Simple quadrature-based quantum feedback of a solid-state qubit

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ABSTRACT

We analyze the newly proposed quantum feedback loop for a solid-state qubit, based on monitoring the quadrature components of the current from a weakly coupled detector, which continuously measures the qubit. Similar to the earlier proposal of the "Bayesian" feedback, the feedback loop is used to maintain the coherent (Rabi) oscillations in a qubit for an arbitrarily long time; however, this is done in a significantly simpler way, which requires much smaller bandwidth of the control circuitry. The price for simplicity is a less-than-ideal operation: the fidelity is limited to about 95%. The feedback loop operation can be experimentally verified by appearance of a positive in-phase component of the detector current relative to an external oscillating signal used for synchronization. The quadrature-based quantum feedback seems to be within the reach of the present-day technology.

Keywords: quantum measurement, qubit, quantum noise, quantum feedback

1. INTRODUCTION

There is a rapid progress in experiments with solid-state qubits[1] (controllable two-level systems) during last years. In particular, quantum coherent (Rabi) oscillations have been demonstrated using superconducting "charge",[2] "flux",[3] and "phase" qubits[4] as well as double-quantum-dot qubits.[5] Successful experiments with two superconducting qubits have also been performed.[6] Even though at present only very basic operations with qubits are experimentally accessible, more advanced experiments are a natural next stage. One of the directions for the advanced qubit control is realization of the quantum feedback control of a solid-state qubit,[7] which can be used in a quantum computer for qubit initialization and is also an important demonstration by itself, clarifying the controversial issue of gradual collapse of a quantum state. (In optics quantum feedback control was proposed more than a decade ago[8] and has been already demonstrated experimentally.[9])

For the analysis of a quantum feedback we have to take into account the process of continuous qubit collapse. Therefore, the conventional approach to continuous quantum measurement[10, 11] is inapplicable, and it is necessary to use the recently developed Bayesian approach[12] or the equivalent (though much different technically) approach of quantum trajectories.[13] The possibility of a quantum feedback is based on the fact that measurement by an ideal solid-state detector (with 100% quantum efficiency η) does not decohere a single qubit,[12] even though it decoheres an ensemble of qubits because each qubit evolves in a different way. The random evolution of a qubit in the process of measurement can be monitored using the noisy detector output, with accuracy depending on η, so that for an ideal detector ($\eta = 1$) even the monitoring of qubit wavefunction is possible. An example of theoretically ideal solid state detector is[12, 14] the quantum point contact (η comparable to 1 has been demonstrated experimentally[15]). The single-electron transistor is significantly nonideal[11, 12, 16] ($\eta \ll 1$) in the semiclassical "orthodox" mode of operation;[17] however, it can reach ideality in some modes based on cotunneling or Cooper pair tunneling.[18]

Monitoring of the quantum state in real time can naturally be used for continuous feedback control of a quantum system. In the setup proposed in Ref. 7 and reviewed here in Section 3 (see Fig. 1 below) the quantum feedback is used to maintain quantum coherent (Rabi[19]) oscillations in a qubit for an arbitrarily long time, synchronizing them with an external classical signal. This is done by measuring the noisy current $I(t)$ in a weakly coupled detector and using the quantum Bayesian equations[12] to translate information contained in $I(t)$ into the evolution of qubit density matrix $\rho(t)$. After that $\rho(t)$ is compared with the desired quantum state $\rho_d(t)$, and the calculated difference is used to control the qubit Hamiltonian in order to decrease the difference. Notice that the measurement back-action necessarily shifts the phase of Rabi oscillations in a random way (adding it to

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dephasing due to environment); however, the information contained in $I(t)$ is sufficient to monitor this change and therefore restore the desired phase.

An important difficulty in such experiment is a necessity to solve the Bayesian equations in real time. Moreover, the bandwidth of the line delivering $I(t)$ to the circuit solving the Bayesian equations (which would most likely be located off-chip), should be significantly wider than the Rabi frequency Ω (otherwise the information contained in the noise is lost). Unfortunately, these conditions are too difficult for the present-day experiments with solid-state qubits.

In this paper we also discuss a much simpler way[20] of processing the information carried by the detector current $I(t)$ (see Fig. 2 below). The idea is to use the fact that besides noise, $I(t)$ contains an oscillating contribution due to Rabi oscillations in the measured qubit. Therefore, if we apply $I(t)$ to a simple tank circuit (which is in resonance with Ω), then the phase of the tank circuit oscillations will depend on the phase of Rabi oscillations. Instead of using the tank circuit, a theoretically almost equivalent procedure is to mix $I(t)$ with the signal from a local oscillator (Fig. 2) in order to determine two quadrature amplitudes of $I(t)$ at frequency Ω, which will carry information on the phase of Rabi oscillations. Since diffusion of the Rabi phase is a slow process (assuming weak coupling to the detector and environment), the further circuitry can be relatively slow, limited by the qubit dephasing rate, but not limited by much higher Rabi frequency. The simplicity of the information processing and relatively small required bandwidth are the main advantages of this proposal in comparison with Ref. 7. (The bandwidth of the line between the detector and mixer should still be much larger than Ω; however, it is not a problem for the on-chip mixer.) The experiment can be realized using either superconductor[2–4, 6, 21] or semiconductor[5, 15] technology.

The idea of this proposal partially stems from the fact that in absence of feedback the qubit Rabi oscillations lead to a noticeable peak in the spectral density $S_I(\omega)$ of the detector current at $\omega \approx \Omega$, with the peak-to-pedestal ratio up to 4 times[22, 23] (somewhat similar experiments have been reported recently[24]). Since 4 is not a big number, one would expect quite inaccurate phase information carried by current quadratures and therefore poor operation of the feedback. Surprisingly, the quantum feedback operates much better than it would be expected from classical analysis.

In the next Section we discuss the model, Section 3 is a review of the Bayesian quantum feedback with some more mathematical details than in the original paper,[7] Section 4 discusses the much simpler quadrature-based feedback, and Section 5 is the conclusion.

2. MODEL

Let us consider a "charge" qubit (either double quantum dot or single Cooper pair box) with Hamiltonian

$$\mathcal{H}_{qb} = \frac{\varepsilon}{2}\,(c_2^\dagger c_2 - c_1^\dagger c_1) + H(c_1^\dagger c_2 + c_2^\dagger c_1), \tag{1}$$

where $c_{1,2}^\dagger$ and $c_{1,2}$ are the creation and annihilation operators in the basis of "localized" (charge) states, ε is their energy asymmetry, and the tunneling

$$H = H_0 + H_{fb}(t) \tag{2}$$

can be controlled by the feedback loop (H_{fb}). We assume the standard coupling[12, 22, 23, 25] between the charge qubit and the detector (quantum point contact or single-electron transistor). Instead of writing Hamiltonians explicitly, we will characterize the measurement by two levels of the average detector current, I_1 and I_2, corresponding to the two charge states, by the detector output noise S_I, and by the total ensemble-averaged qubit dephasing rate Γ due to detector back-action and environment. Assuming sufficiently large detector voltage and quasicontinuous detector current $I(t)$, we describe the qubit evolution by the Bayesian equations[12] (in Stratonovich form[26])

$$\frac{d}{dt}\,\rho_{11} = -2H\,\mathrm{Im}\,\rho_{12} + 2\rho_{11}\rho_{22}\,\frac{\Delta I}{S_I}\,[I(t) - I_0]\,, \tag{3}$$

$$\frac{d}{dt}\,\rho_{12} = i\varepsilon\rho_{12} + iH(\rho_{11} - \rho_{22}) - \gamma\rho_{12} - (\rho_{11} - \rho_{22})\rho_{12}\,\frac{\Delta I}{S_I}\,[I(t) - I_0]\,, \tag{4}$$

where $\hbar = 1$, $\Delta I = I_1 - I_2$, $I_0 = (I_1 + I_2)/2$, and $\gamma = \Gamma - (\Delta I)^2/4S_I$. The qubit decoherence rate $\gamma = \gamma_d + \gamma_e$ is due to detector nonideality, $\gamma_d = (\eta^{-1} - 1)(\Delta I)^2/S_I$, and due to additional coupling with environment (γ_e). The current

$$I(t) = I_0 + \frac{\Delta I}{2}(\rho_{11} - \rho_{22}) + \xi(t) \tag{5}$$

contains the noise component $\xi(t)$ with the flat (white) spectral density S_I. Notice that in the case $\varepsilon = 0$ (which is assumed unless mentioned otherwise), we can disregard the evolution of $\mathrm{Re}\rho_{12}$ (it becomes zero at $t \gg \Gamma^{-1}$), so only two degrees of freedom are left, which may be parameterized as

$$\rho_{11} - \rho_{22} = P\cos(\Omega t + \phi), \qquad 2\mathrm{Im}\rho_{12} = P\sin(\Omega t + \phi), \tag{6}$$

where the feedback-maintained frequency Ω is assumed to be equal (unless stated otherwise) to the bare Rabi frequency $\Omega_0 = (4H_0^2 + \varepsilon^2)^{1/2} = 2H_0$. Moreover, in the ideal case $\gamma = 0$ the state eventually becomes pure,[12] so that $P = 1$ and the evolution can be described by only one parameter $\phi(t)$, which characterizes the phase deviation from the desired perfect oscillations (see below).

We characterize coupling between qubit and detector by the dimensionless constant

$$\mathcal{C} = \frac{(\Delta I)^2}{S_I H_0} \tag{7}$$

(assuming $H > 0$) and concentrate on the case of weak coupling $\mathcal{C} < 1$ (notice that $\mathcal{C} = 1$ can still be considered a weak coupling since the quality factor of oscillations in presence of measurement[22] is $8\eta/\mathcal{C}$ for $\varepsilon = 0$).

The goal of the analyzed quantum feedback is to maintain the qubit evolution as close as possible to the desired evolution

$$\rho_{d,11} = \frac{1 + \cos\Omega t}{2}, \qquad \rho_{d,12} = i\,\frac{\sin\Omega t}{2}, \tag{8}$$

which corresponds to unperturbed Rabi oscillations running for arbitrarily long time. This is done by controlling the tunneling strength H [Eq. (2)]. In principle, there are many possible algorithms ("controllers") of such control, which obviously differ in performance. From the mathematical point of view, two feedback setups discussed in Sections 3 and 4 are two examples of such controllers.

We will characterize the feedback efficiency by the "synchronization degree" D defined as averaged over time scalar product of two vectors on the Bloch sphere corresponding to the desired and actual states of the qubit. Equivalent definitions are

$$D = 2\langle \mathrm{Tr}\rho\rho_d \rangle - 1 = \langle P(t)\cos\phi(t) \rangle, \tag{9}$$

where $\langle .. \rangle$ denotes averaging over time. Perfect feedback operation corresponds to $D = 1$ (notice that ρ_d is a pure state). Feedback efficiency D can be easily translated into fidelity[27] $\mathcal{F} = \langle \rho(t)\rho_d(t) \rangle$ as $\mathcal{F} = (D+1)/2$. We prefer to use D instead of fidelity because $D = 0$ in absence of feedback (when ρ and ρ_d are completely uncorrelated), while the fidelity $\mathcal{F}$ is non-zero.

3. BAYESIAN FEEDBACK

Theoretically, the most natural way to design the qubit feedback[7] is using a "processor" which solves the quantum Bayesian equations (3)–(4) in real time (Fig. 1). If we neglect the effect of finite bandwidth[7,28] of the line carrying the detector signal, and also neglect the signal delay in the feedback loop, then the qubit density matrix $\rho(t)$ is monitored exactly. Hence, the actual qubit evolution can be compared with the desired evolution (8), and the difference signal can be used to control the qubit parameter H in order to decrease the difference.

The simplest feedback controller proposed and analyzed in Ref. 7 is the linear rule:

$$H_{fb} = -FH_0\phi, \tag{10}$$

$$\phi = \arctan\left[\frac{2\,\mathrm{Im}\rho_{12}}{\rho_{11} - \rho_{22}}\right] + \frac{\pi}{2}\left[1 - \mathrm{sgn}(\rho_{11} - \rho_{22})\right] - \Omega t \pmod{2\pi}, \tag{11}$$

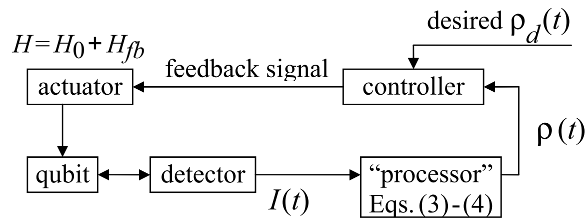

Figure 1. Schematic of the Bayesian quantum feedback loop which maintains quantum coherent oscillations in a qubit for arbitrarily long time. Real-time solution of quantum Bayesian equations by the "processor" provides exact monitoring of the qubit density matrix ρ. Qubit tunneling amplitude H is controlled depending on the difference between $\rho(t)$ and the desired qubit evolution $\rho_d(t)$.

where the phase difference ϕ between the actual and desired evolutions is defined as $|\phi| \leq \pi$, and F is a dimensionless feedback factor. The controller (10) is supposed to decrease the phase difference $|\phi|$ (negative feedback): if $\phi(t)$ is positive (qubit oscillation phase is ahead of the desired value), then H_{fb} is negative, that slows down the qubit oscillations and decreases the phase shift; if $\phi(t)$ is negative (qubit phase is behind the desired value), the oscillation frequency increases to catch up.

Here we analyze the basic ideal case of $\eta = 1$ (quantum-limited detector, e.g. QPC), absence of extra environment ($\gamma_e = 0$), and symmetric qubit ($\varepsilon = 0$). The analytical results for this case have been presented in Ref. 7; here we discuss the derivation in more detail.

Since $\eta = 1$ and $\gamma_e = 0$, so that there is no dephasing term in Eq. (4), the qubit density matrix ρ becomes pure in the process of measurement.[12] Because of the energy symmetry, $\varepsilon = 0$, the real part of ρ_{12} eventually becomes zero. This happens because the product $(\rho_{11} - \rho_{22})(I - I_0)$ affecting the evolution of $\mathrm{Re}\rho_{12}$ in Eq. (4) is on average positive. Therefore, after a transient period the evolution of the density matrix ρ can be parameterized as $\rho_{11} = [1 + \cos(\Omega t + \phi)]/2$, $\rho_{12} = i[\sin(\Omega t + \phi)]/2$ with only one parameter $\phi(t)$.

The evolution equation for phase ϕ can be easily derived from Eq. (4) as

$$\frac{d}{dt}\,\phi = -\frac{\Delta I}{S_I}\,[I(t) - I_0]\sin(\Omega t + \phi) + 2H_{fb} + \Omega_0 - \Omega, \tag{12}$$

so in the case $\Omega = \Omega_0$ the phase difference ϕ evolves as

$$\frac{d}{dt}\,\phi = -\sin(\Omega_0 t + \phi)\,\frac{\Delta I}{S_I}\left[\frac{\Delta I}{2}\,\cos(\Omega_0 + \phi) + \xi\right] - F\Omega_0\,\phi. \tag{13}$$

(All equations are in the Stratonovich form, so we use the usual rules for derivatives.[26]) Notice that because of our definition $|\phi| \leq \pi$, the phase difference jumps by $\pm 2\pi$ at the borders of $\pm\pi$ interval.

For weak coupling ($\mathcal{C}/8 \ll 1$) the qubit oscillations are only slightly perturbed by measurement and corresponding phase diffusion is relatively slow. Assuming that the feedback control is also slow on the timescale of oscillations ($|H_{fb}| \ll H_0$), we can average Eq. (13) over relatively fast oscillations. Then the first term in the brackets is averaged to zero and averaging of the term $-\sin(\Omega_0 t + \phi)(\Delta I/S_I)\xi(t)$ leads to the effective noise $\tilde{\xi}(t)$ with spectral density $S_{\tilde{\xi}} = (\Delta I)^2/2S_I$, so that the remaining slow evolution of phase difference is

$$\frac{d}{dt}\,\phi = \tilde{\xi} - F\Omega_0\,\phi. \tag{14}$$

To find the feedback efficiency $D = \langle\cos\phi(t)\rangle$ analytically, let us also assume that feedback performance is good enough to keep the phase difference ϕ well inside the $\pm\pi$ interval, so that the phase slips (jumps of ϕ by $\pm 2\pi$) occur sufficiently rare. In this case we can consider Eq. (14) on the infinite interval of ϕ. The corresponding Fokker-Planck-Kolmogorov equation for the probability density $\sigma(\phi)$

$$\frac{d\sigma}{dt} = \frac{d}{d\phi}\,(\sigma\,F\Omega_0\,\phi) + \frac{1}{4}\,\frac{d^2(S_{\tilde{\xi}}\sigma)}{d\phi^2} \tag{15}$$

has the Gaussian stationary solution $\sigma_{st}(\phi) = (2\pi\mathcal{V})^{-1/2}\exp[-\phi^2/2\mathcal{V}]$ with variance $\mathcal{V} = S_{\tilde{\xi}}/4F\Omega_0 = \mathcal{C}/16F$. Therefore, $\langle\cos\phi\rangle = \exp(-\mathcal{V}/2)$, and so the feedback efficiency is[7]

$$D = \exp\left(-\frac{\mathcal{C}}{32F}\right) \tag{16}$$

in the case of weak coupling and sufficiently efficient feedback (crudely, this means $\mathcal{C} < 1$ and $D > 1/2$).

Notice that $|H_{fb}|/H_0 < \pi F$, and F scales with coupling $\mathcal{C}$. Therefore, in the experimentally realistic case $\mathcal{C} \ll 1$, a typical amount of the parameter H change due to feedback is small, $|H_{fb}| \ll H_0$. Hence, we should not worry about unnatural assumption of using control equation (10) even when H becomes negative.

Besides analyzing feedback efficiency D, let us also calculate the qubit correlation function $K_z(\tau) = \langle z(t+\tau)\, z(t)\rangle$ where $z = \rho_{11} - \rho_{22}$. In the case of practically harmonic (weakly disturbed) oscillations, the correlation function $K_z(\tau) = \langle\cos[\Omega_0 t + \phi(t+\tau)]\cos[\Omega_0 t + \phi(t)]\rangle$ is equal to $\langle\cos[\Omega_0\tau + \delta\phi(\tau)]\rangle/2$ where $\delta\phi(\tau) = \phi(t+\tau) - \phi(t)$ is the phase deviation during time τ. Since in our case $\langle\sin\delta\phi(\tau)\rangle = 0$ because of the symmetry of Eq. (14), the correlation function is reduced to

$$K_z(\tau) = \frac{\cos\Omega_0\tau}{2}\,\langle\cos\delta\phi(\tau)\rangle. \tag{17}$$

We can find $\langle\cos\delta\phi(\tau)\rangle$ using exact solution of the Fokker-Planck-Kolmogorov equation (15) with a localized (δ-function) initial condition $\sigma(\phi, 0) = \delta(\phi - \phi_0)$:

$$\sigma(\phi, \tau|\phi_0) = \frac{\exp[-(\phi - \phi_0 e^{-F\Omega_0\tau})^2/2\mathcal{V}(\tau)]}{\sqrt{2\pi\mathcal{V}(\tau)}}, \tag{18}$$

$$\mathcal{V}(\tau) = \frac{S_{\tilde{\xi}}}{4F\Omega_0}\left(1 - e^{-2F\Omega_0\tau}\right). \tag{19}$$

Calculating $\langle\cos\delta\phi(\tau)\rangle$ as $\int_{-\infty}^{\infty}\cos(\phi - \phi_0)\sigma(\phi, \tau|\phi_0)\,\sigma_{st}(\phi_0)\,d\phi_0$, we finally find the qubit correlation function

$$K_z(\tau) = \frac{\cos\Omega_0\tau}{2}\,\exp\left[\frac{\mathcal{C}}{16F}\left(e^{-F\Omega_0\tau} - 1\right)\right]. \tag{20}$$

The validity range of this result is the same as for Eq. (16) (crudely, $\mathcal{C} < 1$ and $16F/\mathcal{C} > 1$). Fourier transform $S_z(\omega) = 2\int_{-\infty}^{\infty} K_z(\tau)e^{i\omega\tau}d\tau$ of Eq. (20) in the case of efficient feedback ($\mathcal{C}/16F < 1$, so the exponent is expanded up to the linear term) gives the oscillation spectrum ($\omega > 0$)

$$S_z(\omega) = \frac{1}{2}\left(1 - \frac{\mathcal{C}}{16F}\right)\delta\left(\frac{\omega - \Omega_0}{2\pi}\right) + \frac{\mathcal{C}}{8\Omega_0}\,\frac{1 + F^2 + (\omega/\Omega_0)^2}{[1 + F^2 - (\omega/\Omega_0)^2]^2 + 4F^2(\omega/\Omega_0)^2}, \tag{21}$$

in which the first term (δ-function) corresponds to synchronized non-decaying oscillations, while the second term describes fluctuations and for $F \ll 1$ is peak-like near $\omega \approx \Omega_0$ with the peak height of $\mathcal{C}/16\Omega_0 F^2$ and half-width at half-height of $F\Omega_0$. [It is easy to check that $\int_0^\infty S_z(\omega)\,d\omega/2\pi = 1/2$.]

For completeness let us also calculate the correlation function of the detector current $K_I(\tau) = \langle[I(t+\tau) - I_0][I(t) - I_0]\rangle$. Following Ref. 22, we use Eq. (5) to get $K_I(\tau) = (\Delta I/2)^2 K_z(\tau) + K_\xi(\tau) + (\Delta I/2)K_{z\xi}(\tau)$, where $K_\xi = (S_I/2)\,\delta(\tau)$ is due to pure noise while the cross-correlation term $K_{z\xi}(\tau)$ is due to quantum back-action, which shifts the phase ϕ by $-\sin(\Omega_0 t + \phi)(\Delta I/S_I)\,\xi(t)\,dt$ as a result of noise ξ acting during infinitesimal time dt [see Eq. (13)]. Because of feedback, the effect of the phase shift decreases with time as $\tilde{\delta}\phi(\tau) = -\exp(-F\Omega_0\tau)\sin[\Omega_0 t + \phi(t)](\Delta I/S_I)\,\xi(t)\,dt$ [see Eq. (18)] and the cross-correlation at $\tau > 0$ can be calculated as $K_{z\xi}(\tau) = \langle z(t+\tau)\,\xi(t)\rangle = \langle\cos[\Omega_0 t + \phi(t) + \Omega_0\tau + \delta\phi(\tau) + \tilde{\delta}\phi(\tau)]\,\xi(t)\rangle$. Expanding cosine up to the linear term in $\tilde{\delta}\phi(\tau)$, we obtain $K_{z\xi}(\tau) = \langle\xi^2(t)\;dt\rangle(\Delta I/S_I)\exp(-F\Omega_0\tau)\langle\sin[\Omega_0 t + \phi(t) + \Omega_0\tau + \delta\phi(\tau)]\;\sin[\Omega_0 t + \phi(t)]\rangle$, where $\langle\xi^2(t)\,dt\rangle = S_I/2$. Using symmetry of fluctuations leading to $\langle\sin\delta\phi(\tau)\rangle = 0$ (as above) and averaging over fast oscillations $\langle\sin[\Omega_0 t + \phi(t) + \Omega_0\tau]\sin[\Omega_0 t + \phi(t)]\rangle = (\cos\Omega_0\tau)/2$, we finally obtain

$$K_{z\xi}(\tau) = \frac{\Delta I}{4}(\cos\Omega_0\tau)\,e^{-F\Omega_0\tau}\langle\cos\delta\phi(\tau)\rangle. \tag{22}$$

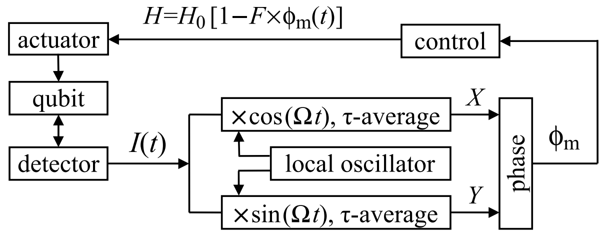

Figure 2. Schematic of the quadrature-based quantum feedback loop. Two quadrature components of the detector current $I(t)$ are used to monitor approximately the phase difference ϕ between Rabi oscillations and a local oscillator, which is used to control the qubit parameter H. (The phase ϕ can also be monitored using a tank circuit.) The difference between actual phase ϕ and its monitored approximate value ϕ_m precludes 100% efficiency of the feedback. Positive average in-phase quadrature $\langle X\rangle$ is an experimental indication of quantum feedback operation.

Since expression for $K_z(\tau)$ has a similar structure [see Eq. (17)], the corresponding terms of $K_I(\tau)$ are combined to yield

$$K_I(\tau) = \frac{S_I}{2}\,\delta(\tau) + \frac{(\Delta I)^2}{4}\left(1+e^{-F\Omega_0\tau}\right) K_z(\tau), \tag{23}$$

where $K_z(\tau)$ is given by Eq. (20).

To calculate the spectral density $S_I(\omega)$ of the detector current, we again expand the outer exponent of Eq. (20) up to the linear term [validity of Eq. (23) requires $16F/\mathcal{C} > 1$]; then the Fourier transform gives

$$S_I(\omega) = S_I + \frac{(\Delta I)^2}{8}\left(1-\frac{\mathcal{C}}{16F}\right)\delta\left(\frac{\omega-\Omega_0}{2\pi}\right) + \frac{S_I}{4}\frac{\mathcal{C}}{F}\frac{F^2[1+F^2+(\omega/\Omega_0)^2]}{[1+F^2-(\omega/\Omega_0)^2]^2+4F^2(\omega/\Omega_0)^2} + \mathrm{T4}, \tag{24}$$

where the last (fourth) term T4 is the same as the previous (third) term but with F replaced by $2F$ and with extra factor $\mathcal{C}/16F$ [actually, higher-order terms of the exponent expansion will lead to extra terms with F replaced by $3F$, $4F$, etc., and will slightly change the coefficients of the existing terms]. Notice that the δ-function in the second term of Eq. (24) is due to synchronized nondecaying oscillations, while the third term at $F \ll 1$ describes a peak with height $(S_I/8)(\mathcal{C}/F)$ and half-width $F\Omega_0$ near $\omega \approx \Omega_0$.

It is easy to check that the integral over all terms in Eq. (24) except pure noise S_I, gives the total variance of the detector current equal to $(\Delta I)^2/4$ [this also follows directly from Eq. (23)]. Similar to the non-feedback case[22,23] this fact resembles the situation of the qubit jumping between the two localized states, instead of oscillating continuously [which would give twice smaller variance $(\Delta I)^2/8$]; in the Bayesian formalism this fact is understood as a consequence of non-classical cross-correlation between output noise and qubit evolution.

Concluding this Section, let us emphasize that in the ideal case the sufficiently strong feedback ($16F/\mathcal{C} \gg 1$) forces the qubit evolution to be arbitrarily close to the perfect coherent oscillations running for arbitrarily long time. In this case the feedback efficiency D approaches 100%, qubit correlation function becomes $K_z(\tau) = (\cos\Omega_0\tau)/2$, and the current spectral density contains (besides the pure noise) the δ-function peak at desired frequency Ω_0 with variance $(\Delta I)^2/8$, and also the narrow peak around Ω_0 (if $\mathcal{C}/16 \ll F \ll 1$) corresponding to same variance $(\Delta I)^2/8$.

4. QUADRATURE-BASED FEEDBACK

Now let us analyze the operation of a much simpler feedback (Fig. 2) which does not require solving the quantum Bayesian equations in real time; instead, some information on the phase of qubit oscillations is extracted from the quadrature components of the detector current. Since the phase ϕ in this case cannot be monitored exactly, we expect that the feedback efficiency should be worse than for the Bayesian feedback considered in the previous Section.

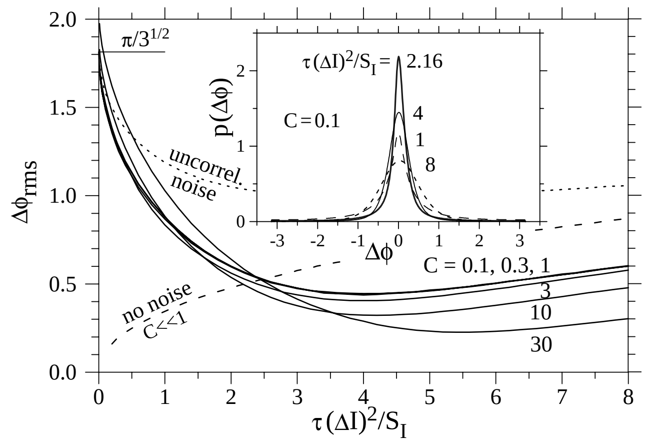

Figure 3. Dependence of monitoring inaccuracy $\Delta\phi_{rms}$ on averaging time τ without feedback for several values of coupling $\mathcal{C}$. Dashed and dotted lines correspond to classical signals. Inset: distribution of $\Delta\phi$ for several τ at weak coupling.

We assume that two quadrature components of the detector current are determined as

$$X(t) = \int_{-\infty}^{t} [I(t') - I_0] \cos(\Omega t') e^{-(t-t')/\tau} dt' \, , \tag{25}$$

$$Y(t) = \int_{-\infty}^{t} [I(t') - I_0] \sin(\Omega t') e^{-(t-t')/\tau} dt' \, , \tag{26}$$

where Ω is the local oscillator frequency applied to the mixer, and τ is the averaging time constant (notice the different meaning of the notation τ in the previous Section). Similar formulas are also applicable to the case of a tank circuit with the resonant frequency Ω and quality factor $Q = \Omega\tau/2$. If the detector current would be a harmonic signal $I(t) = I_0 + P(\Delta I/2)\cos(\Omega t + \phi_0)$, then $\phi_0 = -\arctan(\langle Y\rangle/\langle X\rangle)$, so it is natural to use

$$\phi_m(t) \equiv -\arctan(Y/X) \tag{27}$$

as a monitored estimate of the phase shift $\phi(t)$ [actually, $\pm\pi$ shift should be added in the case $X < 0$ similar to that in Eq. (11); we choose $|\phi_m| \leq \pi$].

Let us assume ideal case $\gamma = 0$ and analyze first how close is the estimate $\phi_m(t)$ to the actual phase $\phi(t)$ without feedback, in which case ϕ evolves in a diffusive manner due to detector back-action. Figure 3 shows the rms phase difference $\Delta\phi_{rms} = \langle(\phi_m - \phi)^2\rangle^{1/2}$ (solid lines) as a function of τ for several values of the dimensionless qubit-detector coupling $\mathcal{C}$, calculated numerically using Monte Carlo simulation of the measurement process.[12] At weak coupling, $\mathcal{C} \leq 1$, the curves practically coincide, and the minimum $\Delta\phi_{rms} \approx 0.44$ is achieved at $\tau \approx 4S_I/(\Delta I)^2 = 1/\Gamma$, as could be expected since Γ determines the phase diffusion:[12, 13, 23] $\langle[\phi(t) - \phi(0)]^2\rangle/t = \Gamma$. At larger τ, ϕ_m includes too much of irrelevant information from distant past, while at smaller τ the quadrature amplitudes suffer too much from the noise. At $\tau \to 0$ (as well as at $\tau \to \infty$) $\Delta\phi_{rms} \to \pi/\sqrt{3} \approx 1.81$ that corresponds to the uniform distribution of $\Delta\phi = \phi_m - \phi$ (complete absence of correlation between ϕ and ϕ_m) within $\pm\pi$ interval (all phases are defined modulo 2π).

It is important to notice that the calculated $\Delta\phi_{rms}$ is significantly smaller than for a naive classical case, in which the noise $\xi(t)$ is not correlated with diffusive evolution of ϕ. The dotted line in Fig. 3 shows the result for such a case at weak coupling, which has a minimum $\Delta\phi_{rms} \approx 1.0$ [actually, for this curve we even increased the signal, assuming $I(t) - I_0 = \sqrt{2}(\Delta I/2)\cos(\Omega t + \phi) + \xi(t)$, which corresponds to correct spectrum[22]]. Even more surprisingly, at $\tau > 2.5S_I/(\Delta I)^2$ the inaccuracy $\Delta\phi_{rms}$ in the quantum case is smaller than for the classical noiseless case, $\xi(t) = 0$ (dashed line), which means that the *noise improves the monitoring accuracy.* This

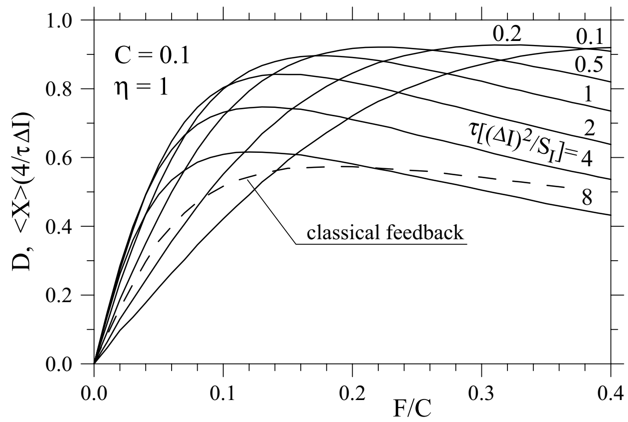

Figure 4. Dependence of the synchronization degree D on the feedback factor F in ideal case ($\gamma = 0$) for several τ. Experimentally D can be measured via average in-phase current quadrature $\langle X \rangle$. Dashed line is for a classical feedback.

quantum behavior can be understood by comparing the actual phase evolution equation (12) (without feedback) to the evolution of the monitored phase derived from Eqs. (25)–(27):

$$\frac{d}{dt}\,\phi_m = \frac{-[I(t)-I_0]\sin(\Omega t+\phi_m)}{\sqrt{X^2+Y^2}}\,. \tag{28}$$

The similarity between Eqs. (12) and (28) shows that the quadrature component of the noise which shifts (increases or decreases) the observed phase ϕ_m, also shifts the actual phase ϕ in the same direction. In other words, when the noise imitates oscillations, it forces the real Rabi oscillations to evolve closer to what is being observed.

Inset in Fig. 3 shows the distribution of $\Delta\phi$ in the weak coupling limit for several values of τ. The distributions are significantly non-Gaussian with the central part significantly narrower then $\Delta\phi_{rms}$. It is interesting that the value $\tau = 4S_I/(\Delta I)^2$ corresponding to the minimum $\Delta\phi_{rms}$, does not provide the highest peak of the $\Delta\phi$ distribution. To find the best τ in this respect, let us compare again Eqs. (12) and (28). We would naturally expect the best approximation of ϕ by ϕ_m when $\langle X^2+Y^2\rangle = (S_I/\Delta I)^2$. Using the definitions (25)–(26) and the current-current correlation function[22] $\langle I(0)I(t)\rangle = (S_I/2)\delta(t) + (\Delta I/2)^2\cos(\Omega t)\exp[-(\Delta I)^2 t/8S_I]$, we obtain $\langle X^2+Y^2\rangle = S_I\tau[1/4+1/(1+8S_I/(\Delta I)^2\tau)]$ at $\Omega\tau \gg 1$ and $\mathcal{C} \ll 1$, so the condition $\langle X^2+Y^2\rangle = (S_I/\Delta I)^2$ is satisfied at $\tau(\Delta I)^2/S_I = (2/5)(\sqrt{41}-1) \approx 2.16$. This indeed corresponds to the largest peak of $\Delta\phi$ distribution (see inset in Fig. 3).

Reasonably small difference between ϕ and ϕ_m in absence of feedback implies that we can expect decent operation of the quantum feedback loop in which the phase estimate ϕ_m is used for determining the feedback action. Let us consider the feedback rule similar to Eq. (10), in which the unknown actual phase ϕ is replaced by its approximately monitored value ϕ_m:

$$H_{fb}/H_0 = -F\phi_m(t), \tag{29}$$

where F is again the dimensionless feedback strength.

Figure 4 shows (solid lines) the dependence of D on the feedback factor F for several time constants τ in the case of weak coupling $\mathcal{C} = 0.1$ and $\gamma = 0$ (we normalize F by $\mathcal{C}$, so the results practically do not depend on $\mathcal{C}$ for $\mathcal{C} < 1$). One can see that each curve has a maximum, so that the "oversteering" effect at larger F makes the feedback performance worse (this is in contrast to the case of Bayesian feedback, in which larger F is always better). Somewhat unexpectedly, $\tau \sim 1/\Gamma = 4S_I/(\Delta I)^2$ is no longer an optimum, and smaller time constants are actually better. It can be shown that the feedback loop can operate even at $\tau \ll \Omega^{-1} \ll \Gamma^{-1}$; however, we are not interested in this regime because it requires a wide bandwidth of the control circuitry. Limiting ourselves to

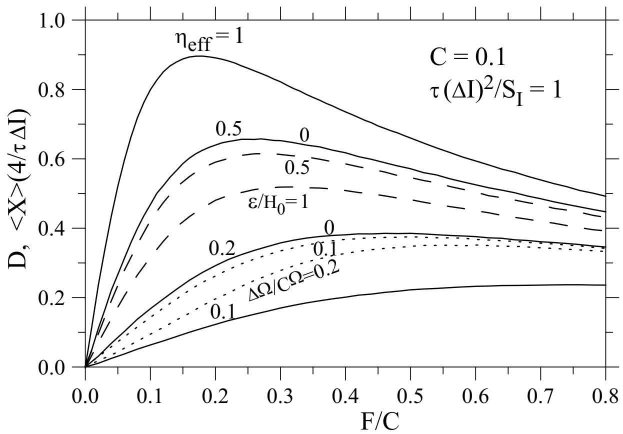

Figure 5. Solid lines: synchronization degree D (and in-phase current quadrature $\langle X\rangle$) as functions of F for several values of the detection efficiency η_{eff}. Dashed and dotted lines illustrate the effects of the energy mismatch ($\varepsilon \neq 0$) and the frequency mismatch ($\Omega \neq \Omega_0$).

$\tau \sim S_I/(\Delta I)^2$, we see that the maximum achievable synchronization degree D_{max} is about 90% (that corresponds to the fidelity $\mathcal{F}$ of about 95%). It is impossible to reach 100% because the monitored simple phase estimate ϕ_m is different from the actual ϕ. It is interesting to note that a very crude estimate of D_{max} as $\cos(\Delta\phi_{rms})$ using $\min(\Delta\phi_{rms}) \simeq 0.44$ from the analysis without feedback, works quite well, $\cos(0.44) = 0.90$ (though for different τ). Dashed line in Fig. 4 shows the feedback performance for a classical signal corresponding to the dotted line in Fig. 3, assuming $\tau(\Delta I)^2/S_I = 1$. As expected, it operates much worse than the quantum feedback because of the reason discussed above. [The crude estimate $D_{max} \sim \cos(\Delta\phi_{rms,\min}) = \cos(1.0) = 0.54$ still works well.]

An important question is how the operation of the quantum feedback loop can be verified experimentally. One of the easiest ways is to check that the average value $\langle X\rangle$ of the in-phase quadrature component $X(t)$ becomes positive, while in absence of feedback ($F = 0$) positive and negative values of X are obviously equally probable. Notice that *any* Hamiltonian control of a qubit which is not based on the information obtained from the detector (i.e. feedback control) cannot provide nonzero $\langle X\rangle$.[29] It is easy to show that $\langle X\rangle = [D + \langle P\cos(2\Omega_m+\phi)\rangle]\tau\Delta I/4$, and since the second term in brackets vanishes at weak coupling (and $\varepsilon = 0$), therefore $\langle X\rangle$ is directly related to D. The numerical results for $\langle X\rangle/(\tau\Delta I/4)$ practically coincide with the curves for D in Fig. 4 (within the thickness of the line).

The ideal case $\gamma = 0$ is obviously not realizable in an experiment because of finite nonideality of a detector ($\eta < 1$) and presence of an extra environment ($\gamma_e > 0$). Both effects can be taken into account simultaneously introducing effective efficiency of quantum detection $\eta_{eff} = [\eta^{-1} + \gamma_e S_I/(\Delta I)^2]^{-1}$. Figure 5 shows (solid lines) the feedback performance for several values of η_{eff} assuming $\tau(\Delta I)^2/S_I = 1$. One can see that $\eta_{eff} \sim 0.1$ is still a sufficient value for a noticeable operation of the quantum feedback loop. Notice that D_{max} is obviously limited by the state purity, $D_{max} < P$, which is[30] $P \approx \sqrt{2\eta_{eff}}$ at $\eta_{eff} \ll 1$ and $\mathcal{C}/\eta \ll 1$ ($D_{max} = P$ can be reached by the Bayesian feedback but not by the quadrature-based feedback).

Finally, let us discuss how accurately the conditions $\Omega = \Omega_0$ and $\varepsilon = 0$ should be satisfied in an experiment. If Ω is different from Ω_0, then without feedback the phase ϕ linearly grows in time [Eq. (12)]. However, if the feedback loop operation is faster than $|\Delta\Omega| = |\Omega - \Omega_0|$, the linear growth of ϕ is stopped by adjusting the Rabi frequency to match the desired frequency Ω. Dotted lines in Fig. 5 show the feedback operation for $\eta_{eff} = 0.2$ and two values of $\Delta\Omega$, confirming still good operation at $|\Delta\Omega| \ll \mathcal{C}\Omega \sim \Gamma \sim \tau^{-1}$. Notice that the frequency mismatch leads to nonzero $\langle\phi_m\rangle$ and therefore can be noticed and corrected. Energy mismatch ($\varepsilon \neq 0$) also worsens the performance of the feedback loop; however, the dashed lines in Fig. 5 ($\eta_{eff} = 0.5$) show that a relatively large mismatch ($\varepsilon < H_0$) can be tolerated.

5. CONCLUSION

In this paper we have discussed the operation of two quantum feedback loops, both designed to maintain unperturbed quantum coherent oscillations in a solid-state qubit for arbitrarily long time. The Bayesian feedback[7] discussed in Section 3 requires fast real-time solution of the quantum Bayesian equations, which depend on noisy current $I(t)$ from a weakly coupled detector. This provides exact monitoring of the qubit density matrix ρ, and as a result the efficiency (fidelity) of the Bayesian feedback can reach 100% in the ideal case. However, experimental realization of the Bayesian feedback at the present-day level of technology is extremely difficult and does not seem to be quite realistic.

An alternative[20] discussed in this paper is a much simpler quantum feedback loop, based on approximate monitoring of the qubit oscillations using two quadrature components of the detector current. Overall, this "simple" feedback loop is quite similar to a usual setup for a classical feedback. Instead of a "processor" solving the Bayesian equations, the quadrature-based feedback can use a mixer or a tank circuit, which can be relatively easily done on-chip, thus solving the difficult requirement of wide signal bandwidth. The required bandwidth for the signal after the mixer (or tank circuit) is limited from below by the qubit dephasing rate, but not by much faster frequency of qubit oscillations.

An anticipated price for simplicity is a decrease of the feedback efficiency compared to the Bayesian feedback. From the previous analysis[22,23] we know that in absence of feedback the spectral peak of the detector current at the Rabi frequency is at most only 4 times higher than the noise pedestal, which means that the quadrature components are very noisy. A classical analysis based on this fact would predict quite inefficient operation of the feedback loop. Surprisingly, the quantum analysis shows much better feedback efficiency. This can be understood as due to the fact that in quantum mechanics "reality follows the observations". The calculated fidelity of the quadrature-based quantum feedback can reach about 95% in the ideal case. Various nonidealities reduce the feedback efficiency; however, as Fig. 5 shows, the feedback efficiency is still pretty good even in presence of significant quantum inefficiency of the detector, extra dephasing due to environment, and deviation of the qubit parameters from the assumed values.

Successful operation of the quantum feedback can be verified experimentally in a straightforward way: the desired synchronization between the qubit oscillations and the local oscillator leads to a nonzero (positive) average value of the in-phase quadrature of the detector current, while in absence of feedback any quadrature is zero on average. Relative simplicity of the quadrature-based quantum feedback of a qubit allows us to hope that it can be realized experimentally within few years.

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The author thanks R. Ruskov for the major contribution to the results discussed in the Section 3 of this paper. The work was supported by NSA and ARDA under ARO grant W911NF-04-1-0204 and by DOD/DARPA/DMEA grant DMEA90-02-2-0216.

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29. Some procedures which include periodic dissipation phase (nonunitary operation) can be used to achieve nonzero $\langle X\rangle$, though much less efficiently than by the quantum feedback.
30. At finite η (assuming $\varepsilon = 0$ and $\Omega = \Omega_0$) the phase equation (12) becomes $\dot{\phi} = -[I(t) - I_0]\sin(\Omega t + \phi)(\Delta I/S_I)/P^2 - (\gamma/2)\sin(2\Omega t + 2\phi)$. At $\mathcal{C}/\eta \ll 1$ the last term can be neglected and state purity $P \equiv [(\rho_{11}-\rho_{22})^2+(2\mathrm{Im}\rho_{12})^2]^{1/2}$ can be approximated using an assumption (not quite accurate) of non-fluctuating P, that leads to $P^2 = 1 + 1/2\eta - \sqrt{(1 + 1/2\eta)^2 - 2}$. In particular, $P \approx \sqrt{2\eta}$ at $\eta \ll 1$.