr_\infty}$ represents a covariance matrix, all the P_{nn_i}'s are positive, and thus we only keep the positive solutions. Finally, substitution in Eqs. (66) and (65) yields

$$P^s_{rr_\infty} = Q_s^{1/2} U \operatorname{diag}\left(\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{1}{\lambda_i}}\right) U^T Q_s^{1/2}$$

and thus

$$\mathbf{P}^{s(0)}_{rr_\infty} = P^s_{rr_\infty} \otimes I_{2\times 2} = \left(Q_s^{1/2} U \operatorname{diag}\left(\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{1}{\lambda_i}}\right) U^T Q_s^{1/2}\right) \otimes I_{2\times 2} \tag{69}$$

Note that the above is one solution to Eq. (63), that was derived based on the assumption of Eq. (67). However, the system of the M robots performing cooperative localization with absolute position measurements is an observable one [9]. Additionally, this is a controllable system, as we can easily verify. Therefore the algebraic Riccati equation in Eq. (63) has a *single* solution [8], the one given by the last expression. For future reference, we note that by application of the properties of the Kronecker product, the matrix $\mathbf{P}_{rr\infty}^{s(0)}$ can be alternatively written as

$$\mathbf{P}_{rr\infty}^{s(0)} = \left(\mathbf{Q}_s^{1/2}\mathbf{U}_s \,\mathrm{diag}\left(\frac{1}{2}+\sqrt{\frac{1}{4}+\frac{1}{\boldsymbol{\lambda}_{s_i}}}\right)\mathbf{U}_s^T\mathbf{Q}_s^{1/2}\right) \tag{70}$$

where the quantities $\mathbf{U}_s$ and $\boldsymbol{\lambda}_{s_i}$ are defined as the modal matrix and the eigenvalues respectively of the matrix

$$\mathbf{C} = \mathbf{Q}_s^{1/2}\mathbf{I}_{rr}\mathbf{Q}_s^{1/2}$$

and satisfy

$$\mathbf{U}_s = U \otimes I_{2\times 2}, \quad \mathrm{diag}(boldsymbol\lambda_{s_i}) = \mathrm{diag}(\lambda_i)\otimes I_{2\times 2}$$

Finally, the steady state solution of the Riccati recursion in Eq. (35) with zero initial condition is given by

$$\mathbf{P}_\infty^{s(0)} = \begin{bmatrix} \mathbf{P}_{rr\infty}^{s(0)} & \mathbf{0}_{2M\times 2N} \\ \mathbf{0}_{2N\times 2M} & \mathbf{0}_{2N\times 2N}\end{bmatrix} = \begin{bmatrix} \left(Q_s^{1/2}U\,\mathrm{diag}\left(\frac{1}{2}+\sqrt{\frac{1}{4}+\frac{1}{\lambda_i}}\right)U^TQ_s^{1/2}\right) & \mathbf{0}_{M\times N} \\ \mathbf{0}_{N\times M} & \mathbf{0}_{N\times N}\end{bmatrix}\otimes I_{2\times 2} \tag{71}$$

3.2.2 Solution with Nonzero Initial Covariance

In this section we determine the steady state solution to Eq. (35), when the initial covariance is a nonzero matrix. Although it is possible to derive a closed-form solution in the general case, in which the initial covariance matrix is an arbitrary positive semidefinite matrix, the resulting expressions are cumbersome, and do not provide intuition about the structure of the problem. Therefore, we here present the analysis for the case in which the estimates about the robots' and landmarks' positions are initially uncorrelated, i.e., the initial covariance matrix is of the form

$$\Pi_0 = \mathbf{P}(0) = \begin{bmatrix} \mathbf{P}_{rr_0} & \mathbf{0}_{2M\times 2N} \\ \mathbf{0}_{2N\times 2M} & \mathbf{P}_{LL_0}\end{bmatrix} \tag{72}$$

where $\mathbf{P}_{rr_0}$ and $\mathbf{P}_{LL_0}$ are arbitrary positive semidefinite matrices. We first derive two necessary intermediate results.

- **Steady State Solution of the Dual Riccati Recursion**

The derivation of the steady state solution to the dual Riccati recursion in Eq. (58) is simplified by exploiting the special structure of the measurement and information matrices. Specifically, we observe that all the matrices that appear in the dual Riccati recursion (Eq. (58)) can be written as the Kronecker product of some matrix with $I_{2\times 2}$, and the initial value of this recursion is zero. Employing the result of Appendix C, we conclude that at any time instant the solution to this recursion will be of the form

$$\mathbf{J}_k = \begin{bmatrix} \mathbf{J}_{rr_k} & \mathbf{J}_{Lr_k}^T \\ \mathbf{J}_{Lr_k} & \mathbf{J}_{LL_k}\end{bmatrix} = J_k \otimes I_{2\times 2} = \begin{bmatrix} J_{rr_k} & J_{Lr_k}^T \\ J_{Lr_k} & J_{LL_k}\end{bmatrix}\otimes I_{2\times 2}$$

The dual Riccati recursion in Eq. (58) leads to the following Riccati for J_k:

$$\begin{aligned}\begin{bmatrix} J_{rr_{k+1}} & J_{Lr_{k+1}}^T \\ J_{Lr_{k+1}} & J_{LL_{k+1}}\end{bmatrix} &= \begin{bmatrix} J_{rr_k} & J_{Lr_k}^T \\ J_{Lr_k} & J_{LL_k}\end{bmatrix} + \begin{bmatrix} I_{rr} & I_{Lr}^T \\ I_{Lr} & I_{LL}\end{bmatrix} \\ &- \begin{bmatrix} J_{rr_k} & J_{Lr_k}^T \\ J_{Lr_k} & J_{LL_k}\end{bmatrix}\begin{bmatrix} \left(Q_s^{-1}+J_{rr_k}\right)^{-1} & \mathbf{0}_{M\times N} \\ \mathbf{0}_{N\times M} & \mathbf{0}_{N\times N}\end{bmatrix}\begin{bmatrix} J_{rr_k} & J_{Lr_k}^T \\ J_{Lr_k} & J_{LL_k}\end{bmatrix}\end{aligned}$$

which can be decomposed in the following recursions:

$$J_{rr_{k+1}} = J_{rr_k} + I_{rr} - J_{rr_k}\left(Q_s^{-1}+J_{rr_k}\right)^{-1}J_{rr_k} \tag{73}$$

$$J_{Lr_{k+1}} = J_{Lr_k} + I_{Lr} - J_{rr_k}\left(Q_s^{-1}+J_{rr_k}\right)^{-1}J_{Lr_k} \tag{74}$$

$$J_{LL_{k+1}} = J_{LL_k} + I_{LL} - J_{Lr_k}\left(Q_s^{-1}+J_{rr_k}\right)^{-1}J_{Lr_k}^T \tag{75}$$

Now we can determine the steady state solution to each of the submatrix elements of J_k independently. Setting $J_{rr_{k+1}} = J_{rr_k} = J_{rr_\infty}$ (i.e., solution at steady state) in Eq. (73) yields

$$I_{rr} = J_{rr_\infty} \left(Q_s^{-1} + J_{rr_\infty}\right)^{-1} J_{rr_\infty} \tag{76}$$

or

$$\begin{aligned}
Q_s^{1/2} I_{rr} Q_s^{1/2} &= Q_s^{1/2} J_{rr_\infty} Q_s^{1/2} Q_s^{-1/2} \left(Q_s^{-1} + J_{rr_\infty}\right)^{-1} Q_s^{-1/2} Q_s^{1/2} J_{rr_\infty} Q_s^{1/2} \Rightarrow \\
U \Lambda U^T &= J_n \left(I_{M\times M} + J_n\right)^{-1} J_n \Rightarrow \\
\Lambda &= U^T J_n U U^T \left(I_{M\times M} + J_n\right)^{-1} U U^T J_n U \Rightarrow \\
\Lambda &= J_{nn} \left(I_{M\times M} + J_{nn}\right)^{-1} J_{nn}
\end{aligned}$$

where we have defined

$$J_n = Q_s^{1/2} J_{rr_\infty} Q_s^{1/2} \tag{77}$$

and

$$J_{nn} = U^T J_n U \tag{78}$$

At this point we assume that J_{nn} is diagonal, i.e., $J_{nn} = \mathrm{diag}\left(J_{nn_i}\right)$, and by solving the set of $2M$ scalar equations

$$\lambda_i = \frac{J_{nn_i}^2}{1 + J_{nn_i}}$$

and back-substituting in Eqs. (77) and (78), we derive the final expression:

$$J_{rr_\infty} = Q_s^{-1/2} U \,\mathrm{diag}\left(\frac{\lambda_i}{2} + \sqrt{\frac{\lambda_i^2}{4} + \lambda_i}\right) U^T Q_s^{-1/2} \tag{79}$$

It is easy to show that the Riccati recursion in Eq. (73) corresponds to a system that is both controllable and observable, and therefore the derived solution is unique. The matrix $\mathbf{J}_{rr_\infty}$ can be written as

$$\begin{aligned}
\mathbf{J}_{rr_\infty} &= \left(Q_s^{-1/2} U \,\mathrm{diag}\left(\frac{\lambda_i}{2} + \sqrt{\frac{\lambda_i^2}{4} + \lambda_i}\right) U^T Q_s^{-1/2}\right) \otimes I_{2\times 2} \\
&= \mathbf{Q}_s^{-1/2} \mathbf{U}_s \,\mathrm{diag}\left(\frac{\boldsymbol{\lambda}_{s_i}}{2} + \sqrt{\frac{\boldsymbol{\lambda}_{s_i}^2}{4} + \boldsymbol{\lambda}_{s_i}}\right) \mathbf{U}_s^T \mathbf{Q}_s^{-1/2}
\end{aligned} \tag{80}$$

In order to derive the steady state value of $J_{Lr_{k+1}}$, we set $J_{Lr_{k+1}} = J_{Lr_k} = J_{Lr_\infty}$ in Eq. (74) and solving for J_{Lr_∞} yields

$$J_{Lr_\infty} = I_{Lr} J_{rr_\infty}^{-1} \left(Q_s^{-1} + J_{rr_\infty}\right) \tag{81}$$

Finally, we note that Eq. (75) can be written as

$$J_{LL_{k+1}} - J_{LL_k} = I_{LL} - J_{Lr_k} \left(Q_s^{-1} + J_{rr_k}\right)^{-1} J_{Lr_k}^T$$

In this expression, the right-hand side is independent of J_{LL_k}, and, after sufficient time, it approaches a constant value given by

$$\begin{aligned}
\lim_{k\to\infty} J_{LL_{k+1}} - J_{LL_k} &= \lim_{k\to\infty} I_{LL} - J_{Lr_k} \left(Q_s^{-1} + J_{rr_k}\right)^{-1} J_{Lr_k}^T \\
&= I_{LL} - J_{Lr_\infty} \left(Q_s^{-1} + J_{rr_\infty}\right)^{-1} J_{Lr_\infty}^T
\end{aligned}$$

We thus conclude that at steady state the term J_{LL_k} increases at a constant rate, i.e.,

$$\begin{aligned}
J_{LL_{ss}} &= \left(I_{LL} - J_{Lr_\infty} \left(Q_s^{-1} + J_{rr_\infty}\right)^{-1} J_{Lr_\infty}^T\right) k + J_c \\
&= J_a k + J_c
\end{aligned} \tag{82}$$

where J_c is a constant term. In the derivations that follow, the exact value of this term is not required. We only require knowledge of the sum of the elements of J_c, i.e., the quantity $\mathbf{1}_{1\times N} J_c \mathbf{1}_{N\times 1}$. This is computed by noting that the sum of all the elements of the matrix J_k is equal to

$$\begin{aligned}
\mathbf{1}_{1\times(M+N)} J_k \mathbf{1}_{1\times(M+N)} &= \mathbf{1}_{1\times M} J_{rr_k} \mathbf{1}_{M\times 1} + 2\mathbf{1}_{1\times N} J_{Lr_k} \mathbf{1}_{M\times 1} + \mathbf{1}_{1\times N} J_{LL_k} \mathbf{1}_{N\times 1} \Rightarrow \\
\mathbf{1}_{1\times N} J_{LL_k} \mathbf{1}_{N\times 1} &= \mathbf{1}_{1\times(M+N)} J_k \mathbf{1}_{1\times(M+N)} - 2\mathbf{1}_{1\times N} J_{Lr_k} \mathbf{1}_{M\times 1} - \mathbf{1}_{1\times M} J_{rr_k} \mathbf{1}_{M\times 1}
\end{aligned}$$

Evaluating this expression at steady state yields

$$\mathbf{1}_{1\times N} \left(J_a k + J_c\right) \mathbf{1}_{N\times 1} = \mathbf{1}_{1\times(M+N)} J_\infty \mathbf{1}_{1\times(M+N)} - 2\mathbf{1}_{1\times N} J_{Lr_\infty} \mathbf{1}_{M\times 1} - \mathbf{1}_{1\times M} J_{rr_\infty} \mathbf{1}_{M\times 1} \tag{83}$$

In Appendix E it is shown that $\mathrm{rank}(J_a) = N - 1$, i.e., J_a is rank deficient, having one eigenvalue equal to zero. The eigenvector associated with the zero eigenvalue is shown to be $\frac{1}{\sqrt{1}}\mathbf{1}_{N\times 1}$, which implies that $\mathbf{1}_{1\times N} J_a \mathbf{1}_{N\times 1} = 0$. Moreover, in Appendix D it is shown that the sum of all the elements of matrix J_k is equal to zero, for *all* $k \geq 0$. Thus Eq. (83) yields

$$\mathbf{1}_{1\times N} J_c \mathbf{1}_{N\times 1} = -2\mathbf{1}_{1\times N} J_{Lr_\infty} \mathbf{1}_{M\times 1} - \mathbf{1}_{1\times M} J_{rr_\infty} \mathbf{1}_{M\times 1} \tag{84}$$

At this point we show that $\mathbf{1}_{1\times N} J_{Lr_\infty} = -\mathbf{1}_{1\times M} J_{rr_\infty}$. Substitution from Eq. (81) yields

$$\mathbf{1}_{1\times N} J_{Lr_\infty} = \mathbf{1}_{1\times N} I_{Lr} J_{rr_\infty}^{-1} \left(Q_s^{-1} + J_{rr_\infty}\right)$$

But in Appendix E (cf Eq. (133)) it is shown that $\mathbf{1}_{1\times N} I_{Lr} = -\mathbf{1}_{1\times M} I_{rr}$ and thus

$$\mathbf{1}_{1\times N} J_{Lr_\infty} = -\mathbf{1}_{1\times M} I_{rr} J_{rr_\infty}^{-1} \left(Q_s^{-1} + J_{rr_\infty}\right) \tag{85}$$

Substitution for Irr from Eq. (76) yields

$$\begin{aligned}
\mathbf{1}_{1\times N} J_{Lr_\infty} &= -\mathbf{1}_{1\times M} J_{rr_\infty} \left(Q_s^{-1} + J_{rr_\infty}\right)^{-1} J_{rr_\infty} J_{rr_\infty}^{-1} \left(Q_s^{-1} + J_{rr_\infty}\right) \\
&= -\mathbf{1}_{1\times M} J_{rr_\infty}
\end{aligned} \tag{86}$$

which is the desired result. Using this property in Eq. (84) we obtain

$$\begin{aligned}
\mathbf{1}_{1\times N} J_c \mathbf{1}_{N\times 1} &= 2\mathbf{1}_{1\times M} J_{rr_\infty} \mathbf{1}_{M\times 1} - \mathbf{1}_{1\times M} J_{rr_\infty} \mathbf{1}_{M\times 1} \\
&= \mathbf{1}_{1\times M} J_{rr_\infty} \mathbf{1}_{M\times 1}
\end{aligned} \tag{87}$$

■

- **Evaluation of the Term $\Phi_p^{(0)}(k+1,0)$ at Steady State**

  Recall that $\Phi_p^{(0)}(k+1,0)$ is defined as

$$\Phi_p^{(0)}(k+1,0) = \left(I_{\xi\times\xi} - \mathbf{P}\mathbf{H}_o'^T \left(\mathbf{R}_s + \mathbf{H}_o'\mathbf{P}\mathbf{H}_o'^T\right)^{-1} \mathbf{H}_o'\right)^{k+1} \left[I_{\xi\times\xi} + \mathbf{P}\mathbf{J}_{k+1}\right]$$

where $\mathbf{P}$ is any solution to the DARE

$$\mathbf{P} = \mathbf{P} - \mathbf{P}\mathbf{H}_o'^T \left(\mathbf{R}_s + \mathbf{H}_o'\mathbf{P}\mathbf{H}_o'^T\right)^{-1} \mathbf{H}_o'\mathbf{P} + \mathbf{G}_o\mathbf{Q}_s\mathbf{G}_o^T$$

It is easy to show, by substitution, that $\mathbf{P}_\infty^{s(0)}$ (cf. Eq. (71)) satisfies this DARE and therefore

$$\Phi_p^{(0)}(k+1,0) = \left(I_{\xi\times\xi} - \mathbf{P}_\infty^{s(0)}\mathbf{H}_o'^T \left(\mathbf{R}_s + \mathbf{H}_o'\mathbf{P}_\infty^{s(0)}\mathbf{H}_o'^T\right)^{-1} \mathbf{H}_o'\right)^{k+1} \left[I_{\xi\times\xi} + \mathbf{P}_\infty^{s(0)}\mathbf{J}_{k+1}\right] \tag{88}$$

Application of the matrix inversion lemma (cf. Appendix G) yields

$$\begin{aligned}
I_{\xi\times\xi} - \mathbf{P}_\infty^{s(0)}\mathbf{H}_o'^T \left(\mathbf{R}_s + \mathbf{H}_o'\mathbf{P}_\infty^{s(0)}\mathbf{H}_o'^T\right)^{-1} \mathbf{H}_o' &= \left(I_{\xi\times\xi} + \mathbf{P}_\infty^{s(0)}\mathbf{H}_o'^T\mathbf{R}_s^{-1}\mathbf{H}_o'\right)^{-1} \\
&= A \otimes I_{2\times 2}
\end{aligned}$$

where

$$A = \begin{bmatrix} I_{M\times M} + P^s_{rr\infty} I_{rr} & P^s_{rr\infty} I^T_{Lr} \\ \mathbf{0}_{N\times M} & I_{N\times N} \end{bmatrix}^{-1}$$

We note that

$$\begin{aligned} C = U\Lambda U^T &= Q_s^{1/2} I_{rr} Q_s^{1/2} \Rightarrow \\ I_{rr} &= Q_s^{-1/2} U\Lambda U^T Q_s^{-1/2} \end{aligned}$$

and thus the $M \times M$ principal diagonal submatrix of A equals

$$\begin{aligned} I_{M\times M} + P^s_{rr\infty} I_{rr} &= I_{M\times M} + P^s_{rr\infty} Q_s^{-1/2} U\Lambda U^T Q_s^{-1/2} \\ &= I_{M\times M} + \left(Q_s^{1/2} U \operatorname{diag}\left(\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{1}{\lambda_i}} \right) U^T Q_s^{1/2} \right) Q_s^{-1/2} U\Lambda U^T Q_s^{-1/2} \\ &= I_{M\times M} + Q_s^{1/2} U \operatorname{diag}\left(\frac{\lambda_i}{2} + \sqrt{\frac{\lambda_i^2}{4} + \lambda_i} \right) U^T Q_s^{-1/2} \\ &= Q_s^{1/2} U \operatorname{diag}\left(1 + \frac{\lambda_i}{2} + \sqrt{\frac{\lambda_i^2}{4} + \lambda_i} \right) U^T Q_s^{-1/2} \\ &= Q_s^{1/2} U \operatorname{diag}\left(f(\lambda_i) \right) U^T Q_s^{-1/2} \end{aligned}$$

where

$$f(\lambda_i) = 1 + \frac{\lambda_i}{2} + \sqrt{\frac{\lambda_i^2}{4} + \lambda_i}$$

At this point we employ the eigendecomposition of matrix A. It is easy to verify, by carrying out the matrix multiplications and applying the formula for the inversion of a partitioned matrix (cf. Appendix H), that

$$A = VLV^{-1}$$

where

$$L = \begin{bmatrix} \operatorname{diag}(f(\lambda_i)) & \mathbf{0}_{N\times M} \\ \mathbf{0}_{N\times M} & I_{N\times N} \end{bmatrix} \text{ and } V = \begin{bmatrix} Q_s^{1/2} U & -I_{rr}^{-1} I_{Lr}^T \\ \mathbf{0}_{N\times M} & I_{N\times N} \end{bmatrix} = \begin{bmatrix} Q_s^{1/2} U & \Phi_o \\ \mathbf{0}_{N\times M} & I_{N\times N} \end{bmatrix}$$

with

$$\Phi_o = -I_{rr}^{-1} I_{Lr}^T$$

We can now write

$$\begin{aligned} \left(I_{\xi\times\xi} - \mathbf{P}_\infty^{s(0)} \mathbf{H}_o'^T \left(\mathbf{R}_s + \mathbf{H}_o' \mathbf{P}_\infty^{s(0)} \mathbf{H}_o'^T \right)^{-1} \mathbf{H}_o' \right)^{k+1} &= \left(A^{-1} \otimes I_{2\times 2} \right)^{k+1} \\ &= A^{-(k+1)} \otimes I_{2\times 2} \\ &= VL^{-(k+1)} V^{-1} \end{aligned}$$

and

$$\begin{aligned} \lim_{k\to\infty} \left(I_{\xi\times\xi} - \mathbf{P}_\infty^{s(0)} \mathbf{H}_o'^T \left(\mathbf{R}_s + \mathbf{H}_o' \mathbf{P}_\infty^{s(0)} \mathbf{H}_o'^T \right)^{-1} \mathbf{H}_o' \right)^{k+1} &= \lim_{k\to\infty} VL^{-(k+1)} V^{-1} \\ &= V \left(\lim_{k\to\infty} L^{-(k+1)} \right) V^{-1} \end{aligned}$$

But

$$\begin{aligned}
\lim_{k\to\infty} L^{-(k+1)} &= \begin{bmatrix} \lim_{k\to\infty} \mathrm{diag}\,(f(\lambda_i))^{-(k+1)} & \mathbf{0}_{N\times M} \\ \mathbf{0}_{N\times M} & I_{N\times N} \end{bmatrix} \\
&= \begin{bmatrix} \lim_{k\to\infty} \mathrm{diag}\left(\frac{1}{f(\lambda_i)}\right)^{(k+1)} & \mathbf{0}_{N\times M} \\ \mathbf{0}_{N\times M} & I_{N\times N} \end{bmatrix} \\
&= \begin{bmatrix} \mathbf{0}_{M\times M} & \mathbf{0}_{N\times M} \\ \mathbf{0}_{N\times M} & I_{N\times N} \end{bmatrix}
\end{aligned}$$

since $f(\lambda_i) > 1$. Using the last result we obtain

$$\begin{aligned}
\lim_{k\to\infty} \left(I_{\xi\times\xi} - \mathbf{P}_\infty^{s(0)} \mathbf{H}_o'^T \left(\mathbf{R}_s + \mathbf{H}_o' \mathbf{P}_\infty^{s(0)} \mathbf{H}_o'^T \right)^{-1} \mathbf{H}_o' \right)^{k+1} &= V \left(\lim_{k\to\infty} L^{-(k+1)} \right) V^{-1} \\
&= \begin{bmatrix} \mathbf{0}_{M\times M} & \Phi_o \\ \mathbf{0}_{N\times M} & I_{N\times N} \end{bmatrix}
\end{aligned}$$

Moreover,

$$\lim_{k\to\infty} \left[I_{\xi\times\xi} + \mathbf{P}_\infty^{s(0)} \mathbf{J}_{k+1} \right] = \begin{bmatrix} I_{N\times N} + P^s_{rr_\infty} J_{rr_\infty} & P^s_{rr_\infty} J^T_{Lr_\infty} \\ \mathbf{0}_{N\times M} & I_{N\times N} \end{bmatrix} \otimes I_{2\times 2}$$

and thus Eq. (88) yields

$$\begin{aligned}
\lim_{k\to\infty} \Phi_p^{(0)}(k+1,0) &= \left(\begin{bmatrix} \mathbf{0}_{M\times M} & \Phi_o \\ \mathbf{0}_{N\times M} & I_{N\times N} \end{bmatrix} \otimes I_{2\times 2} \right) \left(\begin{bmatrix} I_{N\times N} + P^s_{rr_\infty} J_{rr_\infty} & P^s_{rr_\infty} J^T_{Lr_\infty} \\ \mathbf{0}_{N\times M} & I_{N\times N} \end{bmatrix} \otimes I_{2\times 2} \right) \\
&= \begin{bmatrix} \mathbf{0}_{M\times M} & \Phi_o \\ \mathbf{0}_{N\times M} & I_{N\times N} \end{bmatrix} \otimes I_{2\times 2} \\
&= \begin{bmatrix} \mathbf{0}_{2M\times 2M} & \Phi_o \otimes I_{2\times 2} \\ \mathbf{0}_{2N\times 2M} & I_{2N\times 2N} \end{bmatrix}
\end{aligned}$$

■

We can now evaluate the steady state value of $\mathbf{P}_k^s$ by computing the limit of the left hand side member of Eq. (56) after sufficient time. We denote

$$\begin{bmatrix} P_a & P_b \\ P_c & P_d \end{bmatrix} = \lim_{k\to\infty} \left[I + \Pi_0 \mathbf{J}_{k+1} \right]^{-1} \Pi_0 \tag{89}$$

and we obtain

$$\begin{aligned}
\lim_{k\to\infty} \left(\mathbf{P}^s_{k+1} - \mathbf{P}^{s(0)}_{k+1} \right) &= \lim_{k\to\infty} \Phi_p^{(0)}(k+1,0) \left[I + \Pi_0 \mathbf{J}_{k+1} \right]^{-1} \Pi_0 \Phi_p^{(0)}(k+1,0)^T \\
&= \lim_{k\to\infty} \Phi_p^{(0)}(k+1,0) \begin{bmatrix} P_a & P_b \\ P_c & P_d \end{bmatrix} \lim_{k\to\infty} \Phi_p^{(0)}(k+1,0)^T \\
&= \begin{bmatrix} \mathbf{0}_{2M\times 2M} & \Phi_o \otimes I_{2\times 2} \\ \mathbf{0}_{2N\times 2M} & I_{2N\times 2N} \end{bmatrix} \begin{bmatrix} P_a & P_b \\ P_c & P_d \end{bmatrix} \begin{bmatrix} \mathbf{0}_{2M\times 2M} & \Phi_o^T \otimes I_{2\times 2} \\ \mathbf{0}_{2N\times 2M} & I_{2N\times 2N} \end{bmatrix} \\
&= \begin{bmatrix} (\Phi_o \otimes I_{2\times 2}) P_d \left(\Phi_o^T \otimes I_{2\times 2} \right) & (\Phi_o \otimes I_{2\times 2}) P_d \\ P_d \left(\Phi_o^T \otimes I_{2\times 2} \right) & P_d \end{bmatrix}
\end{aligned} \tag{90}$$

From the last expression we conclude that only the P_d submatrix is required in order to determine the steady state value of $\mathbf{P}_k^s$. Substituting from Eqs. (72) and (59) yields

$$\lim_{k\to\infty} \left[I + \Pi_0 \mathbf{J}_{k+1} \right]^{-1} \Pi_0 = \lim_{k\to\infty} \left(I_{\xi\times\xi} + \begin{bmatrix} \mathbf{P}_{rr_0} & \mathbf{0}_{2M\times 2N} \\ \mathbf{0}_{2N\times 2M} & \mathbf{P}_{LL_0} \end{bmatrix} \begin{bmatrix} \mathbf{J}_{rr_k} & \mathbf{J}^T_{Lr_k} \\ \mathbf{J}_{Lr_k} & \mathbf{J}_{LL_k} \end{bmatrix} \right)^{-1} \begin{bmatrix} \mathbf{P}_{rr_0} & \mathbf{0}_{2M\times 2N} \\ \mathbf{0}_{2N\times 2M} & \mathbf{P}_{LL_0} \end{bmatrix}$$

$$
\begin{aligned}
&= \lim_{k\to\infty} \left[\begin{array}{cc} I_{2M\times 2M} + \mathbf{P}_{rr_0}\mathbf{J}_{rr_k} & \mathbf{P}_{rr_0}\mathbf{J}_{Lr_k}^T \\ \mathbf{P}_{LL_0}\mathbf{J}_{Lr_k} & I_{2N\times 2N} + \mathbf{P}_{LL_0}\mathbf{J}_{LL_k} \end{array} \right]^{-1} \left[\begin{array}{cc} \mathbf{P}_{rr_0} & \mathbf{0}_{2M\times 2N} \\ \mathbf{0}_{2N\times 2M} & \mathbf{P}_{LL_0} \end{array} \right] \\
&= \lim_{k\to\infty} \left[\begin{array}{cc} A_1 & A_2 \\ A_3 & A_4 \end{array} \right] \left[\begin{array}{cc} \mathbf{P}_{rr_0} & \mathbf{0}_{2M\times 2N} \\ \mathbf{0}_{2N\times 2M} & \mathbf{P}_{LL_0} \end{array} \right] \\
&= \lim_{k\to\infty} \left[\begin{array}{cc} A_1\mathbf{P}_{rr_0} & A_2\mathbf{P}_{LL_0} \\ A_3\mathbf{P}_{rr_0} & A_4\mathbf{P}_{LL_0} \end{array} \right]
\end{aligned}
$$

Thus $P_d = \lim_{k\to\infty} A_4\mathbf{P}_{LL_0}$ where the matrix A_4 can be computed by application of the formula for the inversion of a partitioned matrix, given in Appendix H. This computation yields

$$
\begin{aligned}
P_d &= \lim_{k\to\infty} A_4\mathbf{P}_{LL_0} \\
&= \lim_{k\to\infty} \left(I_{2N\times 2N} + \mathbf{P}_{LL_0}\mathbf{J}_{LL_k} - \mathbf{P}_{LL_0}\mathbf{J}_{Lr_k} \left(I_{2M\times 2M} + \mathbf{P}_{rr_0}\mathbf{J}_{rr_k}\right)^{-1} \mathbf{P}_{rr_0}\mathbf{J}_{Lr_k}^T \right)^{-1} \mathbf{P}_{LL_0}
\end{aligned}
$$

But as $k \to \infty$,

$$
\mathbf{J}_{LL_k} \to \mathbf{J}_{LL_{ss}} = \mathbf{J}_a k + \mathbf{J}_c
$$

At this point, in order to simplify the following derivations, we assume that $\mathbf{P}_{LL_0}$ is invertible, although a (considerably more involved) solution can also be derived in the case that $\mathbf{P}_{LL_0}$ is singular. Thus we can write

$$
\begin{aligned}
P_d &= \lim_{k\to\infty} \left(I_{2N\times 2N} + \mathbf{P}_{LL_0}\mathbf{J}_{LL_k} - \mathbf{P}_{LL_0}\mathbf{J}_{Lr_k} \left(I_{2M\times 2M} + \mathbf{P}_{rr_0}\mathbf{J}_{rr_k}\right)^{-1} \mathbf{P}_{rr_0}\mathbf{J}_{Lr_k}^T \right)^{-1} \mathbf{P}_{LL_0} \\
&= \lim_{k\to\infty} \left(\mathbf{P}_{LL_0}^{-1} + \mathbf{J}_{LL_k} - \mathbf{J}_{Lr_k} \left(I_{2M\times 2M} + \mathbf{P}_{rr_0}\mathbf{J}_{rr_k}\right)^{-1} \mathbf{P}_{rr_0}\mathbf{J}_{Lr_k}^T \right)^{-1} \\
&= \lim_{k\to\infty} \left(\mathbf{P}_{LL_0}^{-1} + \mathbf{J}_a k + \mathbf{J}_c - \mathbf{J}_{Lr_k} \left(I_{2M\times 2M} + \mathbf{P}_{rr_0}\mathbf{J}_{rr_k}\right)^{-1} \mathbf{P}_{rr_0}\mathbf{J}_{Lr_k}^T \right)^{-1} \\
&= \lim_{k\to\infty} \left(\mathbf{J}_a k + \mathbf{D}_k\right)^{-1} \qquad (91)
\end{aligned}
$$

where we have denoted

$$
\mathbf{D}_k = \mathbf{P}_{LL_0}^{-1} + \mathbf{J}_c - \mathbf{J}_{Lr_k} \left(I_{2M\times 2M} + \mathbf{P}_{rr_0}\mathbf{J}_{rr_k}\right)^{-1} \mathbf{P}_{rr_0}\mathbf{J}_{Lr_k}^T
$$

In Appendix E it is shown that $\mathbf{J}_a = J_a \otimes I_{2\times 2}$ is singular, and thus computation of the limit in Eq. (91) is not trivial. However, we can now employ the following lemma, whose proof can be found in Appendix F, to determine the limit in Eq. (91).

Lemma 3.5 *If Y is a symmetric square matrix, whose singular value decomposition is denoted as $Y = U\Lambda U^T$, and B_k is a matrix of compatible dimensions whose limit as $k \to \infty$ exists, then*

$$
\lim_{k\to\infty} (Yk + B_k)^{-1} = U_N (U_N^T B_\infty U_N)^{-1} U_N^T \qquad (92)
$$

if the matrix $U_N^T B_\infty U_N$ is inveritble. In the last expression U_N is a matrix whose column vectors form a basis of the nullspace of Y.

The column vectors of the matrix $\frac{1}{\sqrt{N}}\mathbf{1}_{N\times 1} \otimes I_{2\times 2}$ form a basis to the nullspace of $\mathbf{J}_a$, and thus, by setting $Y \to \mathbf{J}_a$ and $B_k \to \mathbf{D}_k$, we can now write

$$
\begin{aligned}
P_d &= \lim_{k\to\infty} \left(\mathbf{J}_a k + \mathbf{D}_k\right)^{-1} \\
&= \left(\frac{1}{\sqrt{N}}\mathbf{1}_{N\times 1} \otimes I_{2\times 2} \right) \left(\left(\frac{1}{\sqrt{N}}\mathbf{1}_{1\times N} \otimes I_{2\times 2} \right) \mathbf{D}_\infty \left(\frac{1}{\sqrt{N}}\mathbf{1}_{N\times 1} \otimes I_{2\times 2} \right) \right)^{-1} \left(\frac{1}{\sqrt{N}}\mathbf{1}_{1\times N} \otimes I_{2\times 2} \right) \\
&= \left(\mathbf{1}_{N\times 1} \otimes I_{2\times 2}\right) \left(\left(\mathbf{1}_{1\times N} \otimes I_{2\times 2}\right) \mathbf{D}_\infty \left(\mathbf{1}_{N\times 1} \otimes I_{2\times 2}\right) \right)^{-1} \left(\mathbf{1}_{1\times N} \otimes I_{2\times 2}\right) \\
&= \mathbf{1}_{N\times N} \otimes \left(\left(\mathbf{1}_{1\times N} \otimes I_{2\times 2}\right) \mathbf{D}_\infty \left(\mathbf{1}_{N\times 1} \otimes I_{2\times 2}\right) \right)^{-1} \\
&= \mathbf{1}_{N\times N} \otimes \mathbf{\Theta}_s^{-1}
\end{aligned}
$$

where $\mathbf{\Theta}_s$ is a 2×2 matrix defined as

$$\begin{aligned}
\mathbf{\Theta}_s &= (\mathbf{1}_{1\times N} \otimes I_{2\times 2})\,\mathbf{D}_\infty\,(\mathbf{1}_{N\times 1} \otimes I_{2\times 2}) \\
&= (\mathbf{1}_{1\times N} \otimes I_{2\times 2})\left(\mathbf{P}_{LL_0}^{-1} + \mathbf{J}_c - \mathbf{J}_{Lr_\infty}\,(I_{2M\times 2M} + \mathbf{P}_{rr_0}\mathbf{J}_{rr_\infty})^{-1}\,\mathbf{P}_{rr_0}\mathbf{J}_{Lr_\infty}^T\right)(\mathbf{1}_{N\times 1} \otimes I_{2\times 2}) \\
&= (\mathbf{1}_{1\times N} \otimes I_{2\times 2})\,\mathbf{P}_{LL_0}^{-1}\,(\mathbf{1}_{N\times 1} \otimes I_{2\times 2}) + (\mathbf{1}_{1\times N} \otimes I_{2\times 2})\,(J_c \otimes I_{2\times 2})\,(\mathbf{1}_{N\times 1} \otimes I_{2\times 2}) \\
&\quad - (\mathbf{1}_{1\times N} \otimes I_{2\times 2})\,\mathbf{J}_{Lr_\infty}\,(I_{2M\times 2M} + \mathbf{P}_{rr_0}\mathbf{J}_{rr_\infty})^{-1}\,\mathbf{P}_{rr_0}\mathbf{J}_{Lr_\infty}^T\,(\mathbf{1}_{N\times 1} \otimes I_{2\times 2}) \\
&= (\mathbf{1}_{1\times N} \otimes I_{2\times 2})\,\mathbf{P}_{LL_0}^{-1}\,(\mathbf{1}_{N\times 1} \otimes I_{2\times 2}) + (\mathbf{1}_{1\times N} J_c \mathbf{1}_{N\times 1}) \otimes I_{2\times 2} \\
&\quad - (\mathbf{1}_{1\times N} \otimes I_{2\times 2})\,\mathbf{J}_{Lr_\infty}\,(I_{2M\times 2M} + \mathbf{P}_{rr_0}\mathbf{J}_{rr_\infty})^{-1}\,\mathbf{P}_{rr_0}\mathbf{J}_{Lr_\infty}^T\,(\mathbf{1}_{N\times 1} \otimes I_{2\times 2})
\end{aligned}$$

Using the result of Eq. (87) the last expression yields

$$\begin{aligned}
\mathbf{\Theta}_s &= (\mathbf{1}_{1\times N} \otimes I_{2\times 2})\,\mathbf{P}_{LL_0}^{-1}\,(\mathbf{1}_{N\times 1} \otimes I_{2\times 2}) + (\mathbf{1}_{1\times N} J_{rr_\infty} \mathbf{1}_{N\times 1}) \otimes I_{2\times 2} \\
&\quad - (\mathbf{1}_{1\times N} \otimes I_{2\times 2})\,\mathbf{J}_{Lr_\infty}\,(I_{2M\times 2M} + \mathbf{P}_{rr_0}\mathbf{J}_{rr_\infty})^{-1}\,\mathbf{P}_{rr_0}\mathbf{J}_{Lr_\infty}^T\,(\mathbf{1}_{N\times 1} \otimes I_{2\times 2}) \\
&= (\mathbf{1}_{1\times N} \otimes I_{2\times 2})\,\mathbf{P}_{LL_0}^{-1}\,(\mathbf{1}_{N\times 1} \otimes I_{2\times 2}) + (\mathbf{1}_{1\times N} \otimes I_{2\times 2})\,\mathbf{J}_{rr_\infty}\,(\mathbf{1}_{N\times 1} \otimes I_{2\times 2}) \\
&\quad - (\mathbf{1}_{1\times N} \otimes I_{2\times 2})\,\mathbf{J}_{Lr_\infty}\,(I_{2M\times 2M} + \mathbf{P}_{rr_0}\mathbf{J}_{rr_\infty})^{-1}\,\mathbf{P}_{rr_0}\mathbf{J}_{Lr_\infty}^T\,(\mathbf{1}_{N\times 1} \otimes I_{2\times 2})
\end{aligned}$$

Employing the property $\mathbf{1}_{1\times N} J_{Lr_\infty} = \mathbf{1}_{1\times M} J_{rr_\infty}$ (cf. Eq. (86)) we obtain

$$\begin{aligned}
\mathbf{\Theta}_s &= (\mathbf{1}_{1\times N} \otimes I_{2\times 2})\,\mathbf{P}_{LL_0}^{-1}\,(\mathbf{1}_{N\times 1} \otimes I_{2\times 2}) + (\mathbf{1}_{1\times N} \otimes I_{2\times 2})\,\mathbf{J}_{rr_\infty}\,(\mathbf{1}_{N\times 1} \otimes I_{2\times 2}) \\
&\quad - (\mathbf{1}_{1\times M} \otimes I_{2\times 2})\,\mathbf{J}_{rr_\infty}\,(I_{2M\times 2M} + \mathbf{P}_{rr_0}\mathbf{J}_{rr_\infty})^{-1}\,\mathbf{P}_{rr_0}\mathbf{J}_{rr_\infty}\,(\mathbf{1}_{M\times 1} \otimes I_{2\times 2}) \\
&= (\mathbf{1}_{1\times N} \otimes I_{2\times 2})\,\mathbf{P}_{LL_0}^{-1}\,(\mathbf{1}_{N\times 1} \otimes I_{2\times 2}) \\
&\quad + (\mathbf{1}_{1\times M} \otimes I_{2\times 2})\left(\mathbf{J}_{rr_\infty} - \mathbf{J}_{rr_\infty}\,(I_{2\times 2} + \mathbf{P}_{rr_0}\mathbf{J}_{rr_\infty})^{-1}\,\mathbf{P}_{rr_0}\mathbf{J}_{rr_\infty}\right)(\mathbf{1}_{M\times 1} \otimes I_{2\times 2}) \\
&= (\mathbf{1}_{1\times N} \otimes I_{2\times 2})\,\mathbf{P}_{LL_0}^{-1}\,(\mathbf{1}_{N\times 1} \otimes I_{2\times 2}) + (\mathbf{1}_{1\times M} \otimes I_{2\times 2})\left(\mathbf{J}_{rr_\infty}^{-1} + \mathbf{P}_{rr_0}\right)^{-1}(\mathbf{1}_{M\times 1} \otimes I_{2\times 2})
\end{aligned}$$

where the Matrix Inversion Lemma (cf. Appendix G) has been employed in the last line. Finally, substitution in Eq. (90), yields

$$\lim_{k\to\infty}\left(\mathbf{P}_{k+1}^s - \mathbf{P}_{k+1}^{s(0)}\right) = \begin{bmatrix} (\Phi_o \otimes I_{2\times 2})\left(\mathbf{1}_{N\times N} \otimes \mathbf{\Theta}_s^{-1}\right)(\Phi_o^T \otimes I_{2\times 2}) & (\Phi_o \otimes I_{2\times 2})\left(\mathbf{1}_{N\times N} \otimes \mathbf{\Theta}_s^{-1}\right) \\ \left(\mathbf{1}_{N\times N} \otimes \mathbf{\Theta}_s^{-1}\right)(\Phi_o^T \otimes I_{2\times 2}) & \mathbf{1}_{N\times N} \otimes \mathbf{\Theta}_s^{-1} \end{bmatrix}$$

Applying the property of the Kronecker product in Eq. (126) we can write

$$\begin{aligned}
\left(\mathbf{1}_{N\times N} \otimes \mathbf{\Theta}_s^{-1}\right)(\Phi_o^T \otimes I_{2\times 2}) &= \left(\mathbf{1}_{N\times N}\Phi_o^T\right) \otimes \mathbf{\Theta}_s^{-1} \\
&= \left(\mathbf{1}_{N\times 1}\mathbf{1}_{1\times N}\Phi_o^T\right) \otimes \mathbf{\Theta}_s^{-1}
\end{aligned}$$

and similarly

$$\begin{aligned}
(\Phi_o \otimes I_{2\times 2})\left(\mathbf{1}_{N\times N} \otimes \mathbf{\Theta}_s^{-1}\right)(\Phi_o^T \otimes I_{2\times 2}) &= \left(\Phi_o\mathbf{1}_{N\times N}\Phi_o^T\right) \otimes \mathbf{\Theta}_s^{-1} \\
&= \left(\Phi_o\mathbf{1}_{N\times 1}\mathbf{1}_{1\times N}\Phi_o^T\right) \otimes \mathbf{\Theta}_s^{-1}
\end{aligned}$$

But

$$\begin{aligned}
\mathbf{1}_{1\times N}\Phi_o^T &= -\mathbf{1}_{1\times N} I_{Lr} I_{rr}^{-1} \\
&= \mathbf{1}_{1\times M} I_{rr} I_{rr}^{-1} \\
&= \mathbf{1}_{1\times M}
\end{aligned}$$

and thus

$$\begin{aligned}
\lim_{k\to\infty}\left(\mathbf{P}_{k+1}^s - \mathbf{P}_{k+1}^{s(0)}\right) &= \begin{bmatrix} \left(\Phi_o\mathbf{1}_{N\times 1}\mathbf{1}_{1\times N}\Phi_o^T\right) \otimes \mathbf{\Theta}_s^{-1} & \left(\Phi_o\mathbf{1}_{N\times 1}\mathbf{1}_{1\times N}\right) \otimes \mathbf{\Theta}_s^{-1} \\ \left(\mathbf{1}_{N\times 1}\mathbf{1}_{1\times N}\Phi_o^T\right) \otimes \mathbf{\Theta}_s^{-1} & \mathbf{1}_{N\times N} \otimes \mathbf{\Theta}_s^{-1} \end{bmatrix} \\
&= \begin{bmatrix} \left(\mathbf{1}_{M\times 1}\mathbf{1}_{1\times M}\right) \otimes \mathbf{\Theta}_s^{-1} & \left(\mathbf{1}_{M\times 1}\mathbf{1}_{1\times N}\right) \otimes \mathbf{\Theta}_s^{-1} \\ \left(\mathbf{1}_{N\times 1}\mathbf{1}_{1\times M}\right) \otimes \mathbf{\Theta}_s^{-1} & \mathbf{1}_{N\times N} \otimes \mathbf{\Theta}_s^{-1} \end{bmatrix} \\
&= \mathbf{1}_{(M+N)\times(M+N)} \otimes \mathbf{\Theta}_s^{-1}
\end{aligned}$$

We synopsize the preceding analysis in the following lemma:

Lemma 3.6 *The steady state value of the covariance matrix in LTI C-SLAM Eq. (53), when the initial value of the covariance matrix is*

$$\mathbf{P}_0 = \begin{bmatrix} \mathbf{P}_{rr_0} & \mathbf{0}_{2M\times 2N} \\ \mathbf{0}_{2N\times 2M} & \mathbf{P}_{LL_0} \end{bmatrix} \tag{93}$$

is given by

$$\mathbf{P}^s_\infty = \begin{bmatrix} \mathbf{Q}_s^{1/2}\mathbf{U}_s \operatorname{diag}\left(\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{1}{\boldsymbol{\lambda}_{s_i}}}\right)\mathbf{U}_s^T\mathbf{Q}_s^{1/2} & \mathbf{0}_{2M\times 2N} \\ \mathbf{0}_{2N\times 2M} & \mathbf{0}_{2N\times 2N} \end{bmatrix} + \mathbf{1}_{(M+N)\times(M+N)} \otimes \boldsymbol{\Theta}_s^{-1} \tag{94}$$

with

$$\boldsymbol{\Theta}_s = (\mathbf{1}_{1\times N}\otimes I_{2\times 2})\,\mathbf{P}_{LL_0}^{-1}\,(\mathbf{1}_{N\times 1}\otimes I_{2\times 2}) + (\mathbf{1}_{1\times M}\otimes I_{2\times 2})\left(\mathbf{J}_{rr_\infty}^{-1} + \mathbf{P}_{rr_0}\right)^{-1}(\mathbf{1}_{M\times 1}\otimes I_{2\times 2}) \tag{95}$$

and

$$\mathbf{J}_{rr_\infty} = \mathbf{Q}_s^{-1/2}\mathbf{U}_s \operatorname{diag}\left(\frac{\boldsymbol{\lambda}_{s_i}}{2} + \sqrt{\frac{\boldsymbol{\lambda}_{s_i}^2}{4} + \boldsymbol{\lambda}_{s_i}}\right)\mathbf{U}_s^T\mathbf{Q}_s^{-1/2} \tag{96}$$

Where we have employed the definition

$$\mathbf{C} = \mathbf{Q}_s^{-1/2}\mathbf{I}_{rr}\mathbf{Q}_s^{-1/2} = \mathbf{U}_s \operatorname{diag}(\boldsymbol{\lambda}_{s_i})\mathbf{U}_s^T$$

with

$$\mathbf{I}_{rr} = \begin{bmatrix} I_{2M\times 2M} & \mathbf{0}_{2M\times 2N} \end{bmatrix} \mathbf{H}_o'^T \mathbf{R}_s^{-1}\mathbf{H}_o' \begin{bmatrix} I_{2M\times 2M} \\ \mathbf{0}_{2N\times 2M} \end{bmatrix} \tag{97}$$

3.3 Steady State Covariance Bounds

In this section we present the main results of this work. It was shown that an upper bound on the uncertainty of C-SLAM is determined from the solution of the Riccati recursion in Eq. (35). At steady state, i.e., after sufficient time, the upper bound on the covariance of C-SLAM is evaluated by employing Lemma 3.6. We note that replacing the matrix $\mathbf{R}_s$ with $\mathbf{R}'_u$ in the definition of $\mathbf{I}_{rr}$ (cf. Eq. (97)) yields

$$\begin{aligned} \begin{bmatrix} I_{2M\times 2M} & \mathbf{0}_{2M\times 2N} \end{bmatrix} \mathbf{H}_o'^T \mathbf{R}_u'^{-1}\mathbf{H}_o' \begin{bmatrix} I_{2M\times 2M} \\ \mathbf{0}_{2N\times 2M} \end{bmatrix} &= \begin{bmatrix} I_{2M\times 2M} & \mathbf{0}_{2M\times 2N} \end{bmatrix} \mathbf{H}_o^T \mathcal{P}^T\mathcal{P}\mathbf{R}_u^{-1}\mathcal{P}^T\mathcal{P}\mathbf{H}_o \begin{bmatrix} I_{2M\times 2M} \\ \mathbf{0}_{2N\times 2M} \end{bmatrix} \\ &= \begin{bmatrix} I_{2M\times 2M} & \mathbf{0}_{2M\times 2N} \end{bmatrix} = \mathbf{H}_o^T \mathbf{R}_u^{-1}\mathbf{H}_o \begin{bmatrix} I_{2M\times 2M} \\ \mathbf{0}_{2N\times 2M} \end{bmatrix} \end{aligned}$$

The following lemma holds:

Lemma 3.7 *When a team of M robots moving in 2D performs C-SLAM with N landmarks and the initial covariance matrix of the position estimates is*

$$\mathbf{P}_0 = \begin{bmatrix} \mathbf{P}_{rr_0} & \mathbf{0}_{2M\times 2N} \\ \mathbf{0}_{2N\times 2M} & \mathbf{P}_{LL_0} \end{bmatrix} \tag{98}$$

the upper bound on the steady state uncertainty of the position estimates is determined by

$$\mathbf{P}^u_\infty = \begin{bmatrix} \mathbf{Q}_u^{1/2}\mathbf{U} \operatorname{diag}\left(\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{1}{\boldsymbol{\lambda}_i}}\right)\mathbf{U}^T\mathbf{Q}_u^{1/2} & \mathbf{0}_{2M\times 2N} \\ \mathbf{0}_{2N\times 2M} & \mathbf{0}_{2N\times 2N} \end{bmatrix} + \mathbf{1}_{(M+N)\times(M+N)} \otimes \boldsymbol{\Theta}_u^{-1} \tag{99}$$

with

$$\boldsymbol{\Theta}_u = (\mathbf{1}_{1\times N}\otimes I_{2\times 2})\,\mathbf{P}_{LL_0}^{-1}\,(\mathbf{1}_{N\times 1}\otimes I_{2\times 2}) + (\mathbf{1}_{1\times M}\otimes I_{2\times 2})\left(\mathbf{J}_{rr_\infty}^{-1} + \mathbf{P}_{rr_0}\right)^{-1}(\mathbf{1}_{M\times 1}\otimes I_{2\times 2}) \tag{100}$$

and

$$\mathbf{J}^u_{rr_\infty} \;=\; \mathbf{Q}_u^{-1/2}\mathbf{U}\,\mathrm{diag}\left(\frac{\boldsymbol{\lambda}_i}{2}+\sqrt{\frac{\boldsymbol{\lambda}_i^2}{4}+\boldsymbol{\lambda}_i}\right)\mathbf{U}^T\mathbf{Q}_u^{-1/2} \tag{101}$$

Where we have employed the definition

$$\mathbf{C} = \mathbf{Q}_u^{-1/2}\mathbf{I}_{rr}\mathbf{Q}_u^{-1/2} = \mathbf{U}\,\mathrm{diag}(\boldsymbol{\lambda}_i)\mathbf{U}^T$$

with

$$\mathbf{I}^u_{rr} = \begin{bmatrix} I_{2M\times 2M} & \mathbf{0}_{2M\times 2N} \end{bmatrix}\mathbf{H}_o^T\mathbf{R}_u^{-1}\mathbf{H}_o\begin{bmatrix} I_{2M\times 2M} \\ \mathbf{0}_{2N\times 2M} \end{bmatrix}$$

The quantities $\mathbf{Q}_u$ and $\mathbf{R}_u$ depend on the accuracy of the robots' sensors, and are defined in Eqs. (39) and (44) respectively.

Similarly, the upper bound on the *expected* steady state covariance of the position estimates in C-SLAM is derived application of Lemma 3.6, for the Riccati recursion in Eq. (36):

Lemma 3.8 *When a team of M robots moving in 2D performs C-SLAM with N landmarks and the initial covariance matrix of the position estimates is*

$$\mathbf{P}_0 = \begin{bmatrix} \mathbf{P}_{rr_0} & \mathbf{0}_{2M\times 2N} \\ \mathbf{0}_{2N\times 2M} & \mathbf{P}_{LL_0} \end{bmatrix} \tag{102}$$

the upper bound on the expected *steady state uncertainty of the position estimates is determined by*

$$\bar{\mathbf{P}}_\infty = \begin{bmatrix} \bar{\mathbf{Q}}_r^{1/2}\bar{\mathbf{U}}\,\mathrm{diag}\left(\frac{1}{2}+\sqrt{\frac{1}{4}+\frac{1}{\bar{\boldsymbol{\lambda}}_i}}\right)\bar{\mathbf{U}}^T\bar{\mathbf{Q}}_r^{1/2} & \mathbf{0}_{2M\times 2N} \\ \mathbf{0}_{2N\times 2M} & \mathbf{0}_{2N\times 2N} \end{bmatrix} + \mathbf{1}_{(M+N)\times(M+N)}\otimes\bar{\boldsymbol{\Theta}}^{-1} \tag{103}$$

with

$$\bar{\boldsymbol{\Theta}} \;=\; (\mathbf{1}_{1\times N}\otimes I_{2\times 2})\,\mathbf{P}_{LL_0}^{-1}\,(\mathbf{1}_{N\times 1}\otimes I_{2\times 2}) + (\mathbf{1}_{1\times M}\otimes I_{2\times 2})\left(\bar{\mathbf{J}}_{rr_\infty}^{-1}+\mathbf{P}_{rr_0}\right)^{-1}(\mathbf{1}_{M\times 1}\otimes I_{2\times 2}) \tag{104}$$

and

$$\bar{\mathbf{J}}_{rr_\infty} \;=\; \bar{\mathbf{Q}}_r^{-1/2}\bar{\mathbf{U}}\,\mathrm{diag}\left(\frac{\bar{\boldsymbol{\lambda}}_i}{2}+\sqrt{\frac{\bar{\boldsymbol{\lambda}}_i^{\,2}}{4}+\bar{\boldsymbol{\lambda}}_i}\right)\bar{\mathbf{U}}^T\bar{\mathbf{Q}}_r^{-1/2} \tag{105}$$

Where we have employed the definition

$$\bar{\mathbf{C}} = \bar{\mathbf{Q}}_r^{-1/2}\bar{\mathbf{I}}_{rr}\bar{\mathbf{Q}}_r^{-1/2} = \bar{\mathbf{U}}\,\mathrm{diag}(\bar{\boldsymbol{\lambda}}_i)\bar{\mathbf{U}}^T$$

with

$$\bar{\mathbf{I}}_{rr} = \begin{bmatrix} I_{2M\times 2M} & \mathbf{0}_{2M\times 2N} \end{bmatrix}\mathbf{H}_o^T\bar{\mathbf{R}}^{-1}\mathbf{H}_o\begin{bmatrix} I_{2M\times 2M} \\ \mathbf{0}_{2N\times 2M} \end{bmatrix}$$

The quantities $\bar{\mathbf{Q}}_r$ and $\bar{\mathbf{R}}$ depend on the accuracy of the robots' sensors, and are defined in Eqs. (45) and (48) respectively.

4 RPMG Reconfigurations

Up to this point, we have assumed that the topology of the RPMG remains constant. However, it is interesting to study the behavior of the covariance matrix of the position estimates in the case of RPMG reconfigurations. In this section, we derive upper bounds for the steady-state covariance matrix of C-SLAM, after the RPMG changes. The following results are only presented for the LTI C-SLAM system model, since their extension for the bounds on the worst-case and average covariance is straightforward.

4.1 Reconfiguration before convergence

We first address the case where the topology of the RPMG changes *before* steady state has been reached. At the time instant when the change in the graph's topology occurs, k_o, the covariance matrix of the position estimates of the robots and landmarks will be a positive definite matrix $\mathbf{P}_{k_o}$. This matrix can be viewed as the initial covariance matrix of C-SLAM, with the new RPMG topology. Thus an analysis similar to that presented in the previous section can be employed, to evaluate the asymptotic uncertainty. Compared to the preceding section, the difference in this case lies in that the initial covariance matrix is not block-diagonal, and thus the value of the matrix P_d, determined by the expression in Eq. (89), should be re-computed. We now obtain

$$\begin{bmatrix} P_a{}' & P_b{}' \\ P_c{}' & P_d{}' \end{bmatrix} = \lim_{k\to\infty} \left[I + \mathbf{P}_{k_o}\mathbf{J}'_{k+1}\right]^{-1}\mathbf{P}_{k_o} = \lim_{k\to\infty}\left[\mathbf{P}_{k_o}^{-1} + \mathbf{J}'_{k+1}\right]^{-1} \tag{106}$$

where the primed quantities refer to the RPMG topology after its reconfiguration. By defining the partitioning

$$\mathbf{P}_{k_o}^{-1} = \begin{bmatrix} \mathbf{W}_{rr} & \mathbf{W}_{rL} \\ \mathbf{W}_{Lr} & \mathbf{W}_{LL} \end{bmatrix}$$

the previous expression can be written as

$$\lim_{k\to\infty}\left[\mathbf{P}_{k_o}^{-1} + \mathbf{J}'_{k+1}\right]^{-1} \quad = \quad \lim_{k\to\infty}\left(\begin{bmatrix} \mathbf{W}_{rr} & \mathbf{W}_{rL} \\ \mathbf{W}_{Lr} & \mathbf{W}_{LL} \end{bmatrix} + \begin{bmatrix} \mathbf{J}'_{rr_k} & \mathbf{J}'^{T}_{Lr_k} \\ \mathbf{J}'_{Lr_k} & \mathbf{J}'_{LL_k} \end{bmatrix}\right)^{-1} \tag{107}$$

Employing the formula for the inversion of a partitioned matrix, we obtain:

$$P'_d \quad = \quad \lim_{k\to\infty}\Big(\mathbf{W}_{LL} + \mathbf{J}'_{LL_k} - \left(\mathbf{W}_{Lr} + \mathbf{J}'_{Lr_k}\right)\left(\mathbf{W}_{rr} + \mathbf{J}'_{rr_k}\right)^{-1}\left(\mathbf{W}_{rL} + \mathbf{J}'^{T}_{Lr_k}\right)\Big)^{-1}$$

As $k \to \infty$,

$$\mathbf{J}'_{LL_k} \to \mathbf{J}'_{LL_{ss}} = \mathbf{J}'_a k + \mathbf{J}'_c$$

Thus we can write

$$P'_d \quad = \quad \lim_{k\to\infty}\left(\mathbf{J}'_a k + \mathbf{D}'_k\right)^{-1} \tag{108}$$

where we have denoted

$$\mathbf{D}_k = \mathbf{W}_{LL} + \mathbf{J}'_c - \left(\mathbf{W}_{Lr} + \mathbf{J}'_{Lr_k}\right)\left(\mathbf{W}_{rr} + \mathbf{J}'_{rr_k}\right)^{-1}\left(\mathbf{W}_{rL} + \mathbf{J}'^{T}_{Lr_k}\right)$$

By application of Lemma 3.5 yields, similarly to the previous section:

$$P'_d \quad = \quad \mathbf{1}_{N\times N} \otimes \mathbf{\Theta}'^{-1}_s \tag{109}$$

where $\mathbf{\Theta}'_s$ is defined as

$$\begin{aligned} \mathbf{\Theta}'_s &= \left(\mathbf{1}_{1\times N} \otimes I_{2\times 2}\right)\mathbf{D}'_\infty\left(\mathbf{1}_{N\times 1} \otimes I_{2\times 2}\right) \\ &= \left(\mathbf{1}_{1\times N} \otimes I_{2\times 2}\right)\Big(\mathbf{W}_{LL} + \mathbf{J}'_c - \left(\mathbf{W}_{Lr} + \mathbf{J}'_{Lr_\infty}\right)\left(\mathbf{W}_{rr} + \mathbf{J}'_{rr_\infty}\right)^{-1}\left(\mathbf{W}_{rL} + \mathbf{J}'^{T}_{Lr_\infty}\right)\Big)\left(\mathbf{1}_{N\times 1} \otimes I_{2\times 2}\right) \\ &= \left(\mathbf{1}_{1\times N} \otimes I_{2\times 2}\right)\mathbf{W}_{LL}\left(\mathbf{1}_{N\times 1} \otimes I_{2\times 2}\right) + \left(\mathbf{1}_{1\times M} \otimes I_{2\times 2}\right)\mathbf{J}'_{rr_\infty}\left(\mathbf{1}_{M\times 1} \otimes I_{2\times 2}\right) \\ &+ \left(\left(\mathbf{1}_{1\times N} \otimes I_{2\times 2}\right)\mathbf{W}_{Lr} - \left(\mathbf{1}_{1\times M} \otimes I_{2\times 2}\right)\mathbf{J}'_{rr_\infty}\right)\left(\mathbf{J}'_{rr_\infty} + \mathbf{W}_{rr}\right)^{-1}\left(\left(\mathbf{1}_{1\times N} \otimes I_{2\times 2}\right)\mathbf{W}_{Lr} - \left(\mathbf{1}_{1\times M} \otimes I_{2\times 2}\right)\mathbf{J}'_{rr_\infty}\right)^T \end{aligned} \tag{110}$$

In the last expression, we have once again used the result of Eq. (87), and the property $\mathbf{1}_{1\times N} J_{Lr_\infty} = \mathbf{1}_{1\times M} J_{rr_\infty}$ (cf. Eq. (86)).

Clearly, all the remaining derivations for the steady-state covariance with the *new* RPMG topology remain unchanged. This is given by:

$$\mathbf{P}^{s\prime}_\infty = \begin{bmatrix} \mathbf{Q}_s^{1/2}\mathbf{U}'_s \operatorname{diag}\left(\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{1}{\boldsymbol{\lambda}'_{s_i}}}\right)\mathbf{U}'^{T}_s\mathbf{Q}_s^{1/2} & \mathbf{0}_{2M\times 2N} \\ \mathbf{0}_{2N\times 2M} & \mathbf{0}_{2N\times 2N} \end{bmatrix} + \mathbf{1}_{(M+N)\times(M+N)} \otimes \mathbf{\Theta}'^{-1}_s \tag{111}$$

where $\mathbf{\Theta}'^{-1}_s$ is defined in Eq. (110), and all the primed quantities correspond to the new RPMG topology.

4.2 Reconfigurations after convergence

A special case of interest arises when the RPMG reconfiguration occurs *after* steady state has been reached. In order to compute the asymptotic covariance after the topology change, we can once again view the covariance matrix $\mathbf{P}_{k_o}$ as the initial covariance matrix of C-SLAM with the new RPMG. Since the covariance has converged to its steady-state value prior to the reconfiguration, we have

$$\begin{aligned}\mathbf{P}_{k_o} &= \begin{bmatrix} \mathbf{Q}_s^{1/2}\mathbf{U}_s \operatorname{diag}\left(\frac{1}{2}+\sqrt{\frac{1}{4}+\frac{1}{\boldsymbol{\lambda}_{s_i}}}\right)\mathbf{U}_s^T\mathbf{Q}_s^{1/2} & \mathbf{0}_{2M\times 2N} \\ \mathbf{0}_{2N\times 2M} & \mathbf{0}_{2N\times 2N}\end{bmatrix} + \mathbf{1}_{(M+N)\times(M+N)}\otimes\boldsymbol{\Theta}_s^{-1} \\ &= \mathbf{P}_\infty^{s(0)} + \mathbf{1}_{(M+N)\times(M+N)}\otimes\boldsymbol{\Theta}_s^{-1} \end{aligned} \tag{112}$$

Thus the new value of the matrix P_d is determined by the equation

$$\begin{aligned}\begin{bmatrix} P_a{}' & P_b{}' \\ P_c{}' & P_d{}' \end{bmatrix} &= \lim_{k\to\infty}\left(I+\mathbf{P}_{k_o}\mathbf{J}'_{k+1}\right)^{-1}\mathbf{P}_{k_o} \\ &= \lim_{k\to\infty}\left(I+\left(\mathbf{P}_\infty^{s(0)}+\mathbf{1}_{(M+N)\times(M+N)}\otimes\boldsymbol{\Theta}_s^{-1}\right)\mathbf{J}'_{k+1}\right)^{-1}\mathbf{P}_{k_o}\end{aligned}$$

where the primed quantities refer to the RPMG topology after its reconfiguration. At this point we note that

$$\begin{aligned}\left(\mathbf{1}_{(M+N)\times(M+N)}\otimes\boldsymbol{\Theta}_s^{-1}\right)\mathbf{J}'_{k+1} &= \left(\mathbf{1}_{(M+N)\times(M+N)}J'_{k+1}\right)\otimes\boldsymbol{\Theta}_s^{-1} \\ &= \mathbf{0}_{\xi\times\xi}\end{aligned}$$

Using this result, the preceding expression simplifies to

$$\begin{bmatrix} P_a{}' & P_b{}' \\ P_c{}' & P_d{}' \end{bmatrix} = \lim_{k\to\infty}\left(I+\mathbf{P}_\infty^{s(0)}\mathbf{J}'_{k+1}\right)^{-1}\mathbf{P}_{k_o}$$

From this expression, we obtain

$$P_d{}' = \mathbf{1}_{M\times M}\otimes\boldsymbol{\Theta}_s^{-1} \tag{113}$$

and thus the asymptotic covariance with the new RPMG topology is given by In this case, the upper bound of the asymptotic covariance after the reconfiguration, is given by [10]:

$$\mathbf{P}_\infty^{s'} = \mathbf{P}_\infty^{s(0)'} + \mathbf{1}_{(M+N)\times(M+N)}\otimes\Theta^{-1} \tag{114}$$

where $\mathbf{P}_\infty^{s(0))'}$ is defined as in Eq. (71), but with all quantities corresponding to the new RPMG, and Θ is defined in Eq. (95).

It should be stressed at this point that, while the upper bound on the robots' uncertainty depends on the structure of the new RPMG, the upper bound on the landmarks' covariance is *identical* to the value of the bound prior to the RPMG topology change. This result implies that once steady state has been reached and in the absence of any new external positioning information (e.g., from GPS), *no measurement strategy can reduce the uncertainty of the map features' positions*. This is a consequence of the fact that, at steady state, the uncertainty of the map lies entirely in the unobservable subspace of the system, whose basis comprises the column vectors of the matrix $\mathbf{1}_{N\times 1}\otimes I_{2\times 2}$. Since the unobservable subspace of the system does not change when the topology of the RPMG changes, unless absolute positioning information becomes available (e.g., in the form of GPS measurements), it is impossible to improve the accuracy of the landmarks' position estimates.

A Upper Bound Riccati Recursion

In this appendix we prove that if $\mathbf{R}'_u \succeq \mathbf{R}'_o(k)$ and $\mathbf{Q}_u \succeq \mathbf{Q}_r(k)$ for all $k \geq 0$, then the solutions to the following two Riccati recursions

$$\mathbf{P}_{k+1} = \mathbf{P}_k - \mathbf{P}_k\mathbf{H}_o'^T \left(\mathbf{H}'_o\mathbf{P}_k\mathbf{H}_o'^T + \mathbf{R}'_o(k+1)\right)^{-1} \mathbf{H}'_o\mathbf{P}_k + \mathbf{G}_o\mathbf{Q}_r(k+1)\mathbf{G}_o^T \tag{115}$$

and

$$\mathbf{P}^u_{k+1} = \mathbf{P}^u_k - \mathbf{P}^u_k\mathbf{H}_o'^T \left(\mathbf{H}'_o\mathbf{P}^u_k\mathbf{H}_o'^T + \mathbf{R}'_u\right)^{-1} \mathbf{H}'_o\mathbf{P}^u_k + \mathbf{G}_o\mathbf{Q}_u\mathbf{G}_o^T \tag{116}$$

with the *same* initial condition, $\mathbf{P}_0$, satisfy $\mathbf{P}^u_k \succeq \mathbf{P}_k$ for all $k \geq 0$. The proof is carried out by induction, and requires the following two intermediate results:

- **Monotonicity with respect to the measurement covariance matrix**

If $\mathbf{R}_1 \succeq \mathbf{R}_2$, then for any $\mathbf{P} \succeq 0$

$$\mathbf{P} - \mathbf{P}\mathbf{H}^T\left(\mathbf{H}\mathbf{P}\mathbf{H}^T + \mathbf{R}_1\right)^{-1}\mathbf{H}\mathbf{P} + \mathbf{Q}_o \succeq \mathbf{P} - \mathbf{P}\mathbf{H}^T\left(\mathbf{H}\mathbf{P}\mathbf{H}^T + \mathbf{R}_2\right)^{-1}\mathbf{H}\mathbf{P} + \mathbf{Q}_o \tag{117}$$

This statement is proven by taking into account the properties of linear matrix inequalities:

$$\begin{aligned}
\mathbf{R}_1 &\succeq \mathbf{R}_2 \Rightarrow \\
\mathbf{H}\mathbf{P}\mathbf{H}^T + \mathbf{R}_1 &\succeq \mathbf{H}\mathbf{P}\mathbf{H}^T + \mathbf{R}_2 \Rightarrow \\
\left(\mathbf{H}\mathbf{P}\mathbf{H}^T + \mathbf{R}_1\right)^{-1} &\preceq \left(\mathbf{H}\mathbf{P}\mathbf{H}^T + \mathbf{R}_2\right)^{-1} \Rightarrow \\
\mathbf{P}\mathbf{H}^T\left(\mathbf{H}\mathbf{P}\mathbf{H}^T + \mathbf{R}_1\right)^{-1}\mathbf{H}\mathbf{P} &\preceq \mathbf{P}\mathbf{H}^T\left(\mathbf{H}\mathbf{P}\mathbf{H}^T + \mathbf{R}_2\right)^{-1}\mathbf{H}\mathbf{P} \Rightarrow \\
-\mathbf{P}\mathbf{H}^T\left(\mathbf{H}\mathbf{P}\mathbf{H}^T + \mathbf{R}_1\right)^{-1}\mathbf{H}\mathbf{P} &\succeq -\mathbf{P}\mathbf{H}^T\left(\mathbf{H}\mathbf{P}\mathbf{H}^T + \mathbf{R}_2\right)^{-1}\mathbf{H}\mathbf{P} \Rightarrow \\
\mathbf{P} - \mathbf{P}\mathbf{H}^T\left(\mathbf{H}\mathbf{P}\mathbf{H}^T + \mathbf{R}_1\right)^{-1}\mathbf{H}\mathbf{P} + \mathbf{Q}_o &\succeq \mathbf{P} - \mathbf{P}\mathbf{H}^T\left(\mathbf{H}\mathbf{P}\mathbf{H}^T + \mathbf{R}_2\right)^{-1}\mathbf{H}\mathbf{P} + \mathbf{Q}_o
\end{aligned}$$

- **Monotonicity with respect to the state covariance matrix**

The solution to the Riccati recursion at time $k+1$ is monotonic with to the solution at time k, i.e., if $\mathbf{P}^{(1)}_k$ and $\mathbf{P}^{(2)}_k$ are two different solutions to the same Riccati recursion at time k, with $\mathbf{P}^{(1)}_k \succeq \mathbf{P}^{(2)}_k$ then $\mathbf{P}^{(1)}_{k+1} \succeq \mathbf{P}^{(2)}_{k+1}$. In order to prove the result in the general case, in which $\mathbf{P}^{(1)}_k$ and $\mathbf{P}^{(2)}_k$ are positive semidefinite, we use the following expression that relates the one-step ahead solutions to two Riccati recursions with identical $\mathbf{H}$, $\mathbf{Q}$ and $\mathbf{R}$ matrices, but different initial values $\mathbf{P}^{(1)}_k$ and $\mathbf{P}^{(2)}_k$ ([8]). It is

$$\mathbf{P}^{(2)}_{k+1} - \mathbf{P}^{(1)}_{k+1} = F_{p,k}\left(\left(\mathbf{P}^{(2)}_k - \mathbf{P}^{(1)}_k\right) - \left(\mathbf{P}^{(2)}_k - \mathbf{P}^{(1)}_k\right)\mathbf{H}^T\left(\mathbf{H}\mathbf{P}^{(2)}_k\mathbf{H}^T + \mathbf{R}\right)\mathbf{H}\left(\mathbf{P}^{(2)}_k - \mathbf{P}^{(1)}_k\right)\right)F^T_{p,k} \tag{118}$$

where $F_{p,k}$ is a matrix whose exact structure is not important for the purposes of this proof. Since we have assumed $\mathbf{P}^{(1)}_k \succeq \mathbf{P}^{(2)}_k$ we can write $\mathbf{P}^{(2)}_k - \mathbf{P}^{(1)}_k \preceq 0$. Additionally, the matrix

$$\left(\mathbf{P}^{(2)}_k - \mathbf{P}^{(1)}_k\right)\mathbf{H}^T\left(\mathbf{H}\mathbf{P}^{(2)}_k\mathbf{H}^T + \mathbf{R}\right)\mathbf{H}\left(\mathbf{P}^{(2)}_k - \mathbf{P}^{(1)}_k\right)$$

is positive semidefinite, and therefore we have

$$\begin{aligned}
-\left(\mathbf{P}^{(2)}_k - \mathbf{P}^{(1)}_k\right)\mathbf{H}^T\left(\mathbf{H}\mathbf{P}^{(2)}_k\mathbf{H}^T + \mathbf{R}\right)\mathbf{H}\left(\mathbf{P}^{(2)}_k - \mathbf{P}^{(1)}_k\right) &\preceq 0 \Rightarrow \\
\left(\mathbf{P}^{(2)}_k - \mathbf{P}^{(1)}_k\right) - \left(\mathbf{P}^{(2)}_k - \mathbf{P}^{(1)}_k\right)\mathbf{H}^T\left(\mathbf{H}\mathbf{P}^{(2)}_k\mathbf{H}^T + \mathbf{R}\right)\mathbf{H}\left(\mathbf{P}^{(2)}_k - \mathbf{P}^{(1)}_k\right) &\preceq 0 \Rightarrow \\
F_{p,k}\left(\left(\mathbf{P}^{(2)}_k - \mathbf{P}^{(1)}_k\right) - \left(\mathbf{P}^{(2)}_k - \mathbf{P}^{(1)}_k\right)\mathbf{H}^T\left(\mathbf{H}\mathbf{P}^{(2)}_k\mathbf{H}^T + \mathbf{R}\right)\mathbf{H}\left(\mathbf{P}^{(2)}_k - \mathbf{P}^{(1)}_k\right)\right)F^T_{p,k} &\preceq 0 \Rightarrow \\
\mathbf{P}^{(2)}_{k+1} - \mathbf{P}^{(1)}_{k+1} &\preceq 0
\end{aligned}$$

The last line implies that $\mathbf{P}_{k+1}^{(1)} \succeq \mathbf{P}_{k+1}^{(2)}$, which is the desired result.

We can now employ induction to prove the main statement of this appendix. Assuming that at some time instant i, $\mathbf{P}_i^u \succeq \mathbf{P}_i$, we can write

$$\begin{aligned}
\mathbf{P}_{i+1}^u &= \mathbf{P}_i^u - \mathbf{P}_i^u \mathbf{H}_o'^T \left(\mathbf{H}_o' \mathbf{P}_i^u \mathbf{H}_o'^T + \mathbf{R}_u'\right)^{-1} \mathbf{H}_o' \mathbf{P}_i^u + \mathbf{G}_o \mathbf{Q}_u \mathbf{G}_o^T \\
&\succeq \mathbf{P}_i - \mathbf{P}_i \mathbf{H}_o'^T \left(\mathbf{H}_o' \mathbf{P}_i \mathbf{H}_o'^T + \mathbf{R}_u'\right)^{-1} \mathbf{H}_o' \mathbf{P}_i + \mathbf{G}_o \mathbf{Q}_u \mathbf{G}_o^T \\
&\succeq \mathbf{P}_i - \mathbf{P}_i \mathbf{H}_o'^T \left(\mathbf{H}_o' \mathbf{P}_i \mathbf{H}_o'^T + \mathbf{R}_u'\right)^{-1} \mathbf{H}_o' \mathbf{P}_i + \mathbf{G}_o \mathbf{Q}_r(k+1) \mathbf{G}_o^T \\
&\succeq \mathbf{P}_i - \mathbf{P}_i \mathbf{H}_o'^T \left(\mathbf{H}_o' \mathbf{P}_i \mathbf{H}_o'^T + \mathbf{R}_o'(k+1)\right)^{-1} \mathbf{H}_o' \mathbf{P}_i + \mathbf{G}_o \mathbf{Q}_r(k+1) \mathbf{G}_o^T = \mathbf{P}_{i+1}
\end{aligned}$$

where the monotonicity of the Riccati recursion with respect to the covariance matrix, the property $\mathbf{Q}_u \succeq \mathbf{Q}_r(k+1)$ and the monotonicity of the Riccati recursion with respect to the measurement covariance matrix have been used in the last three lines. Thus $\mathbf{P}_i^u \succeq \mathbf{P}_i \Rightarrow \mathbf{P}_{i+1}^u \succeq \mathbf{P}_{i+1}$. For $i = 0$ the condition $\mathbf{P}_i^u \succeq \mathbf{P}_i$ holds with equality, and therefore for any $i > 0$, the solution to the Riccati recursion in Eq. (115) is an upper bound to the solution of the recursion in Eq. (116).

B Riccati Recursion for the Upper Bound on the Average Covariance

In this appendix we prove that if $\bar{\mathbf{R}}'$ and $\bar{\mathbf{Q}}_r$ are matrices such that $\bar{\mathbf{R}}' = E\{\mathbf{R}_o'(k)\}$ and $\bar{\mathbf{Q}}_r = \{\mathbf{Q}_r(k)\}$ for all $k \geq 0$, then the solutions to the following two Riccati recursions

$$\mathbf{P}_{k+1} = \mathbf{P}_k - \mathbf{P}_k \mathbf{H}_o'^T \left(\mathbf{H}_o' \mathbf{P}_k \mathbf{H}_o'^T + \mathbf{R}_o'(k+1)\right)^{-1} \mathbf{H}_o' \mathbf{P}_k + \mathbf{G}_o \mathbf{Q}_r(k+1) \mathbf{G}_o \tag{119}$$

and

$$\bar{\mathbf{P}}_{k+1} = \bar{\mathbf{P}}_k - \bar{\mathbf{P}}_k \mathbf{H}_o'^T \left(\mathbf{H}_o' \bar{\mathbf{P}}_k \mathbf{H}_o'^T + \bar{\mathbf{R}}'\right)^{-1} \mathbf{H}_o' \bar{\mathbf{P}}_k + \mathbf{G}_o \bar{\mathbf{Q}}_r \mathbf{G}_o^T \tag{120}$$

with the *same* initial condition, $\mathbf{P}_0$, satisfy $\bar{\mathbf{P}}_k \succeq E\{\mathbf{P}_k\}$ for all $k \geq 0$. We first prove a useful intermediate result:

- **Concavity of the Riccati recursion**

We note that the Riccati recursion

$$P_{k+1} = P_k - P_k H^T \left(H P_k H^T + R_{k+1}\right)^{-1} H P_k + G Q_{k+1} G \tag{121}$$

can equivalently be written as

$$\begin{aligned}
P_{k+1} = &\begin{bmatrix} I & \mathbf{0} \end{bmatrix} \begin{bmatrix} P_k & \mathbf{0} \\ \mathbf{0} & R_{k+1} \end{bmatrix} \begin{bmatrix} I \\ \mathbf{0} \end{bmatrix} \\
&- \begin{bmatrix} I & \mathbf{0} \end{bmatrix} \begin{bmatrix} P_k & \mathbf{0} \\ \mathbf{0} & R_{k+1} \end{bmatrix} \begin{bmatrix} H^T \\ \mathbf{0} \end{bmatrix} \left(\begin{bmatrix} H & I \end{bmatrix} \begin{bmatrix} P_k & \mathbf{0} \\ \mathbf{0} & R_{k+1} \end{bmatrix} \begin{bmatrix} H^T \\ \mathbf{I} \end{bmatrix} \right)^{-1} \begin{bmatrix} H & \mathbf{0} \end{bmatrix} \begin{bmatrix} P_k & \mathbf{0} \\ \mathbf{0} & R_{k+1} \end{bmatrix} \begin{bmatrix} I \\ \mathbf{0} \end{bmatrix} \\
&+ G Q_{k+1} G
\end{aligned}$$

our goal is to show that the above expression is concave with respect to the matrix

$$\begin{bmatrix} P_k & \mathbf{0} \\ \mathbf{0} & R_{k+1} \end{bmatrix}$$

A sufficient condition for this is that the function

$$f(X) = AXB \left(CXC^T\right)^{-1} B^T X A^T \tag{122}$$

is convex with respect to the positive semidefinite matrix X, when A,B,C are arbitrary matrices of compatible dimensions. This is equivalent to proving the convexity of the function of the scalar variable t

$$f_t(t) = A(X_o + tZ_o)B\left(C(X_o + tZ_o)C^T\right)^{-1} B^T(X_o + tZ_o)A^T \tag{123}$$

with domain those values of t for which $X_o + tZ_o \succeq 0, X_o \succeq 0$ is convex [11]. $f_t(t)$ is convex if and only if the scalar function

$$f_t'(t) = z^T A(X_o + tZ_o)B\left(C(X_o + tZ_o)C^T\right)^{-1} B^T(X_o + tZ_o)A^T z \tag{124}$$

is convex for any vector z of appropriate dimensions [11]. Moreover, it is well known that a function is convex if and only if its epigraph is a convex set, and therefore we obtain the following convexity condition for $f(X)$:

$$f(X) \text{ convex } \Leftrightarrow \{s,t|z^T A(X_o + tZ_o)B\left(C(X_o + tZ_o)C^T\right)^{-1} B^T(X_o + tZ_o)A^T z \leq s\} \text{ is convex}$$

However, from the properties of Schur complements it is well known that if $A_o \succ 0$ then

$$\begin{bmatrix} A_o & B_o \\ B_o^T & C_o \end{bmatrix} \succeq 0 \Leftrightarrow C_o - B_o^T C_o^{-1} B \succeq 0$$

In our problem, the matrix $C(X_o + tZ_o)C^T$ is clearly positive definite, and thus we can write

$$z^T A(X_o{+}tZ_o)B\left(C(X_o + tZ_o)C^T\right)^{-1} B^T(X_o{+}tZ_o)A^T z \leq s \Leftrightarrow \begin{bmatrix} C(X_o + tZ_o)C^T & B^T(X_o + tZ_o)A^T z \\ z^T A(X_o + tZ_o)B & s \end{bmatrix} \succeq 0$$

However, the defining matrix inequality of the epigraph is equivalent to

$$\begin{bmatrix} CX_oC^T & B^T X_o A^T z \\ z^T A X_o B & 0 \end{bmatrix} + t \begin{bmatrix} CZ_oC^T & B^T Z_o A^T z \\ z^T A Z_o B & 0 \end{bmatrix} + s \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix} \succeq 0$$

which defines a convex set in (s, t) [11].

Thus, by the preceding analysis $f(X)$ is a convex function, and consequently P_{k+1} is a concave function of the matrix

$$\begin{bmatrix} P_k & \mathbf{0} \\ \mathbf{0} & R_{k+1} \end{bmatrix}$$

■

We now employ this result to prove the main result of this appendix. The proof is carried out by induction. Assuming that at time step k the inequality $\bar{\mathbf{P}}_k \succeq E\{\mathbf{P}_k\}$ holds, we will show that it also holds for the time step $k+1$. We have

$$\begin{aligned}
\mathbf{P}_{k+1} &= \mathbf{P}_k - \mathbf{P}_k\mathbf{H}_o'^T\left(\mathbf{H}_o'\mathbf{P}_k\mathbf{H}_o'^T + \mathbf{R}_o'(k+1)\right)^{-1}\mathbf{H}_o'\mathbf{P}_k + \mathbf{G}_o\mathbf{Q}(k+1)\mathbf{G}_o^T \Rightarrow \\
E\{\mathbf{P}_{k+1}\} &= E\{\mathbf{P}_k - \mathbf{P}_k\mathbf{H}_o'^T\left(\mathbf{H}_o'\mathbf{P}_k\mathbf{H}_o'^T + \mathbf{R}_o'(k+1)\right)^{-1}\mathbf{H}_o'\mathbf{P}_k + \mathbf{G}_o\mathbf{Q}(k+1)\mathbf{G}_o^T\} \\
&= E\{\mathbf{P}_k - \mathbf{P}_k\mathbf{H}_o'^T\left(\mathbf{H}_o'\mathbf{P}_k\mathbf{H}_o'^T + \mathbf{R}_o'(k+1)\right)^{-1}\mathbf{H}_o'\mathbf{P}_k\} + \mathbf{G}_oE\{\mathbf{Q}(k+1)\}\mathbf{G}_o^T \\
&\preceq E\{\mathbf{P}_k\} - E\{\mathbf{P}_k\}\mathbf{H}_o'^T\left(\mathbf{H}_o'E\{\mathbf{P}_k\}\mathbf{H}_o'^T + E\{\mathbf{R}_o'(k+1)\}\right)^{-1}\mathbf{H}_o'E\{\mathbf{P}_k\} + \mathbf{G}_oE\{\mathbf{Q}(k+1)\}\mathbf{G}_o^T
\end{aligned}$$

where in the last line the concavity of Jensen's inequality was applied [11], in order to exploit tht concavity of the Riccati. By assumption, $\bar{\mathbf{P}}_k \succeq E\{\mathbf{P}_k\}$ and employing the property of the monotonicity of the Riccati with respect to the covariance matrix (cf. Appendix A), we can write

$$\begin{aligned}
E\{\mathbf{P}_{k+1}\} &\preceq \bar{\mathbf{P}}_k - \bar{\mathbf{P}}_k\mathbf{H}_o'^T\left(\mathbf{H}_o'\bar{\mathbf{P}}_k\mathbf{H}_o'^T + E\{\mathbf{R}_o'(k+1)\}\right)^{-1}\mathbf{H}_o'\bar{\mathbf{P}}_k + \mathbf{G}_oE\{\mathbf{Q}(k+1)\}\mathbf{G}_o^T \\
&= \bar{\mathbf{P}}_k - \bar{\mathbf{P}}_k\mathbf{H}_o'^T\left(\mathbf{H}_o'\bar{\mathbf{P}}_k\mathbf{H}_o'^T + \bar{\mathbf{R}}'\}\right)^{-1}\mathbf{H}_o'\bar{\mathbf{P}}_k + \mathbf{G}_o\bar{\mathbf{Q}}_r\mathbf{G}_o^T \\
&= \bar{\mathbf{P}}_{k+1}
\end{aligned}$$

Thus, $\bar{\mathbf{P}}_k \succeq E\{\mathbf{P}_k\} \Rightarrow \bar{\mathbf{P}}_{k+1} \succeq E\{\mathbf{P}_{k+1}\}$. For $k = 0$ the condition $\bar{\mathbf{P}}_k \succeq E\{\mathbf{P}_k\}$ holds with equality, and the proof is complete.

C A Special Case of the Riccati Recursion

In this appendix we prove the following lemma:

Lemma C.1 *Consider the following general form of the Riccati recursion*

$$\mathbf{X}_{k+1} = \mathbf{X}_k + \mathbf{A} - \mathbf{X}_k\mathbf{B}^T\left(\mathbf{B}\mathbf{X}_k\mathbf{B}^T + \mathbf{C}\right)^{-1}\mathbf{B}\mathbf{X}_k$$

with $\mathbf{X}_k, \mathbf{A} \in \mathcal{R}^{2n\times 2n}$, $\mathbf{B} \in \mathcal{R}^{2m\times 2n}$, *and* $\mathbf{C} \in \mathcal{R}^{2m\times 2m}$. *If the initial value of this recursion is of the form*

$$\mathbf{X}_0 = X_0 \otimes I_{2\times 2}$$

and additionally

$$\mathbf{A} = A \otimes I_{2\times 2}, \quad \mathbf{B} = B \otimes I_{2\times 2}, \quad \mathbf{C} = C \otimes I_{2\times 2},$$

with $X_0, A \in \mathcal{R}^{n\times n}$, $B \in \mathcal{R}^{m\times n}$, *and* $C \in \mathcal{R}^{m\times m}$ *then the solution for all* $k > 0$ *can be expressed as*

$$\mathbf{X}_k = X_k \otimes I_{2\times 2}$$

with $X_k \in \mathcal{R}^{n\times n}$.

Proof *We prove this statement by induction. Assuming that for some* $i \geq 0$, $\mathbf{X}_i = X_i \otimes I_{2\times 2}$, *then we can write*

$$\begin{aligned}
\mathbf{X}_{i+1} &= \mathbf{X}_i + \mathbf{A} - \mathbf{X}_i\mathbf{B}^T\left(\mathbf{B}\mathbf{X}_i\mathbf{B}^T + \mathbf{C}\right)^{-1}\mathbf{B}\mathbf{X}_i \\
&= X_i \otimes I_{2\times 2} + A \otimes I_{2\times 2} \\
&- \big(X_i \otimes I_{2\times 2}\big)\big(B^T \otimes I_{2\times 2}\big)\Big(\big(B \otimes I_{2\times 2}\big)\big(X_i \otimes I_{2\times 2}\big)\big(B^T \otimes I_{2\times 2}\big) + \big(C \otimes I_{2\times 2}\big)\Big)^{-1}\big(B \otimes I_{2\times 2}\big)\big(X_i \otimes I_{2\times 2}\big)
\end{aligned}$$

At this point we use the following properties of the Kronecker Product:

$$W \otimes D + Y \otimes D = (W + Y) \otimes D \tag{125}$$

$$(W \otimes D)(Y \otimes E) = (WY) \otimes (DE) \tag{126}$$

$$(W \otimes D)^{-1} = W^{-1} \otimes D^{-1} \tag{127}$$

Applying Eqs. (125) and (126) with $D = E = I_{2\times 2}$, *yields*

$$\big(B \otimes I_{2\times 2}\big)\big(X_i \otimes I_{2\times 2}\big)\big(B^T \otimes I_{2\times 2}\big) + \big(C \otimes I_{2\times 2}\big) = \big(BX_iB^T + C\big) \otimes I_{2\times 2}$$

Applying the property of Eq. (127), we obtain $\big((BX_iB^T + C) \otimes I_{2\times 2}\big)^{-1} = \big(BX_iB^T + C\big)^{-1} \otimes I_{2\times 2}$. *Hence,*

$$\begin{aligned}
\mathbf{X}_{i+1} &= X_i \otimes I_{2\times 2} + A \otimes I_{2\times 2} - \big(X_i \otimes I_{2\times 2}\big)\big(B^T \otimes I_{2\times 2}\big)\big((BX_iB^T + C)^{-1} \otimes I_{2\times 2}\big)\big(B \otimes I_{2\times 2}\big)\big(X_i \otimes I_{2\times 2}\big) \\
&= X_i \otimes I_{2\times 2} + A \otimes I_{2\times 2} - \big(X_iB^T(BX_iB^T + C)^{-1}BX_i\big) \otimes I_{2\times 2} \\
&= \big(X_i + A - X_iB^T(BX_iB^T + C)^{-1}BX_i\big) \otimes I_{2\times 2}
\end{aligned}$$

Thus, whenever $\mathbf{X}_i$ *can be written as the Kronecker product of some matrix with* $I_{2\times 2}$, $\mathbf{X}_{i+1}$ *retains the same special structure. For* $i = 0$, $\mathbf{X}_0$ *is by assumption of the form* $X_0 \otimes I_{2\times 2}$, *and therefore, the proof is complete.*

D Sum of the Elements of $\mathbf{J}_k$

In this appendix we prove that the sum of all elements of J_k is equal to zero for all $k \geq 0$, when $J_0 = \mathbf{0}_{(M+N)\times(M+N)}$. For this purpose we employ the method of induction. Assume that at time step i, $\mathbf{1}_{1\times(M+N)}J_i\mathbf{1}_{(M+N)\times 1} = 0$. Then from Eq. (58) we obtain

$$\mathbf{1}_{1\times(M+N)}J_{i+1}\mathbf{1}_{(M+N)\times 1} = \mathbf{1}_{1\times(M+N)}\left(J_i + H_o^TR_s^{-1}H_o - J_iG\left(Q_s^{-1} + G^TJ_iG\right)^{-1}G^TJ_i\right)\mathbf{1}_{(M+N)\times 1}$$

where

$$G = \begin{bmatrix} I_{M\times M} \\ \mathbf{0}_{N\times N} \end{bmatrix}$$

But the sum of the elements of J_i is zero, and additionally $\mathbf{1}_{1\times(M+N)} H_o^T R_s^{-1} H_o \mathbf{1}_{(M+N)\times 1} = 0$ since the vector $\mathbf{1}_{(M+N)\times 1}$ belongs in the nullspace of the matrix H_o. Therefore, we can write

$$\mathbf{1}_{1\times(M+N)} J_{i+1} \mathbf{1}_{(M+N)\times 1} \quad = \quad -\mathbf{1}_{1\times(M+N)} J_i G \left(Q_s^{-1} + G^T J_i G\right)^{-1} G^T J_i \mathbf{1}_{(M+N)\times 1}$$

But J_{i+1} is a positive semidefinite matrix, therefore the sum of its elements, $\mathbf{1}_{1\times(M+N)} J_{i+1} \mathbf{1}_{(M+N)\times 1}$ cannot be negative. The right hand side of the above equation is a non-positive number, since the matrix $\left(Q_s^{-1} + G^T J_i G\right)^{-1}$ is positive definite, and we conclude that $\mathbf{1}_{1\times(M+N)} J_{i+1} \mathbf{1}_{(M+N)\times 1} = 0$. We have thus shown that $\mathbf{1}_{1\times(M+N)} J_i \mathbf{1}_{(M+N)\times 1} = 0 \Rightarrow \mathbf{1}_{1\times(M+N)} J_{i+1} \mathbf{1}_{(M+N)\times 1} = 0$. For $i = 0$ the statement we seek to prove is trivially true, and thus the proof is complete.

E Rank of the matrix J_a

In this appendix we prove that the $N \times N$ matrix

$$J_a = I_{LL} - J_{Lr_\infty} \left(Q_s^{-1} + J_{rr_\infty}\right)^{-1} J_{Lr_\infty}^T$$

is of rank $N-1$, and that $\frac{1}{\sqrt{1}} \mathbf{1}_{N\times 1}$ is its nullvector. Substitution for the value of J_{Lr_∞} from Eq. (81) yields

$$\begin{aligned} J_a &= I_{LL} - J_{Lr_\infty} \left(Q_s^{-1} + J_{rr_\infty}\right)^{-1} J_{Lr_\infty}^T \\ &= I_{LL} - I_{Lr} J_{rr_\infty}^{-1} \left(Q_s^{-1} + J_{rr_\infty}\right) J_{rr_\infty}^{-1} I_{Lr}^T \end{aligned} \tag{128}$$

and using the result of Eq. (76) we obtain the simple expression

$$J_a = I_{LL} - I_{Lr} I_{rr}^{-1} I_{Lr}^T \tag{129}$$

In order to compute the rank of this matrix, we note that J_a is the Schur complement of I_{rr} in the matrix

$$H_o'^T R_s^{-1} H_o' = \begin{bmatrix} I_{rr} & I_{Lr}^T \\ I_{Lr} & I_{LL} \end{bmatrix} \tag{130}$$

But the matrix H_o' is identical to the incidence matrix of the RPMG describing the relative position measurements. Since the RPMG is assumed to be a connected graph, H_o' is of rank $M+N-1$ [12]. As a result $H_o'^T R_s^{-1} H_o'$ is of rank $M+N-1$ [13]. Moreover, the invertibility of I_{rr} enables us to apply to following property of the Schur complement:

$$\begin{aligned} \text{rank}\left(H_o'^T R_s^{-1} H_o'\right) &= \text{rank}(I_{rr}) + \text{rank}(I_{LL} - I_{Lr} I_{rr}^{-1} I_{Lr}^T) \Rightarrow \\ M + N - 1 &= M + \text{rank}(I_{LL} - I_{Lr} I_{rr}^{-1} I_{Lr}^T) \Rightarrow \\ \text{rank}(I_{LL} - I_{Lr} I_{rr}^{-1} I_{Lr}^T) &= N - 1 \Rightarrow \\ \text{rank}(J_a) &= N - 1 \end{aligned} \tag{131}$$

We next show that $\frac{1}{\sqrt{1}} \mathbf{1}_{N\times 1}$ is the nullvector of J_a. For this purpose it suffices to show that

$$\begin{aligned} \mathbf{1}_{1\times N} \left(I_{LL} - I_{Lr} I_{rr}^{-1} I_{Lr}^T\right) \mathbf{1}_{N\times 1} = 0 \Rightarrow \\ \mathbf{1}_{1\times N} I_{LL} \mathbf{1}_{N\times 1} - \mathbf{1}_{1\times N} I_{Lr} I_{rr}^{-1} I_{Lr}^T \mathbf{1}_{N\times 1} = 0 \end{aligned} \tag{132}$$

But from the structure of the measurement equations, it is easy to see that $\mathbf{1}_{1\times N} H_2^T = -\mathbf{1}_{1\times M} H_1^T$, and also $\mathbf{1}_{1\times M} H_R^T = \mathbf{0}_{1\times M_{RR}}$. Therefore

$$\begin{aligned} \mathbf{1}_{1\times N} I_{Lr} &= \mathbf{1}_{1\times N} \left(H_2^T F_2^T H_R + H_2^T F_4 H_1\right) \\ &= -\mathbf{1}_{1\times M} \left(H_1^T F_2^T H_R + H_1^T F_4 H_1\right) \\ &= -\mathbf{1}_{1\times M} \left(H_1^T F_2^T H_R + H_1^T F_4 H_1\right) - \mathbf{1}_{1\times M} \left(H_R^T F_1 H_R + H_R^T F_2 H_1\right) \\ &= -\mathbf{1}_{1\times M} \left(H_R^T F_1 H_R + H_1^T F_2^T H_R + H_R^T F_2 H_1 + H_1^T F_4 H_1\right) \\ &= -\mathbf{1}_{1\times M} I_{rr} \end{aligned} \tag{133}$$

In a similar way we can show that $\mathbf{1}_{1\times N} I_{LL} \mathbf{1}_{N\times 1} = \mathbf{1}_{1\times M} I_{rr} \mathbf{1}_{M\times 1}$, and thus Eq. (132) can be rewritten as

$$\begin{aligned} \mathbf{1}_{1\times M} I_{rr} \mathbf{1}_{M\times 1} - \mathbf{1}_{1\times M} I_{rr} I_{rr}^{-1} I_{rr} \mathbf{1}_{M\times 1} &= 0 \Rightarrow \\ \mathbf{1}_{1\times M} I_{rr} \mathbf{1}_{M\times 1} - \mathbf{1}_{1\times M} I_{rr} \mathbf{1}_{M\times 1} &= 0 \end{aligned}$$

which holds trivially.

Finally, the rank of the matrix $\mathbf{J}_a = J_a \otimes I_{2\times 2}$ can be computed by application of the properties of the Kronecker product. Specifically,

$$\operatorname{rank}(\mathbf{J}_a) = \operatorname{rank}(J_a \otimes I_{2\times 2}) = \operatorname{rank}(J_a)\operatorname{rank}(I_{2\times 2}) = 2N-2$$

Thus the $\mathbf{J}_a$ has two eigenvalues equal to zero. The eigenvectors corresponding to these eigenvalues can be easily determined by noting that

$$\mathbf{J}_a \left(\mathbf{1}_{N\times 1} \otimes I_{2\times 2}\right) = \left(J_a 1_{N\times 1}\right) \otimes I_{2\times 2} = \mathbf{0}_{2N\times 2}$$

Therefore we conclude that two basis vectors for the nullspace of $\mathbf{J}_a$ are given by the column vectors of the matrix $\frac{1}{\sqrt{N}}\mathbf{1}_{N\times 1}$.

F Proof of Lemma 3.5

We denote $U = [V \;\; U_N]$, where V is a matrix whose column vectors form a basis of the range of Y, while the columns of U_N form a basis of the nullspace of Y. Assuming that $Y \in \mathcal{R}^{n\times n}$, and that the nullspace of Y is of dimension m, then $V \in \mathcal{R}^{n\times p}$, with $p = n - m$ and $U_N \in \mathcal{R}^{n\times m}$. We can thus write

$$\begin{aligned} (Yk + B_k)^{-1} &= (U\Lambda U^T k + B_K)^{-1} \\ &= U(\Lambda k + U^T B_K U)^{-1} U^T \\ &= \begin{bmatrix} V & U_N \end{bmatrix} \left(\begin{bmatrix} \Lambda_o k & \mathbf{0}_{p\times m} \\ \mathbf{0}_{m\times p} & \mathbf{0}_{m\times m} \end{bmatrix} + \begin{bmatrix} V^T \\ U_N^T \end{bmatrix} B_k \begin{bmatrix} V & U_N \end{bmatrix} \right)^{-1} \begin{bmatrix} V^T \\ U_N^T \end{bmatrix} \end{aligned}$$

where Λ_o denotes a $p \times p$ diagonal matrix, whose diagonal elements are the nonzero singular values of Y. Carrying out the matrix operations yields

$$(Yk + B_k)^{-1} = \begin{bmatrix} V & U_N \end{bmatrix} \begin{bmatrix} \Lambda_o k + V^T B_k V & V^T B_k U_N \\ U_N^T B_k V & U_N^T B_k U_N \end{bmatrix}^{-1} \begin{bmatrix} V^T \\ U_N^T \end{bmatrix}$$

Employing the formula for the inversion of a partitioned matrix (cf. Appendix H) yields

$$(Yk + B_k)^{-1} = \begin{bmatrix} V & U_N \end{bmatrix} \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \begin{bmatrix} V^T \\ U_N^T \end{bmatrix} \tag{134}$$

with

$$\begin{aligned} A_1 &= \left(\Lambda_o k + V^T B_k V - V^T B_k U_N \left(U_N^T B_k U_N\right)^{-1} U_N^T B_k V\right)^{-1} \\ A_2 &= -\left(\Lambda_o k + V^T B_k V - V^T B_k U_N \left(U_N^T B_k U_N\right)^{-1} U_N^T B_k V\right)^{-1} V^T B_k U_N \left(U_N^T B_k U_N\right)^{-1} \\ &= A_1 V^T B_k U_N \left(U_N^T B_k U_N\right)^{-1} \\ A_3 &= -\left(U_N^T B_k U_N\right)^{-1} U_N^T B_k V \left(\Lambda_o k + V^T B_k V - V^T B_k U_N \left(U_N^T B_k U_N\right)^{-1} U_N^T B_k V\right)^{-1} \\ &= -\left(U_N^T B_k U_N\right)^{-1} U_N^T B_k V \\ A_4 &= \left(U_N^T B_k U_N - U_N^T B_k V \left(\Lambda_o k + V^T B_k V\right)^{-1} V^T B_k U_N\right)^{-1} \end{aligned}$$

Computation of the limits of these matrices as $k \to \infty$ is now possible. We have

$$\begin{aligned} \lim_{k\to\infty} A_1 &= \lim_{k\to\infty} \left(\Lambda_o k + V^T B_k V - V^T B_k U_N \left(U_N^T B_k U_N\right)^{-1} U_N^T B_k V\right)^{-1} \\ &= \lim_{k\to\infty} \frac{1}{k} \left(\Lambda_o + \frac{1}{k} V^T B_k V - \frac{1}{k} V^T B_k U_N \left(U_N^T B_k U_N\right)^{-1} U_N^T B_k V\right)^{-1} \end{aligned}$$

But assuming that $\lim_{k\to\infty} B_k = B_\infty$ exists (i.e, it is finite), we have

$$\lim_{k\to\infty} \frac{1}{k} V^T B_k V - \frac{1}{k} V^T B_k U_N \left(U_N^T B_k U_N\right)^{-1} U_N^T B_k V = \mathbf{0}_{p\times p}$$

and thus

$$\lim_{k\to\infty} A_1 \quad = \quad \lim_{k\to\infty} \frac{1}{k} \Lambda_o^{-1} = \mathbf{0}_{p\times p}$$

As a consequence $\lim_{k\to\infty} A_2 = \mathbf{0}_{p\times m}$ and $\lim_{k\to\infty} A_3 = \mathbf{0}_{m\times p}$. Finally, we have that

$$\begin{aligned} \lim_{k\to\infty} \left(\Lambda_o k + V^T B_k V\right)^{-1} &= \lim_{k\to\infty} \frac{1}{k} \left(\Lambda_o + \frac{1}{k} V^T B_k V\right)^{-1} \\ &= \mathbf{0}_{p\times p} \end{aligned}$$

and therefore

$$\lim_{k\to\infty} A_4 \quad = \quad (U_N^T B_\infty U_N)^{-1}$$

Substitution in Eq. (134) yields

$$\begin{aligned} (Yk + B_k)^{-1} &= \begin{bmatrix} V & U_N \end{bmatrix} \begin{bmatrix} \mathbf{0}_{p\times p} & \mathbf{0}_{p\times m} \\ \mathbf{0}_{m\times p} & (U_N^T B_\infty U_N)^{-1} \end{bmatrix} \begin{bmatrix} V^T \\ U_N^T \end{bmatrix} \\ &= U_N (U_N^T B_\infty U_N)^{-1} U_N^T \end{aligned} \tag{135}$$

which is the desired result.

G Matrix Inversion Lemma

If A is $n \times n$, B is $n \times m$, C is $m \times m$ and D is $m \times n$ then:

$$(A^{-1} + BC^{-1}D)^{-1} = A - AB(DAB + C)^{-1}DA \tag{136}$$

H Inversion of a Partitioned Matrix

Let a $(m + n) \times (m + n)$ matrix K be partitioned as

$$K = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

Where the $m \times m$ matrix A and the $n \times n$ matrix D are invertible. Then the inverse matrix of K can be written as

$$\begin{bmatrix} X & Y \\ Z & U \end{bmatrix} = \begin{bmatrix} (A - BD^{-1}C)^{-1} & -A^{-1}B(D - CA^{-1}B)^{-1} \\ -D^{-1}C(A - BD^{-1}C)^{-1} & (D - CA^{-1}B)^{-1} \end{bmatrix} \tag{137}$$

References

[1] A. I. Mourikis and S. I. Roumeliotis, "Analysis of Positioning Uncertainty in Reconfigurable Networks of Heterogeneous Mobile Robots," in *Proceedings of the 2004 IEEE International Conference on Robotics and Automation*, New Orleans, LA, USA, Apr. 26 - May 1 2004, pp. 572–579.

[2] R. Kurazume, S. Nagata, and S. Hirose, "Cooperative positioning with multiple robots," in *Proceedings of the IEEE International Conference in Robotics and Automation*, Los Alamitos, CA, May 8-13 1994, pp. 1250–1257.

[3] I. M. Rekleitis, G. Dudek, and E. Milios, "Multi-robot cooperative localization: A study of trade-offs between efficiency and accuracy," in *Proceedings of the IEEE/RSJ International Conference on Intelligent Robots and Systems*, Lausanne, Switzerland, Sep.30-Oct.4 2002, pp. 2690–2695.

[4] J. W. Fenwick, P. M. Newman, and J. J. Leonard, "Cooperative Concurrent Mapping and Localization," in *Proceedings of the 2002 IEEE International Conference on Robotics and Automation*, Washington D.C., 11-15 May 2002, pp. 1810–1817.

[5] A. Kelly, "General solution for linearized systematic error propagation in vehicle odometry," in *Proceedings of the IEEE/RSJ International Conference on Robots and Systems (IROS)*, Maui, HI, Oct.29-Nov.3 2001, pp. 1938–45.

[6] R. C. Smith, M. Self, and P. Cheeseman, *Autonomous Robot Vehicles*. Springer-Verlag, 1990, ch. Estimating Uncertain Spatial Relationships in Robotics, pp. 167–193.

[7] P. Moutarlier and R. Chatila, "Stochastic multisensory data fusion for mobile robot location and environment modeling," in *Fifth International Symposium of Robotics Research*, H. Miura and S. Arimoto, Eds., Tokyo, Japan, 28-31 Aug. 1989, pp. 85–94.

[8] B. Hassibi, "Indefinite metric spaces in estimation, control and adaptive filtering," Ph.D. dissertation, Stanford University, August 1996.

[9] S. I. Roumeliotis and G. A. Bekey, "Distributed multirobot localization," *IEEE Transactions on Robotics and Automation*, vol. 18, no. 5, pp. 781–795, Oct. 2002.

[10] A. I. Mourikis and S. I. Roumeliotis, "Performance bounds for cooperative simultaneous localization and mapping C-SLAM," Dept. of Computer Science and Engineering, University of Minnesota, Tech. Rep., 2004, http://www.cs.umn.edu/∼mourikis/TR_MultiSLAM.pdf.

[11] S. Boyd and L. Vandenberghe, *Convex Optimization*. Cambridge University Press, 2004.

[12] C. D. Meyer, *Matrix Analysis and Applied Linear Algebra*. Society for Industrial and Applied Mathematics, 2000.

[13] A. I. Mourikis and S. I. Roumeliotis, "Analysis of positioning uncertainty in reconfigurable networks of heterogeneous mobile robots," Dept. of Computer Science and Engineering, University of Minnesota, Tech. Rep., 2003, www.cs.umn.edu/∼mourikis/TR_multi.pdf.