Analysis of Positioning Uncertainty in Simultaneous Localization and Mapping (SLAM)

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Abstract— **This paper studies the time evolution of the covariance of the position estimates in single-robot Simultaneous Localization And Mapping (SLAM). A closed-form expression is derived, that establishes a functional relation between the noise parameters of the robot's proprioceptive and exteroceptive sensors, the number of features being mapped, and the attainable accuracy of SLAM. Furthermore, it is demonstrated how prior information about the spatial density of landmarks can be utilized in order to compute a tight upper bound on the expected covariance of the positioning errors. The derived closed-form expressions enable the prediction of SLAM positioning performance, without resorting to extensive simulations, and thus offer an analytical tool for determining the sensor characteristics required to achieve a desired degree of accuracy. Simulation experiments are conducted, that corroborate the presented theoretical analysis.**

I. Introduction

Mobile robots that operate autonomously within an area must be able to determine their position with respect to a global frame of reference. In an ideal scenario, a robot equipped with a GPS receiver would have direct access to measurements of its absolute position. In a number of situations this is not feasible since GPS signals are not available everywhere (e.g., indoors), or, triangulation techniques based on these may provide erroneous results due to multiple reflections (e.g., in the vicinity of tall structures and buildings). Cost, size, weight, and power constraints may also prohibit reliance on GPS. These limitations suggest that alternative means are required for *aiding* odometry when mobile robots localize.

In certain cases, when details about the structure of the area are available, a robot can localize by detecting previously mapped features. Relative position measurements to known landmarks, received by exteroceptive sensors such as a laser scanner or a camera, can be processed in order to update the estimates for the position of the robot. In most cases, however, compiling a detailed map of the environment is a tedious and time consuming process, and robots must localize while building a map of their surroundings. This introduces the problem of Simultaneous Localization And Mapping (SLAM) that has recently attracted the interest of many researchers. The number of potential applications that require robots to perform SLAM is immense and continuously grows as autonomous vehicles are employed for tasks ranging from planetary exploration and environmental monitoring, to construction and transportation.

Recent research on SLAM has primarily focused on developing algorithms that can be used in real-time implementations. The proposed methods have often traded accuracy, robustness, and realization simplicity for speed. This is justified in practice since the quadratic, in the number of mapped features, computational complexity of SLAM prohibits robots from localizing within large-scale environments. However, the theoretical analysis of positioning accuracy in SLAM remains an open issue to date, and only few cases exist in the literature where the properties of the time evolution of covariance in SLAM have been studied (e.g., [1], [2], [3]).

This paper presents the first derivation of analytical *upper bounds* on the SLAM positioning uncertainty for a mobile robot navigating within a 2D environment populated with point features. The closed-form expression of Lemma 4.1 establishes a *functional relation* between the noise parameters of the robot's sensors and the accuracy of SLAM. Furthermore, the result of Lemma 4.2 demonstrates how prior information about the spatial density of landmarks can be utilized in order to compute a tight upper bound on the *expected* covariance of the positioning errors. The proposed bounds constitute, to the best of our knowledge, the only existing *analytical tool* for predicting the attainable mapping precision as well as the accuracy of the robot's localization in a given SLAM application. Hence, they facilitate the selection of the sensor parameters, in order to satisfy task-imposed performance constraints.

In this work, SLAM is considered within the Stochastic Mapping framework [4], [5]. We assume that a robot moves continuously and randomly in a planar environment, and at each time instant measures the relative position (i.e., range and bearing) of N stationary landmarks. The metric used to describe the localization uncertainty is the covariance matrix associated with the errors in the position estimates for the robot and the mapped features. In order to facilitate the required derivations, an Extended Kalman Filter (EKF) estimator is selected, since it provides a well-studied mechanism for propagating and updating the covariance matrix through time.

The remainder of the paper is structured as follows: In the next section, the most prominent approaches to SLAM are briefly outlined and a more detailed description of existing work on the study of covariance in SLAM is provided. In Section III, the problem formulation is introduced. Section IV presents the main theoretical results of the paper, synopsized in Lemmas 4.1 and 4.2. In Section V, the validity of the analysis is illustrated with simulation results, and finally, in Section VI the conclusions of this work are drawn.

II. Related Work

Most of the existing approaches to SLAM have been inspired by the seminal papers of Moutarlier and Chatila [5] and of Smith, Self, and Cheeseman [4], [6] that introduced the notion of the *Stochastic Map*. This work has emphasized the importance of properly accounting for the correlations between the state estimates of the robot's and landmarks' positions, and laid down a complete and rigorous theoretical framework for the study of the SLAM problem.

The main limitation of maintaining all the cross-correlation elements of the covariance matrix in EKF-based SLAM is that it results in algorithms which have complexity quadratic in the number of features. This leads to a prohibitively large computational load in cases when online estimation of a large map is necessary. This problem has recently received growing attention and there have been numerous attempts to produce scalable SLAM algorithms, without significant loss of accuracy. For example, particle filtering [7], [8], use of local submaps [9], [10], covariance intersection techniques [11], and approximations to the extended information filter [12] are only few of the proposed approaches. A second computational bottleneck in SLAM arises from the need to perform robust data association, for large numbers of landmarks and observations, in the presence of uncertainty. An excellent overview of existing techniques can be found in [7]. In this paper, we do not address any of the aforementioned implementation issues. We assume perfect data association and seek to characterize the theoretically attainable estimation accuracy in SLAM, by providing bounds for the covariance of the position estimates.

At this point, we present previous work that aims at describing the time evolution of the covariance in SLAM. In [2], the authors consider the one-dimensional problem, in which the robot and landmarks are all situated along a single coordinate axis. In this case, both the state propagation and measurement models are linear time-invariant. Under the additional assumptions that (i) the initial covariance of each of the features on the map is equal to the covariance of the measurement associated with it, and (ii) the robot has perfect initial knowledge of its position, a closed form solution for the time evolution of the covariance is derived. A limitation of this approach is that the realistic case of infinite initial uncertainty for the features (i.e., unknown initial map) is not treated. Furthermore, these results are only valid for motion in 1D which is of limited practical importance.

The covariance convergence properties of SLAM have also been studied in [1], [13], [14]. The authors assume linear time-invariant models for both the propagation and measurement equations and provide proofs for the following statements: (i) The covariance of the landmarks' position estimates decreases monotonically, and (ii) At steady state, the landmark position estimates become fully correlated. Additionally, a lower bound for the steady state uncertainty is derived by considering the restrictive case of the robot remaining *static* while recording measurements of the landmarks' positions. However, the proposed *lower* bound cannot be employed for determining the performance of SLAM in the case of a robot *in motion* exploring an unknown area. In such a scenario, the global coordinate frame can be arbitrarily defined, thus the robot has perfect knowledge of its initial position, and the described lower bound reduces to zero.

The approach presented in [1] is extended to the case of cooperative Concurrent Mapping and Localization in [3], [15]. By employing similar assumptions, of linear propagation and measurement models, it is shown that at steady state, all of the vehicle and feature position estimates become fully correlated. Finally, when no loss of information occurs in the system, (i.e., the robots receive noise-free odometry measurements), lower bounds for the covariance of all vehicles and features are derived in a manner similar to [1].

The main contribution of the work presented in this paper is a characterization of the accuracy of the position estimates in SLAM. This is achieved by deriving *analytical expressions* for the maximum value of the estimates' covariance at *any* time instant after the commencement of the exploration task (cf. Eq. (25)). Furthermore, by obtaining the limit value of the derived expression after sufficient time, the maximum asymptotic (steady state) position uncertainty is derived (cf. Lemma. 4.1). Finally, a method for incorporating prior information about the spatial distribution of the features in the environment is presented that yields an improved description of the estimation performance (cf. Lemma. 4.2). What distinguishes these results from previous ones is that the analysis is based on the *actual* (non-linear) system and measurement equations for a robot navigating in 2D.

III. Problem Formulation

Consider a mobile robot moving on a planar surface, while observing N landmarks in the environment. The robot uses proprioceptive measurements (e.g., from an odometric or inertial sensor) to propagate its state estimates and exteroceptive measurements (e.g., from a laser range finder) to measure the relative positions of the map features with respect to itself. These measurements are fused using an Extended Kalman Filter (EKF) in order to produce estimates of the position of the robot and the landmarks. In our formulation, it is assumed that an upper bound for the variance of the errors in the robot's orientation estimates can be determined a priori. This allows us to decouple the task of position estimation from that of orientation estimation and facilitates the derivation of a closed-form expression for an upper bound on the positioning uncertainty.

The robot's orientation uncertainty is bounded when, for example, absolute orientation measurements from a compass or a sun sensor are available, or when perpendicularity of the walls in an indoor environment is used to infer orientation. In cases where neither approach is possible, our analysis still holds under the condition that a conservative upper bound for the orientation uncertainty is determined by alternative means, e.g., by estimating the maximum orientation error accumulated, over a certain period of time, due to the integration of noise in the odometric measurements [16]. It should be noted that the requirement for bounded orientation error covariance is not too restrictive: In the EKF framework, the nonlinear state propagation and measurement equations are linearized around the estimates of the robot's orientation. If the errors in these estimates are allowed to increase unbounded, the linearization will unavoidably become erroneous, and the estimates will diverge. Thus, in the vast majority of practical situations, provisions are made in order to constrain the robot's orientation uncertainty within given limits.

The metric that is employed to describe the attainable estimation accuracy in SLAM is the covariance matrix of the position errors, $P(t)$. The time evolution of this matrix is described by the following Riccati Differential Equation (RDE):

$$\begin{aligned}\dot{P}(t) &= F(t)P(t)+P(t)F^T(t)+G(t)Q(t)G^T(t)\\ &\quad - P(t)H^T(t)R^{-1}(t)H(t)P(t)\end{aligned} \tag{1}$$

where $F(t)$ is the state transition matrix, the quantity $G(t)Q(t)G^T(t)$ accounts for the influx of uncertainty due to the noise in the odometric measurements used for propagation, and the term $H^T(t)R^{-1}(t)H(t)$ represents the information input to the system by the exteroceptive (relative position) measurements.

The kinematic equations of a robot moving in 2D are nonlinear and thus the matrices involved in the RDE are time-varying. In this case, a general closed-form solution for $P(t)$ does not exist. We therefore resort to deriving an *upper bound* for $P(t)$. The basis of our approach is the fact that the solution of the RDE in Eq. (1) is a monotonically increasing function of the *input data matrix* [17]:

$$E(t) = \begin{bmatrix} G(t)Q(t)G^T(t) & -F(t) \\ -F^T(t) & -H^T(t)R^{-1}(t)H(t)\end{bmatrix} \tag{2}$$

This implies, that by solving a different RDE, for which the input data matrix, $E_o(t)$, satisfies $E_o(t) \succeq E(t)$, we can derive an upper bound for the covariance of the position estimates. A rigorous proof of this statement can be found in [18]. In the following subsections, the exact form of the matrices appearing in Eq. (1), along with appropriate bounds for them, are derived. These results will enable us to formulate a simpler differential equation, whose solution exists in closed form, and is the sought upper bound for the uncertainty in SLAM.

A. Position propagation

The continuous-time kinematic equations for a robot moving in 2D are

$$\dot{x}_R(t) = V(t)\cos(\phi(t)) \tag{3}$$

$$\dot{y}_R(t) = V(t)\sin(\phi(t)) \tag{4}$$

$$\dot{\phi}(t) = \omega(t) \tag{5}$$

where $V(t)$ and $\omega(t)$ are the linear and rotational velocity of the robot at time t. Since in our formulation position estimation is decoupled from orientation estimation, the robot's state comprises only of its x and y coordinates, while orientation is considered an input, of which only noisy measurements, i.e., the orientation estimates $\hat{\phi}(t)$, are available. Clearly, the motion model is nonlinear in the orientation and time-varying. Linearization of Eqs. (3) and (4) yields the position error propagation equations for the robot:[1]

$$\begin{aligned}\begin{bmatrix}\dot{\tilde{x}}_R(t)\\ \dot{\tilde{y}}_R(t)\end{bmatrix} &= \begin{bmatrix}\cos(\hat{\phi}(t)) & -V_m(t)\sin(\hat{\phi}(t))\\ \sin(\hat{\phi}(t)) & V_m(t)\cos(\hat{\phi}(t))\end{bmatrix}\begin{bmatrix} w_V(t)\\ \tilde{\phi}(t)\end{bmatrix}\\ \Leftrightarrow \dot{\tilde{X}}_R(t) &= F_R(t)\tilde{X}_R(t)+G_R(t)W(t)\end{aligned} \tag{6}$$

where $F_R(t) = \mathbf{0}_{2\times 2}$, $V_m(t)$ are the robot's velocity measurements, corrupted by a white Gaussian noise process $w_V(t)$, with power σ_V^2, and $\tilde{\phi}(t)$ is the error in the robot's orientation estimate, modeled as a white Gaussian noise process with power σ_ϕ^2.

The landmarks are modeled as static points in 2D space, and thus the state and error propagation equations are: $\dot{X}_{L_i}(t) = \mathbf{0}_{2\times 1}$ and $\dot{\tilde{X}}_{L_i}(t) = \mathbf{0}_{2\times 1}$, respectively, for $i = 1..N$. The state vector for the entire system, X, is defined as the stacked vector of the position of the robot and landmarks. Therefore the state transition matrix for the entire system is $\mathbf{F} = \mathbf{0}_{(2N+2)\times(2N+2)}$, while the system noise covariance matrix is:

$$\mathbf{G}(t)\mathbf{Q}(t)\mathbf{G}^T(t) = \begin{bmatrix} G_R(t)Q_R(t)G_R^T(t) & \mathbf{0}_{2\times 2N}\\ \mathbf{0}_{2N\times 2} & \mathbf{0}_{2N\times 2N}\end{bmatrix} \tag{7}$$

where, for $Q_R(t) = E\{W(t)W^T(t)\}$ and using the expressions in Eq. (6), it is:

$$G_R(t)Q_R(t)G_R^T(t) = C(\hat{\phi}(t))\begin{bmatrix}\sigma_V^2 & 0\\ 0 & \sigma_\phi^2 V_m^2(t)\end{bmatrix} C^T(\hat{\phi}(t))$$

with $C(\hat{\phi}(t)) = \begin{bmatrix}\cos\hat{\phi}(t) & -\sin\hat{\phi}(t)\\ \sin\hat{\phi}(t) & \cos\hat{\phi}(t)\end{bmatrix}$ denoting the rotational matrix associated with $\hat{\phi}(t)$. From the properties of rotational matrices, it follows that the eigenvalues of $G_R(t)Q_R(t)G_R^T(t)$ are σ_V^2 and $\sigma_\phi^2 V_m^2$, and thus $G_R(t)Q_R(t)G_R^T(t) \preceq \max(\sigma_V^2, \sigma_\phi^2 V_m^2) I_{2\times 2}$. Consequently, we can write

$$\mathbf{G}(t)\mathbf{Q}(t)\mathbf{G}^T(t) \preceq \mathbf{Q}_o = \begin{bmatrix} qI_{2\times 2} & \mathbf{0}_{2\times 2N}\\ \mathbf{0}_{2N\times 2} & \mathbf{0}_{2N\times 2N}\end{bmatrix} = q\mathbf{Q}_n \tag{8}$$

where

$$q = \max(\sigma_V^2, \sigma_\phi^2 V_m^2) \simeq \max(\sigma_V^2, \sigma_\phi^2 V^2) \tag{9}$$

[1]Due to space limitations many of the details of the derivations have been omitted. The interested reader is referred to [18] for a thorough description of the intermediate steps. Throughout this paper $\mathbf{0}_{m\times n}$ denotes the $m\times n$ matrix of zeros, $\mathbf{1}_{m\times n}$ denotes the $m\times n$ matrix of ones, and $I_{n\times n}$ denotes the $n\times n$ identity matrix.

B. Relative Position Measurement Model

At every time instant, the robot measures the relative position of each of the N landmarks in the environment. The measurement equation for the relative position of the ith landmark is given by:

$$z_i(t) = C^T(\phi(t))\left(X_{L_i}(t) - X_R(t)\right) + n_{z_i}(t) \qquad (10)$$

where $n_{z_i}(t)$ is the noise affecting this measurement. By linearizing Eq. (10), the measurement error equation is obtained:

$$\widetilde{z}_i(t) = z_i(t) - \hat{z}_i(t) \simeq H_i(t)\widetilde{X}(t) + \Gamma_i(t)n_i(t)$$

where

$$\begin{aligned} H_i(t) &= C^T(\hat{\phi}(t))\ H_{o_i} \qquad (11)\\ H_{o_i} &= \begin{bmatrix} -I_{2\times 2} & \mathbf{0}_{2\times 2} & .. & \underbrace{I_{2\times 2}}_{i} & .. & \mathbf{0}_{2\times 2} \end{bmatrix}\\ \widetilde{X} &= \begin{bmatrix} \widetilde{X}_R^T & \widetilde{X}_{L_1}^T & \dots & \widetilde{X}_{L_i}^T & \dots & \widetilde{X}_{L_N}^T \end{bmatrix}^T\\ \Gamma_i(t) &= \begin{bmatrix} I_{2\times 2} & -C^T(\hat{\phi}(t))J\widehat{\Delta p}_i(t) \end{bmatrix}\\ J &= \begin{bmatrix} 0 & -1\\ 1 & 0 \end{bmatrix},\quad n_i(t) = \begin{bmatrix} n_{z_i}(t)\\ \widetilde{\phi}(t) \end{bmatrix}\\ \widehat{\Delta p}_i(t) &= \hat{X}_{L_i}(t) - \hat{X}_R(t) \end{aligned}$$

The covariance of the measurement error is given by:

$$\begin{aligned} R_{ii}(t) &= \Gamma_i(t)E\{n_i(t)n_i^T(t)\}\Gamma_i^T(t)\\ &= R_{z_i}(t) + R_{\phi_i}(t) \qquad (12) \end{aligned}$$

This expression encapsulates all sources of noise and uncertainty that contribute to the measurement error, i.e., the covariance $R_{z_i}(t)$ of the noise $n_{z_i}(t)$ in the recorded relative position measurement, and the additional covariance term $R_{\phi_i}(t)$ due to the error $\tilde{\phi}_i(t)$ in the orientation estimate of the robot.

Assuming that each exteroceptive measurement consists of a range measurement ρ_i and a bearing measurement θ_i, whose errors n_{ρ_i} and n_{θ_i} are uncorrelated, the term $R_{z_i}(t)$ can be expressed as [18]:

$$R_{z_i}(t) = C(\theta_i(t))\begin{bmatrix} \sigma_\rho^2 & 0\\ 0 & \hat{\rho}_i^2(t)\sigma_\theta^2 \end{bmatrix} C^T(\theta_i(t)) \qquad (13)$$

where σ_ρ and σ_θ are the standard deviations of the white zero-mean Gaussian noise processes affecting the range and bearing measurements, and $C(\theta_i(t))$ is the rotational matrix associated with the bearing angle of the relative position measurement, expressed in the robot's coordinate frame.

The existence of errors in the orientation estimates of the robot introduces an additional component to the measurement error of the relative position of each landmark. This causes the measurements of all landmarks to be correlated. It can be shown [18] that the additional covariance term for each measurement is equal to:

$$R_{\phi_i}(t) = \sigma_\phi^2 C^T(\hat{\phi})J\widehat{\Delta p}_i\widehat{\Delta p}_i^T J^T C(\hat{\phi}) \qquad (14)$$

while the correlation matrix between the errors in the measurements $z_i(t)$ and $z_j(t)$ is

$$\begin{aligned} R_{ij}(t) &= \Gamma_i(t)E\{n_i(t)n_j^T(t)\}\Gamma_j^T(t)\\ &= \sigma_\phi^2 C^T(\hat{\phi})J\widehat{\Delta p}_i\widehat{\Delta p}_j^T J^T C(\hat{\phi}) \qquad (15) \end{aligned}$$

Using these results, we can evaluate the covariance matrix $\mathbf{R}(t)$ for all measurements gathered by the robot at each time instant. This is a matrix whose $R_{ii}(t)$ and $R_{ij}(t)$, $i, j = 1\dots N$, block elements are defined in Eqs. (12) and (15), respectively. Hence, the information contributed by the exteroceptive measurements is $\mathbf{H}^T(t)\mathbf{R}^{-1}(t)\mathbf{H}(t)$, where $\mathbf{H}$ is the corresponding measurement matrix, i.e., a matrix whose block rows are $H_i(t),\ j = 1..N$ (cf. Eq. (11)).

By considering the special structure of the matrices $\mathbf{H}(t)$ and $\mathbf{R}(t)$, it can be shown [18] that

$$\mathbf{H}^T(t)\mathbf{R}^{-1}(t)\mathbf{H}(t) = \mathbf{H}_o^T\mathbf{R}_o^{-1}(t)\mathbf{H}_o$$

where

$$\mathbf{H}_o = [-\mathbf{1}_{N\times 1} \quad I_{N\times N}] \otimes I_{2\times 2} \qquad (16)$$

and

$$\begin{aligned} \mathbf{R}_o^{-1}(t) = \frac{1}{\sigma_\rho^2}D_1\,\mathrm{diag}\left(\frac{1}{\hat{\rho}_i^2}\right)D_1^T + \frac{1}{\sigma_\theta^2}D_2\,\mathrm{diag}\left(\frac{1}{\hat{\rho}_i^4}\right)D_2^T\\ - \frac{1}{\sigma_\eta^2}D_2\,\mathrm{diag}\left(\frac{1}{\hat{\rho}_i^2}\right)\mathbf{1}_{N\times N}\,\mathrm{diag}\left(\frac{1}{\hat{\rho}_i^2}\right)D_2 \qquad (17) \end{aligned}$$

In these expressions $\otimes$ denotes the Kronecker matrix product, D_1 is a $2N \times N$ block diagonal matrix whose ith block diagonal element is the 2×1 vector $\widehat{\Delta p}_i(t)$, D_2 is a $2N \times N$ block diagonal matrix whose ith block diagonal element is $J\widehat{\Delta p}_i(t)$, and $\sigma_\eta^2 = \frac{\sigma_\theta^4}{\sigma_\phi^2} + N\sigma_\theta^2$.

In the derivation of an upper bound on the uncertainty of the position estimates for the robot and landmarks, we will employ a lower bound of the matrix $\mathbf{H}_o^T\mathbf{R}_o^{-1}(t)\mathbf{H}_o$. In [18] it is shown that

$$\mathbf{R}_o(t) \preceq \left(\sigma_\rho^2 + \rho_o^2\sigma_\theta^2 + N\rho_o^2\sigma_\phi^2\right) I_{2N\times 2N} = rI_{2N\times 2N} \qquad (18)$$

where ρ_o is the maximum possible distance between the robot and any landmark, and thus

$$\mathbf{H}_o^T\mathbf{R}_o^{-1}(t)\mathbf{H}_o \succeq \frac{1}{r}\mathbf{H}_o^T\mathbf{H}_o \qquad (19)$$

C. Transition from the discrete to the continuous-time model

In the preceding sections, the continuous-time motion and measurement models have been presented. Their use greatly simplifies the ensuing analysis since it allows for the use of the differential, rather than difference, Riccati equation to describe the time evolution of uncertainty. However, in a real implementation, measurements are available at discrete time instants and are corrupted by discrete-time noise processes, whose variances are determined by the characteristics of the sensors. Given the properties of the noise in the actual, discrete-time system, we can construct an equivalent continuous-time system model, by selecting the parameters of the noise in continuous-time so that the influx of uncertainty over a given time interval for both systems is identical [19]. If all measurements are available every δt seconds and the standard deviations of the velocity and orientation errors in discrete time are σ_{V_d} and σ_{ϕ_d}, respectively, then selecting

$$\sigma_V = \sqrt{\delta t}\sigma_{V_d}, \text{ and } \sigma_\phi = \sqrt{\delta t}\sigma_{\phi_d} \tag{20}$$

yields an equivalent continuous-time model [18]. Similarly, if the covariance matrix of the exteroceptive measurements in discrete time is R, the equivalent continuous-time measurements' covariance matrix function is $R\delta t \cdot \delta(t-\tau)$, where $\delta(t-\tau)$ is the Dirac delta function. The time step duration, δt, can be seen as a normalizing factor to ensure that the information influx in the system is appropriately scaled with the sampling frequency of the measurements.

IV. Maximum Covariance of the SLAM Position Estimates

In this section we formulate a RDE whose solution is the upper bound on the covariance of the SLAM position estimates. The input data matrix for the system under consideration is (cf. Eq. (2)):

$$\mathbf{E}(t) = \begin{bmatrix} \mathbf{G}(t)\mathbf{Q}(t)\mathbf{G}^T(t) & \mathbf{0}_{(2N+2)\times(2N+2)} \\ \mathbf{0}_{(2N+2)\times(2N+2)} & -\mathbf{H}_o^T\mathbf{R}_o^{-1}(t)\mathbf{H}_o \end{bmatrix} \tag{21}$$

where the matrices appearing in the last expression are defined in Eqs. (7), (16) and (17). Using the bounds for the quantities $\mathbf{G}(t)\mathbf{Q}(t)\mathbf{G}^T(t)$ and $\mathbf{H}_o^T\mathbf{R}_o^{-1}(t)\mathbf{H}_o$, derived in the previous sections (Eqs. (8) and (19), respectively), we can write

$$\mathbf{E}(t) \preceq \begin{bmatrix} \mathbf{Q}_o & \mathbf{0}_{(2N+2)\times(2N+2)} \\ \mathbf{0}_{(2N+2)\times(2N+2)} & -\frac{1}{r}\mathbf{H}_o^T\mathbf{H}_o \end{bmatrix} \tag{22}$$

Hence, following the discussion in Section III, we deduce that the solution of the following RDE is an upper bound on the covariance of the position estimates in SLAM:

$$\begin{aligned} \dot{\bar{\mathbf{P}}}(t) &= \mathbf{Q}_o - \frac{1}{r}\bar{\mathbf{P}}(t)\mathbf{H}_o^T\mathbf{H}_o\bar{\mathbf{P}}(t) \\ &= q\mathbf{Q}_n - \frac{1}{r}\bar{\mathbf{P}}(t)\mathbf{H}_o^T\mathbf{H}_o\bar{\mathbf{P}}(t) \end{aligned} \tag{23}$$

In [18], the solution of this differential equation for the most general case, in which the initial covariance matrix is any positive semi-definite matrix, is derived. However, in SLAM, it is usually assumed that the robot starts operating in a totally unknown area, and thus it can arbitrarily define the origin of the global coordinate frame. In this case, the initial uncertainty about the position of the robot is zero, while the uncertainty about the landmarks' positions is infinite. In what follows, we will solve Eq. (23) for the corresponding initial condition:

$$\bar{\mathbf{P}}(0) = \mathbf{P}(0) = \begin{bmatrix} \mathbf{0}_{2\times 2} & \mathbf{0}_{2\times 2N} \\ \mathbf{0}_{2N\times 2} & \mathbf{P}_{LL} \end{bmatrix} \tag{24}$$

where $\mathbf{P}_{LL}$ is an arbitrary positive definite matrix. In order to model the total lack of initial knowledge about the landmarks' positions, we will then let $\mathbf{P}_{LL} = \mu I_{2N\times 2N},\ \mu \to \infty$.

The constant-coefficient RDE in Eq. (23), whose solution is the *maximum possible uncertainty* in SLAM, can be solved by decomposing $\bar{\mathbf{P}}(t)$ into the product of two matrices and forming the Hamiltonian matrix [20]. The resulting expression is [18]:

$$\begin{aligned} \bar{\mathbf{P}}(t) &= q\mathbf{U}^{-T}\mathbf{Q}_n\mathbf{L}\mathbf{K}^{-1}\mathbf{U}^{-1} \\ &\quad + q\mathbf{U}^{-T}\mathbf{K}^{-1}\mathbf{P}_0(\mathbf{K}+\mathbf{U}^{-1}\mathbf{C}\mathbf{U}^{-T}\mathbf{L}\mathbf{P}_0)^{-1}\mathbf{U}^{-1} \\ &= \bar{\mathbf{P}}_1(t) + \bar{\mathbf{P}}_2(t) \end{aligned} \tag{25}$$

where $\mathbf{P}_0 = \frac{1}{q}\mathbf{U}^T\bar{\mathbf{P}}(0)\mathbf{U}$,

$$\mathbf{C} = \frac{q}{r}\begin{bmatrix} NI_{2\times 2} & -\mathbf{J}^T \\ -\mathbf{J} & I_{2N\times 2N} \end{bmatrix},\ \mathbf{U} = \begin{bmatrix} I_{2\times 2} & \mathbf{0}_{2\times 2N} \\ -\frac{1}{N}\mathbf{J} & I_{2N\times 2N} \end{bmatrix}$$

with $\mathbf{J} = \mathbf{1}_{N\times 1} \otimes I_{2\times 2}$,

$$\mathbf{K} = \begin{bmatrix} \frac{e^{\lambda t}+e^{-\lambda t}}{2}I_{2\times 2} & \mathbf{0}_{2\times 2N} \\ \mathbf{0}_{2N\times 2} & I_{2N\times 2N} \end{bmatrix}$$

$$\mathbf{L} = \begin{bmatrix} \frac{e^{\lambda t}-e^{-\lambda t}}{2\lambda}I_{2\times 2} & \mathbf{0}_{2\times 2N} \\ \mathbf{0}_{2N\times 2} & t\ I_{2N\times 2N} \end{bmatrix}$$

and $\lambda = \sqrt{\frac{Nq}{r}}$. At this point, we should note that the first of the two terms, in Eq. (25), comprising $\bar{\mathbf{P}}(t)$ is independent of the initial uncertainty, while the second one captures the effect of the initial condition.

The expression in Eq. (25) can be used to determine an upper bound on the covariance of the position estimates at *any* time instant during the run of the SLAM algorithm. It is well known [1] that the covariance of the landmarks monotonically decreases and asymptotically assumes a steady state value. By evaluating the limit of $\bar{\mathbf{P}}(t)$ after sufficient time, i.e., as $t \to \infty$, we can obtain an upper bound for the steady state covariance $\mathbf{P}_{ss}$ of the map features and the robot. Substituting for the values of $\mathbf{U}$, $\mathbf{Q}_n$, $\mathbf{K}$, $\mathbf{L}$, and applying the limiting operation, the steady state value of $\bar{\mathbf{P}}_1(t)$ is easily found to be:

$$\bar{\mathbf{P}}_{1_{ss}} = \lim_{t\to\infty}\bar{\mathbf{P}}_1(t) = \sqrt{\frac{qr}{N}}\begin{bmatrix} I_{2\times 2} & \mathbf{0}_{2\times 2N} \\ \mathbf{0}_{2N\times 2} & \mathbf{0}_{2N\times 2N} \end{bmatrix} \tag{26}$$

The computation of the steady state value of $\bar{\mathbf{P}}_2(t)$ is significantly more cumbersome, and cannot be included in this paper due to space limitations. We only present the final expression here, and the interested reader is referred to [18] for the details of the proof.

When the initial covariance is given by Eq. (24), the steady state value of $\bar{\mathbf{P}}_2(t)$ is:

$$\bar{\mathbf{P}}_{2_{ss}} = \lim_{t\to\infty}\bar{\mathbf{P}}_2(t) = \mathbf{1}_{(N+1)\times(N+1)} \otimes M$$

where the 2×2 matrix M is given by

$$M = \left(\mathbf{J}^T\mathbf{P}_{LL}^{-1}\mathbf{J} + \sqrt{\frac{N}{qr}}I_{2\times 2}\right)^{-1} \tag{27}$$

We now apply this result, which was derived for an arbitrary initial landmark covariance matrix $\mathbf{P}_{LL}$, to the following two cases of interest.

A. Unknown Landmark Distribution

In order to compute the maximum steady state covariance in SLAM, when the positions of the landmarks are initially unknown, we set $\mathbf{P}_{LL} = \mu I_{2N\times 2N}$ and let $\mu \rightarrow \infty$. In this case, it is trivial to show that $\lim_{\mu\rightarrow\infty} M = \sqrt{\frac{qr}{N}} I_{2\times 2}$, and therefore we can state the following lemma:

Lemma 4.1: The maximum steady state covariance matrix in SLAM, when the robot builds a map of an initially unknown area containing N landmarks is

$$\bar{\mathbf{P}}_{ss} = \sqrt{\frac{qr}{N}} \begin{bmatrix} 2 & \mathbf{1}_{1\times N} \\ \mathbf{1}_{N\times 1} & \mathbf{1}_{N\times N} \end{bmatrix} \otimes I_{2\times 2} \qquad (28)$$

where q and r are defined in Eqs. (9) and (18), respectively.

It is clear that in deriving this upper bound *no assumption* on the distribution of the landmarks in space was introduced, except for the realistic requirement of limited maximum distance between the robot and landmarks. This value can be computed, for example, based on the maximum sensing range of the device used for measuring the landmarks' relative positions. Hence, the worst-case positioning accuracy in SLAM can be computed as a *functional relationship* of the characteristics of the robot's sensors, and the number of landmarks. It is clear that the steady state positioning accuracy improves with increasing map size, as well as with increasing accuracy of the robot's sensing devices.

B. Known Spatial Density of Landmarks

The expression in Eq. (28) provides an upper bound on the worst-case performance of SLAM, under any possible placement of the landmarks in space. However, when the features of the environment to be treated as landmarks are selected (e.g., visual features, prominent geometric features), it is beneficial to choose them so that they are abundant in the environment and evenly distributed throughout it. This way, a more detailed map of an area can be created. In such cases, the density of landmarks in the environment can be *a priori* modeled, for example, by a uniform probability density function (pdf), and this information can be exploited in order to compute a tighter upper bound for the *expected* steady state covariance of the position estimates. Specifically, the time instant right after the first update (due to observations of the positions of the landmarks) and before the robot moves, the covariance matrix has the form of Eq. (24) with $\mathbf{P}_{LL} = \mathbf{R}_o(0)$ [18]. Substituting from Eq. (17) into Eq. (27) and applying simple algebraic manipulations, results in the following closed form expression for the matrix M:

$$M = \frac{1}{\det A} A, \quad \text{with} \quad A = \begin{bmatrix} \alpha & \beta \\ \beta & \gamma \end{bmatrix} \qquad (29)$$

and

$$\alpha = \sum_{i=1}^{N} \frac{\widehat{\Delta x_i}^2}{\sigma_\rho^2 \hat{\rho}_i^2} + \sum_{i=1}^{N} \frac{\widehat{\Delta y_i}^2}{\sigma_\theta^2 \hat{\rho}_i^4} - \left(\sum_{i=1}^{N} \frac{\widehat{\Delta y_i}}{\sigma_\eta \hat{\rho}_i^2} \right)^2 + \sqrt{\frac{Nq}{r}}$$

$$\beta = \sum_{i=1}^{N} \frac{\widehat{\Delta x_i}}{\sigma_\eta \hat{\rho}_i^2} \sum_{i=1}^{N} \frac{\widehat{\Delta y_i}}{\sigma_\eta \hat{\rho}_i^2} - \sum_{i=1}^{N} \frac{\widehat{\Delta x_i}\widehat{\Delta y_i}}{\sigma_\rho^2 \hat{\rho}_i^2} - \sum_{i=1}^{N} \frac{\widehat{\Delta x_i}\widehat{\Delta y_i}}{\sigma_\theta^2 \hat{\rho}_i^4}$$

$$\gamma = \sum_{i=1}^{N} \frac{\widehat{\Delta y_i^2}}{\sigma_\rho^2 \hat{\rho}_i^2} + \sum_{i=1}^{N} \frac{\widehat{\Delta x_i^2}}{\sigma_\theta^2 \hat{\rho}_i^4} - \left(\sum_{i=1}^{N} \frac{\widehat{\Delta x_i}}{\sigma_\eta \hat{\rho}_i^2} \right)^2 + \sqrt{\frac{Nq}{r}}$$

At this point, we note that $\mathbf{P}_{ss} \preceq \bar{\mathbf{P}}_{ss} \Rightarrow E\{\mathbf{P}_{ss}\} \preceq E\{\bar{\mathbf{P}}_{ss}\}$, where the quantity $E\{\bar{\mathbf{P}}_{ss}\}$ depends on the mean value of matrix M. Using the previous closed-form expression for M, the computation of $E\{M\}$ for a given pdf of the landmarks' positions can be trivially performed through Monte Carlo simulations. We now state the following lemma:

Lemma 4.2: The maximum expected steady state covariance of the position estimates in SLAM, when the spatial density of landmarks is described by a known pdf, is given by

$$E\{\mathbf{P}_{ss}\} \preceq \begin{bmatrix} \sqrt{\frac{qr}{N}} I_{2\times 2} & \mathbf{0}_{2\times 2N} \\ \mathbf{0}_{2N\times 2} & \mathbf{0}_{2N\times 2N} \end{bmatrix} + \mathbf{1}_{(N+1)\times(N+1)} \otimes \bar{M}$$

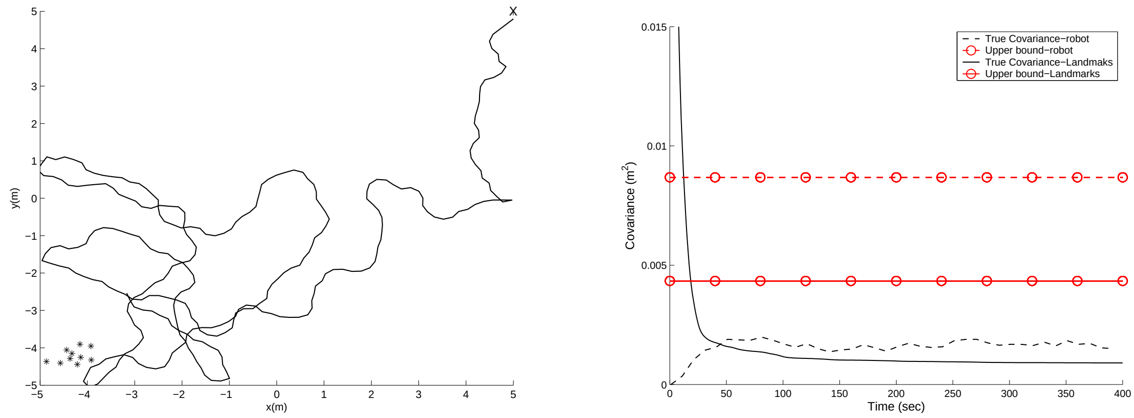
where $\bar{M} = E\{M\}$ can be computed using Eq. (29).

The simulation results presented in the next section demonstrate that the availability of additional information (i.e., known spatial density of landmarks), results in a substantially tighter bound.

V. Simulation Results

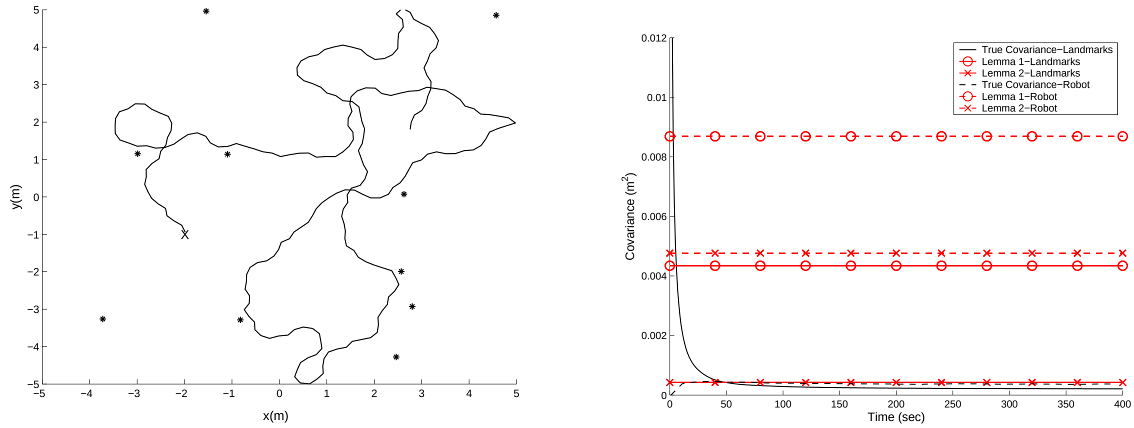
A series of experiments in simulation were conducted for validating the preceding theoretical analysis. The simulated robot moves in an arena of dimensions 10×10m, in which point landmarks are located. The velocity of the robot is kept constant, at $V = 0.3$m/s, while its orientation changes randomly, using samples drawn from a uniform distribution. In order to account for practical considerations, we impose a minimum distance constraint for the range between the robot and the landmarks, so that the robot does not move closer than 20cm to any of the landmarks. The standard deviation of the velocity measurement noise is $\sigma_V = 0.05V$, while the standard deviation of the errors in the orientation estimates is $\sigma_\phi = 2^{\text{o}}$. The values selected for the standard deviations of the exteroceptive (range and bearing) measurements of the robot are $\sigma_\theta = 2^{\text{o}}$ and $\sigma_\rho = 0.05$m, respectively.

In Fig 1(b), the theoretical upper bound (red lines with circles) for the steady state covariance of the robot (dashed line) and the landmarks (solid line) is compared against the "true" covariance of the position estimates, as this is computed by the EKF. This figure corresponds to a scenario in which the robot starts its motion at one corner of the arena, while all the landmarks form a cluster at the opposite corner (cf. Fig 1(a)). In this simulation, the map comprises of $N = 10$ landmarks. It is clear that the worst-case performance bound for the steady state covariance, computed by Eq. (28) in Lemma 4.1, is indeed larger than the covariance of the position estimates. At this point, we should note that the scenario depicted in Fig. 1(a) is a



(a) (b)

Fig. 1. (a) The trajectory of the robot and landmark locations for an adverse scenario. Landmark positions are denoted by asterisks and the initial position of the robot is marked with an X. (b) True value and theoretical upper bound for the covariance of the robot and landmarks for the scenario depicted in (a). The average covariance along the two coordinate axes is plotted for the robot, while for the landmarks averaging is performed along both axes and over all landmarks.



(a) (b)

Fig. 2. (a) A typical example of the arena with uniformly distributed landmarks. Landmark positions are denoted by asterisks and the initial position of the robot is marked with an X. (b) Average true covariance vs. theoretical bounds (from Lemmas 4.1 and 4.2) for SLAM with 10 uniformly distributed landmarks.

particularly adverse one, since, due to the placement of the landmarks, the exteroceptive measurements provide only a small amount of positioning information during the crucial first few updates.

Although the bound of Lemma 4.1 accounts for the worst-case of SLAM accuracy, it does not yield a sufficient performance description for cases in which the map features are more evenly distributed in space. In order to demonstrate this, in Fig. 2(b) the average value of the robot's and landmarks' covariance over 20 runs of SLAM is plotted (black lines), and compared against the bounds computed using Lemma 4.1 (lines with circles) and Lemma 4.2 (lines with asterisks). In generating this plot, the locations of the 10 landmarks were selected using samples from a uniform distribution for each run of the algorithm (cf. Fig. 2(a)). We observe that the worst-case performance bound of Eq. (28) is a quite loose one. When instead, the available information about the distribution of the landmarks is exploited, i.e., by employing the expressions from Lemma 4.2, a better characterization of the expected accuracy of the position estimates is achieved.

No. of Landmarks	1	5	10	20	50
Worst Case	7.3	17.78	23.59	35.83	66.58
Known Distribution	3.62	3.07	2.34	1.99	1.56

TABLE I

RATIO BETWEEN THE AVERAGE STEADY STATE LANDMARK COVARIANCE AND THE CORRESPONDING UPPER BOUNDS.

In Table I, the ratio of the presented bounds for the landmarks' covariance, compared to the average landmark covariance for various map sizes is shown. Each entry represents the sample mean of the corresponding ratio, computed from 20 simulation experiments in which the landmark positions are uniformly-distributed random variables. These numerical values reveal that the gain from utilizing information about the landmarks' spatial distribution becomes more significant for large maps.

VI. Conclusions

In this paper we have presented a method for predicting the positioning performance in SLAM, without the need to resort to extensive simulations or experimentation. This was achieved through a theoretical study of the time evolution of the position estimates' covariance, that allowed for the derivation of an *analytical upper bound* for the positioning uncertainty. The closed-form expression of Lemma 4.1 establishes a functional relation between the noise parameters of the robot's sensors, the number of features being mapped, and the accuracy of SLAM. Moreover, in Lemma 4.2 it is shown that when prior information, in the form of a model for the density of landmarks in the area, is available, we can determine a tighter upper bound for the *expected* value of the steady state covariance of the errors for both the robot and the map features. Thus, a powerful design tool is made available that enables the prediction of the performance of a robot in a mapping application. This can be employed to determine the required accuracy of the robot's sensors, in order to meet task-dependent specifications. Certainly, the most restrictive assumption employed in the current work is that the robot can see all landmarks simultaneously. Although this is not possible in most real-world applications, the presented analysis can serve as a basis for extensions to more realistic scenarios, where only subsets of the map are visible at each time instant.

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