Noisy quantum measurement of solid-state qubits

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ABSTRACT

The quantum evolution of an individual solid-state qubit during the process of its continuous measurement can be described by the recently developed Bayesian formalism. In contrast to the conventional ensemble-averaged formalism, it takes into account the measurement record (in a way similar to the standard Bayesian analysis) and therefore is able to consider individual realizations of the measurement process. The formalism provides testable experimental predictions and can be used for the analysis of a quantum feedback control of solid-state qubits. The Bayesian formalism can be also applied to the continuous measurement of entangled qubits; in particular, it shows how to create a fully entangled pair of qubits without their direct interaction, just by measuring them with an equally coupled detector.

Keywords: quantum measurement, qubit, quantum noise, quantum feedback, entanglement

1. INTRODUCTION

The approach of Bayesian analysis to the problem of continuous quantum measurement is a relatively new subject for the solid-state community, even though this approach has a long history[1] as a general quantum framework and is rather well developed, for example, in quantum optics[2,3] (for more references, see Ref.[4]) The main problem considered in this paper is a very simple question: *what is the evolution of a quantum two-level system (qubit) during the process of its measurement by a solid-state detector* (Fig. 1)? In spite of simplicity of the question, the answer is not trivial.

The textbook "orthodox" quantum mechanics[5] says that the measurement should instantly collapse the qubit state, so that after the measurement the qubit state is either $|1\rangle$ or $|2\rangle$, depending on the measurement outcome. [The measurement basis is obviously defined by the detector; in particular, it is a charge basis for the examples of Figs. 1(a) and 1(b).] Such answer is sufficient for typical optical experiments when the measurement is instantaneous (a scintillator flash or a photocounter click). However, for typical solid-state setups (as well as for some more advanced setups in quantum optics[2,3]) the instantaneous collapse is not a sufficient answer. In particular, in the examples of Fig. 1 typically the detector is weakly coupled to the qubit, so the measurement process can take a significant time and therefore the collapse should be considered as a continuous process. The notion of a continuous evolution due to measurement is well accepted in the solid-state community and is usually considered within the framework of the Leggett's formalism.[6] This formalism gives the decoherence-based answer to the question posed above. It says that the nondiagonal matrix elements of the qubit density matrix (obtained by tracing over the detector degrees of freedom) gradually decay to zero, while the diagonal matrix elements do not evolve (assuming that the qubit does not oscillate by itself, $H = 0$, where H describes the tunneling between $|1\rangle$ and $|2\rangle$). So, after the completed measurement we have an incoherent mixture of the states $|1\rangle$ and $|2\rangle$.

Let us notice that these two answers to our question obviously *contradict* each other and the "orthodox" answer cannot be obtained as some limiting case of the decoherence answer. The resolution of the apparent contradiction is simple: two approaches consider different objects. The decoherence approach describes the *average evolution of the ensemble* of qubits, while the "orthodox" quantum mechanics is designed to treat a *single quantum system.* This difference also explains the inability of the decoherence formalism to take the measurement outcome into account.

Obviously, it is desirable to have a formalism which would combine advantages of the two approaches and describe the *continuous measurement of a single qubit.* Then the "orthodox" result would be a limiting case for very fast (and "strong") measurement, while the decoherence result could be obtained by an ensemble averaging. The Bayesian formalism[4,7] which is the subject of this paper has been developed exactly for that purpose

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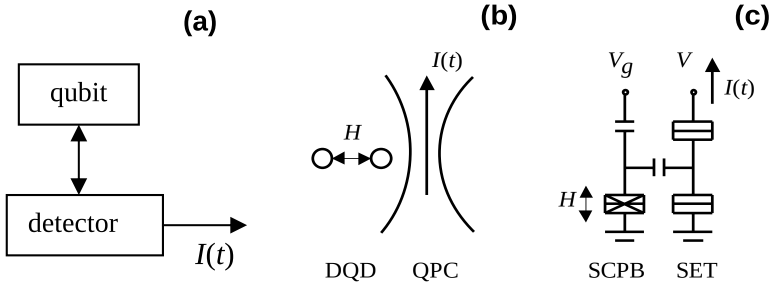

Figure 1. (a) General schematic of a continuously measured solid-state qubit and two particular realizations of the setup: (b) a qubit made of double quantum dot (DQD) measured by a quantum point contact (QPC) and (c) a qubit based on single-Cooper-pair box (SCPB) measured by a single-electron transistor (SET).

(some extensions of the Bayesian formalism will be discussed later). Notice that the formalism has been also reproduced[8] in a somewhat different language (using the terminology of quantum trajectories, quantum jumps, and quantum state diffusion) by the team which has previously developed[2] a similar theory in quantum optics. It is important to stress that the Bayesian approach is not a phenomenological theory which just correctly describes two previously known cases. It claims the description of a real and experimentally verifiable evolution of a single qubit in a process of measurement.

Simply speaking, the Bayesian formalism gives the following answer to the question posed above (for $H = 0$). During the measurement process the diagonal matrix elements of the qubit density matrix evolve according to the classical Bayes formula[9,10] which takes into account the noisy detector output [$I(t)$ in Fig. 1] and describes a gradual qubit localization into one of the states $|1\rangle$ or $|2\rangle$, depending on $I(t)$. The evolution of nondiagonal matrix elements can be easily calculated using somewhat surprising result that a good (ideal) detector preserves the purity of the qubit state, so that the decoherence is actually just a consequence of averaging over different detector outcomes $I(t)$ for different members of the ensemble. (Nonideal detectors also produce some amount of qubit decoherence, which is calculated within the formalism.)

Notice that in the case of an ideal detector, our result can actually be considered as a simple consequence of the so-called Quantum Bayes Theorem (we borrow this name from the book on quantum noise by Gardiner,[11] even though it is not a theorem in a mathematical sense). However, the application of this "theorem" is not always straightforward, so instead of applying it as an ansatz, we derive the Bayesian formalism for particular measurement setups, starting from the textbook quantum mechanics.

It is difficult to avoid philosophical questions discussing a problem related to quantum measurements. In brief, the Bayesian approach philosophy is exactly the philosophy of the "orthodox" quantum mechanics. A minor technical difference is that instead of assuming instantaneous information on measurement result corresponding to instantaneous "orthodox" collapse, we consider a more realistic case of continuous information flow.

Finally, let us mention that the problem of solid-state qubit evolution due to continuous measurement has been recently a subject of theoretical study by many groups (see, e.g.[12–22]). However, most of the obtained results are limited by the assumption of ensemble averaging (except the results obtained by Australian group[8,18] which also studies single realizations of the measurement process).

2. SIMPLE MODEL

Even though Bayesian approach is applicable to a broad range of measurement setups, let us start with a particularly simple setup [Fig. 1(b)] consisting of a double quantum dot occupied by a single electron, the position of which is measured by a low-transparency Quantum Point Contact (QPC) or (which is almost the same) by just a tunnel junction [Fig. 2(a)]. Basically following the model of Ref.[12] we assume that the detector barrier height depends on the location of the electron in either dot 1 or 2; then the current through the tunnel junction (which is the detector output) is sensitive to the electron location.

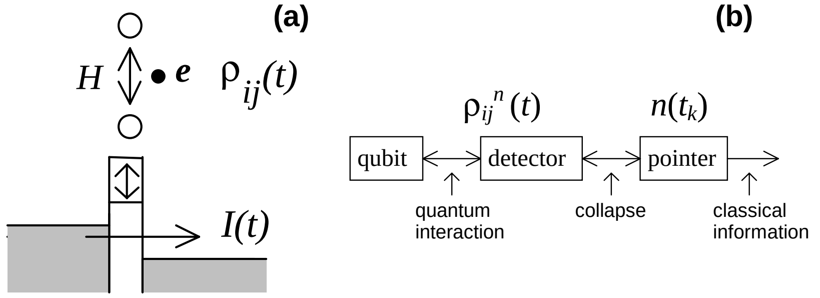

Figure 2. (a) Tunnel junction as a detector of the double-quantum-dot qubit. The electron location in the DQD affects the detector barrier height. The noisy current $I(t)$ (detector output) reflects the evolution of the qubit density matrix $\rho_{ij}(t)$. (b) Idea of the Bayesian formalism derivation via Bloch equations. The number n of electrons passed through the detector is periodocally collapsed (forced to choose a definite value) at moments t_k.

The Hamiltonian of the system, $\mathcal{H} = \mathcal{H}_{QB} + \mathcal{H}_{DET} + \mathcal{H}_{INT}$, consists of terms describing the double-dot qubit, the detector, and their interaction. The qubit Hamiltonian,

$$\mathcal{H}_{QB} = \frac{\varepsilon}{2}\,(c_2^\dagger c_2 - c_1^\dagger c_1) + H\,(c_1^\dagger c_2 + c_2^\dagger c_1), \tag{1}$$

is characterized by the energy asymmetry ε between two dots and the tunneling strength H (we assume real H without loss of generality). The detector and interaction Hamiltonians can be written as $\mathcal{H}_{DET} = \sum_l E_l a_l^\dagger a_l + \sum_r E_r a_r^\dagger a_r + \sum_{l,r} T(a_r^\dagger a_l + a_l^\dagger a_r)$, $\mathcal{H}_{INT} = \sum_{l,r}(\Delta T/2)(c_1^\dagger c_1 - c_2^\dagger c_2)(a_r^\dagger a_l + a_l^\dagger a_r)$, where both T and ΔT are assumed real and their dependence on the states in electrodes (l, r) is neglected. For simplicity we assume zero temperature (Bayesian formalism at finite temperatures has been considered in Refs.[4, 8, 23, 24]). If the electron occupies dot 1, then the average current through the detector is $I_1 = 2\pi(T + \Delta T/2)^2 \rho_l \rho_r e^2 V/\hbar$ (V is the voltage across the tunnel junction and $\rho_{l,r}$ are the densities of states in the electrodes), while if the measured electron is in the dot 2, the average current is $I_2 = 2\pi(T - \Delta T/2)^2 \rho_l \rho_r e^2 V/\hbar$.

The difference between the currents,

$$\Delta I \equiv I_1 - I_2, \tag{2}$$

determines the detector response to the electron position. Because of the detector shot noise, the two states cannot be distinguished instantaneously and the signal-to-noise ratio (S/N) gradually improves with the increase of the measurement duration. The S/N becomes close to unity after the "measurement" time

$$\tau_m = \frac{(\sqrt{S_1} + \sqrt{S_2})^2}{2(\Delta I)^2}, \tag{3}$$

where the spectral densities S_1 and S_2 of the detector shot noise for states $|1\rangle$ and $|2\rangle$ are given by the Schottky formula, $S_{1,2} = 2eI_{1,2}$. [Actually, Eq. (3) exactly corresponds to S/N=1 for $S_1 = S_2$, while for S/N$\neq$1 it gives the proper asymptotic scaling at $t \gg \tau_m$.] To avoid an explicit account of the detector quantum noise we will consider only processes at frequencies $\omega \ll eV/\hbar$; in particular, we assume $\max(\hbar\tau_m^{-1}, |\varepsilon|, |H|) \ll eV$.

Notice that due to electron charge discreteness and stochastic nature of tunneling, the total number $n(t)$ of electrons passed through the detector is sometimes a more convenient magnitude to work with than the current $I(t) = e\,dn(t)/dt$. In particular, we will use $n(t)$ instead of $I(t)$ for the Bayesian formalism derivation in the next section.

3. DERIVATION OF THE BAYESIAN FORMALISM VIA "BLOCH" EQUATIONS

The "conventional" ensemble-averaged equations for the qubit density matrix $\rho_{ij}(t)$,

$$\dot\rho_{11} = -\dot\rho_{22} = -2\,\frac{H}{\hbar}\,\mathrm{Im}\,\rho_{12}, \tag{4}$$

$$\dot\rho_{12} = \mathrm{i}\,\frac{\varepsilon}{\hbar}\,\rho_{12} + \mathrm{i}\,\frac{H}{\hbar}\,(\rho_{11} - \rho_{22}) - \Gamma_d\,\rho_{12}, \tag{5}$$

do not take into account any information about the detector outcome and describe the effect of continuous measurement by the ensemble decoherence rate[12] $\Gamma_d = (\sqrt{I_1} - \sqrt{I_2})^2/2e$. (Notice a relation $\Gamma_d \tau_m = 1/2$; as will be seen later, this means that the detector is ideal).

Equations (4)–(5) imply tracing over all detector degrees of freedom, including the measurement outcome. An important step towards taking into account the measurement record was a derivation[12] of "Bloch" equations for the density matrix $\rho_{ij}^n(t)$ which is divided into components with different number n of electrons passed through the detector:

$$\dot\rho_{11}^n = -\frac{I_1}{e}\,\rho_{11}^n + \frac{I_1}{e}\,\rho_{11}^{n-1} - 2\,\frac{H}{\hbar}\,\mathrm{Im}\,\rho_{12}^n\,, \quad \dot\rho_{22}^n = -\frac{I_2}{e}\,\rho_{22}^n + \frac{I_2}{e}\,\rho_{22}^{n-1} + 2\,\frac{H}{\hbar}\,\mathrm{Im}\,\rho_{12}^n\,, \tag{6}$$

$$\dot\rho_{12}^n = \mathrm{i}\,\frac{\varepsilon}{\hbar}\,\rho_{12}^n + \mathrm{i}\,\frac{H}{\hbar}\,(\rho_{11}^n - \rho_{22}^n) - \frac{I_1+I_2}{2e}\,\rho_{12}^n + \frac{\sqrt{I_1 I_2}}{e}\,\rho_{12}^{n-1}\,. \tag{7}$$

Eqs. (4)–(5) can be obtained from the Bloch equations using summation over n and relation $\rho_{ij} = \sum_n \rho_{ij}^n$. (Absence of nondiagonal in n matrix elements ρ_{ij}^{nm} is related to the assumption of large detector voltage.[12])

Despite the Bloch equations carry the total number n of electrons passed through the detector, they cannot take into account the whole measurement record $n(t)$ for a particular realization of measurement process. We should make a simple but important step for that: *we should introduce a sufficiently frequent collapse of $n(t)$ corresponding to a particular realization of the measurement record.*[4] This idea is illustrated in Fig. 2(b). Including "detector" into the quantum part of the setup, we anyway have to deal with a classical information, so we introduce a classical "pointer" which periodically (at times t_k) forces the system "qubit+detector" to choose a definite value for $n(t_k)$. An obvious drawback of such construction is that it is absolutely not clear what should be a sequence of t_k (in other words, how strongly the detector and pointer should be coupled). Fortunately, for a model described in the previous section the results do not depend on the choice of t_k if $\Delta t_k \equiv t_k - t_{k-1}$ are sufficiently small, so the natural choice is $\Delta t_k \to 0$.

Technically the procedure is the following. During the time between t_{k-1} and t_k the "qubit+detector" evolves according to the Bloch equations (6)–(7), while at time t_k the number n is collapsed in the "orthodox" way.[5] This means that the probability $P(n)$ of getting some $n(t_k)$ is equal to

$$P(n) = \rho_{11}^n(t_k) + \rho_{22}^n(t_k), \tag{8}$$

while after a particular n_k is picked, the density matrix ρ_{ij}^n is immediately updated (collapsed):

$$\rho_{ij}^n(t_k+0) = \delta_{n,n_k}\,\rho_{ij}(t_k+0), \quad \rho_{ij}(t_k+0) = \frac{\rho_{ij}^{n_k}(t_k-0)}{\rho_{11}^{n_k}(t_k-0) + \rho_{22}^{n_k}(t_k-0)}, \tag{9}$$

where δ_{n,n_k} is the Kronecker symbol. After that the evolution is again described by Eqs. (6)–(7) with n shifted by n_k until the next collapse occurs at $t = t_{k+1}$, and so on. *This procedure is the main step* in the derivation of the Bayesian formalism.

Let us discuss now the relation of this procedure to the classical Bayes theorem[9,10] which says that a posteriori probability $P(B_i|A)$ of a hypothesis B_i after learning an information A (B_i form a complete set of mutually exclusive hypotheses) is equal to

$$P(B_i|A) = \frac{P(B_i)P(A|B_i)}{\sum_k P(B_k)P(A|B_k)} \tag{10}$$

where $P(B_i)$ is a priori probability of the hypothesis B_i (before learning information A) and $P(A|B_i)$ is the conditional probability of the event A under hypothesis B_i.

Assuming for a moment $H = 0$ and $\varepsilon = 0$ in the qubit Hamiltonian (so that the qubit evolution is due to measurement only), it is easy to find[4] that Eqs. (6)–(7) and our procedure (9) lead to the qubit evolution as

$$\rho_{ii}(t_k) = \frac{\rho_{ii}(t_{k-1})P_i(\Delta n_k)}{\rho_{11}(t_{k-1})P_1(\Delta n_k) + \rho_{22}(t_{k-1})P_2(\Delta n_k)}, \tag{11}$$

$$\rho_{12}(t_k) = \rho_{12}(t_{k-1})\frac{[\rho_{11}(t_k)\rho_{22}(t_k)]^{1/2}}{[\rho_{11}(t_{k-1})\rho_{22}(t_{k-1})]^{1/2}}, \tag{12}$$

where $\Delta n_k = n_k - n_{k-1}$ is the number of electrons passed through the detector during time Δt_k and

$$P_i(n) = \frac{(I_i \Delta t_k / e)^n}{n!} \exp(-I_i \Delta t_k / e) \tag{13}$$

is the classical Poisson distribution for this number assuming either qubit state $|1\rangle$ or $|2\rangle$. One can see that the diagonal matrix elements ρ_{ii} *exactly obey the classical Bayes formula* (10), i.e. *as if* the qubit is really either in the state $|1\rangle$ or $|2\rangle$, but not in both simultaneously. Actually, this fact is not much surprising because at least in some sense ρ_{ii} are the probabilities. Equation (12) is a little more surprising and says that the measurement preserves the degree of qubit purity $\rho_{12}/(\rho_{11}\rho_{22})^{1/2}$; for instance, *a pure state remains pure during the whole measurement process.*

After introducing the main procedure (9), further derivation of the Bayesian formalism is pretty simple and depends on whether we want to consider finite detector response, $|\Delta I| \sim I_0 \equiv (I_1 + I_2)/2$ or a weak response, $|\Delta I| \ll I_0$. In the first case each event of tunneling through the detector carries significant information and significantly affects the qubit state, so a reasonable "experimental" setup implies recording the time of each tunneling event. Then during the time periods when no electrons are passing through the detector, the evolution is essentially described by the Bloch equations (6)–(7) with $n = 0$, while the frequent collapses $[\Delta t_k \ll \min(e/I_1, e/I_2, \hbar/H, \hbar/\varepsilon)]$ just restore the density matrix normalization, leading to the continuous (but not unitary!) qubit evolution[4,8]:

$$\dot\rho_{11} = -\dot\rho_{22} = -2\frac{H}{\hbar}\,\mathrm{Im}\,\rho_{12} - \frac{\Delta I}{e}\,\rho_{11}\rho_{22}, \quad \dot\rho_{12} = \frac{\mathrm{i}\varepsilon}{\hbar}\,\rho_{12} + \frac{\mathrm{i}H}{\hbar}\,(\rho_{11} - \rho_{22}) + \frac{\Delta I}{2e}(\rho_{11} - \rho_{22})\rho_{12}. \tag{14}$$

However, at moments when one electron passes through the detector, the qubit state changes abruptly (corresponding to $\Delta n_k = 1$ and $\Delta t_k \to 0$ in Eqs. (11)–(12)):

$$\rho_{11}(t+0) = \frac{I_1 \rho_{11}(t-0)}{I_1\rho_{11}(t-0) + I_2\rho_{22}(t-0)}\,, \quad \rho_{22}(t+0) = 1 - \rho_{11}(t+0), \tag{15}$$

$$\rho_{12}(t+0) = \rho_{12}(t-0) \left[\frac{\rho_{11}(t+0)\,\rho_{22}(t+0)}{\rho_{11}(t-0)\,\rho_{22}(t-0)} \right]^{1/2}. \tag{16}$$

These abrupt changes are usually called "quantum jumps".[8] Notice that for $I_1 > I_2$ the jumps always shift the qubit state closer to $|1\rangle$ (because detector tunneling is "more likely" for state $|1\rangle$), while continuous nonunitary evolution shifts the state towards $|2\rangle$. On average the evolution is still given by conventional Eqs. (4)–(5).

The case of a weak detector response, $|\Delta I| \ll I_0$, is more realistic from the experimental point of view. In this case the measurement time τ_m as well as the ensemble decoherence time Γ_d^{-1} are much longer than the average time e/I_0 between electron passages in the detector. If the tunneling in the qubit is also sufficiently slow, $\hbar/H \gg e/I_0$, we can completely disregard individual events in the detector and consider the detector current $I(t)$ as quasicontinuous. Then Eq. (11) for the evolution due to measurement only (neglecting H and ε) transforms into the equation which again has clear Bayesian interpretation:

$$\rho_{ii}(t+\tau) = \frac{\rho_{ii}(t) P_i(\overline{I})}{\rho_{11}(t)P_1(\overline{I}) + \rho_{22}(t)P_2(\overline{I})}, \quad \overline{I} \equiv \frac{1}{\tau}\int_t^{t+\tau} I(t')\,dt' \tag{17}$$

where $\overline{I}$ is the detector current averaged over the time interval $(t, t+\tau)$ and $P_i(\overline{I})$ are its classical Gaussian probability distributions for two qubit states:

$$P_i(\overline{I}) = \frac{1}{(2\pi D)^{1/2}} \exp[-\frac{(\overline{I} - I_i)^2}{2D}], \;\; D = S_0/2\tau, \tag{18}$$

(here $S_0 = 2eI_0$ is the detector noise), while Eq. (12) essentially does not change.

Differentiating Eqs. (17) and (12) over time and including additional evolution due to H and ε, we obtain the *main equations of the Bayesian formalism*:

$$\dot\rho_{11} = -\dot\rho_{22} = -2\frac{H}{\hbar}\,\mathrm{Im}\,\rho_{12} + \rho_{11}\rho_{22}\,\frac{2\Delta I}{S_0}\,[I(t) - I_0], \tag{19}$$

$$\dot\rho_{12} = \mathrm{i}\,\frac{\varepsilon}{\hbar}\,\rho_{12} + \mathrm{i}\,\frac{H}{\hbar}\,(\rho_{11} - \rho_{22}) - (\rho_{11} - \rho_{22})\frac{\Delta I}{S_0}\,[I(t) - I_0]\,\rho_{12}. \tag{20}$$

In each realization of measurement the noisy detector outcome $I(t)$ is different; however, for each realization we can precisely monitor the evolution of the qubit density matrix, plugging experimental $I(t)$ into Eqs. (19)–(20). Let us stress again that these equations show the absence of any qubit decoherence during the process of measurement. Pure initial state remains pure; moreover, initially mixed state gradually purifies during the measurement if $H \neq 0$.[4, 7] The gradual state purification is essentially due to acquiring more and more information about the qubit state from the measurement record.

While the qubit state does not decohere in each individual realization of the measurement, different members of the ensemble evolve differently because of random $I(t)$. Averaging Eqs. (19)–(20) over random $I(t)$ and using the relation [which follows from Eqs. (8), (11), and (13)]

$$I(t) - I_0 = \frac{\Delta I}{2} (\rho_{11} - \rho_{22}) + \xi(t), \tag{21}$$

where $\xi(t)$ is a zero-correlated ("white") random process with the same spectral density as the detector noise, $S_\xi = S_0$, we obtain conventional Eqs. (4)–(5). Therefore, the ensemble-averaged decoherence in our model is just a consequence of averaging over the measurement outcome.

Notice that since $I(t)$ contains the white noise contribution, Eqs. (19)–(20) are nonlinear stochastic differential equations[25] and dealing with them requires a special care. The problem is that their analysis depends on the choice of the derivative definition. Two mainly used definitions are the symmetric derivative: $\dot\rho(t) \equiv \lim_{\tau\to 0}[\rho(t+\tau/2) - \rho(t-\tau/2)]/\tau$ which leads to the so-called Stratonovich interpretation of the stochastic differential equations, and the forward derivative: $\dot\rho(t) \equiv \lim_{\tau\to 0}[\rho(t+\tau) - \rho(t)]/\tau$ (Itô interpretation). Usual calculus rules remain valid only in the Stratonovich form,[25] so the physical intuition works better when using Stratonovich definition. However, Itô interpretation allows simple averaging over the noise and because of that is usually preferred by mathematicians. Since we derived Eqs. (19)–(20) by a simple first-order differentiation, we automatically obtained them in the Stratonovich form (keeping the second-order terms in the expansion, we can obtain different equations, depending on the definition of the derivative). Since sometimes Itô form is more preferable, let us translate Bayesian equations into Itô form using the following general rule.[25] For an arbitrary system of equations

$$\dot x_i(t) = G_i(\mathbf{x}, t) + F_i(\mathbf{x}, t)\, \xi(t) \tag{22}$$

in Stratonovich interpretation, the corresponding Itô equation which has the same solution is

$$\dot x_i(t) = G_i(\mathbf{x}, t) + \frac{S_\xi}{4} \sum_k \frac{\partial F_i(\mathbf{x}, t)}{\partial x_k} F_k(\mathbf{x}, t) + F_i(\mathbf{x}, t)\, \xi(t)\,, \tag{23}$$

where $x_i(t)$ are the components of the vector $\mathbf{x}(t)$, G_i and F_i are arbitrary functions, and the constant S_ξ is the spectral density of the white noise process $\xi(t)$. Applying this transformation to Eqs. (19)–(20), we get the following equations in Itô interpretation:

$$\dot\rho_{11} = -\dot\rho_{22} = -2\,\frac{H}{\hbar}\, \mathrm{Im}\, \rho_{12} + \rho_{11}\rho_{22}\, \frac{2\Delta I}{S_0}\, \xi(t)\,, \tag{24}$$

$$\dot\rho_{12} = \mathrm{i}\,\frac{\varepsilon}{\hbar}\, \rho_{12} + \mathrm{i}\,\frac{H}{\hbar}\, (\rho_{11} - \rho_{22}) - (\rho_{11} - \rho_{22}) \frac{\Delta I}{S_0}\, \rho_{12}\, \xi(t) - \frac{(\Delta I)^2}{4 S_0}\, \rho_{12}\,, \tag{25}$$

while the relation between pure noise $\xi(t)$ and the current $I(t)$ is still given by Eq. (21). Notice that the last term in Eq. (25) does not actually mean the single qubit decoherence (pure state remains pure), but is just a feature of the Itô form [it directly corresponds to the ensemble decoherence after averaging over $\xi(t)$].

4. DERIVATION OF THE FORMALISM VIA CORRESPONDENCE PRINCIPLE

Another derivation[7] of the Bayesian formalism for a single qubit can be based on the logical use of the correspondence principle,[5] classical Bayes formula, and results of the conventional ensemble-averaged formalism. Even though this way lacks some advantages of the "microscopic" derivation discussed in the previous section, it can be applied to a broader class of solid-state detectors, in particular, to the finite-transparency quantum point contact and (with some extension) to the single-electron transistor and SQUID. In this section we will assume a double-dot qubit measured by a finite-transparency QPC and treat the detector current $I(t)$ as a quasicontinuous noisy signal that implies weak detector response, $|\Delta I| \ll I_0$.

Let us start with a completely *classical* case when the electron is actually located in one of two dots (and does not move), but we do not know in which dot, so the measurement gradually reveals the actual electron location. This is a well studied problem of the probability theory. The measurement process can be described as an evolution of probabilities (we still call them ρ_{11} and ρ_{22}) which reflect our knowledge about the system state. Then for arbitrary τ (which can be comparable to τ_m) the current $\overline{I}$ averaged over time interval $(t, t+\tau)$ [see Eq. (17)] has the probability distribution

$$P(\overline{I}) = \rho_{11}(t)P_1(\overline{I}) + \rho_{22}(t)P_2(\overline{I}), \tag{26}$$

where P_i are given by Eq. (18) and depend on the detector white noise spectral density S_0 which should not necessarily satisfy Schottky formula. After the measurement during time τ the information about the system state has increased and the probabilities ρ_{11} and ρ_{22} should be updated using the measurement result $\overline{I}$ and the Bayes formula (17). This completely describes the classical measurement process.

The next step in the derivation is an important assumption: in the quantum case with $H = 0$ the evolution of ρ_{11} and ρ_{22} is still given by Eq. (17) because there is no possibility to distinguish between classical and quantum cases, performing only this kind of measurement. Even though this assumption is quite obvious, it is not derived formally but should rather be regarded as a *consequence of the correspondence principle.*

The correspondence with classical measurement cannot describe the evolution of ρ_{12}; however, there is still an upper limit: $|\rho_{12}| \leq [\rho_{11}\rho_{22}]^{1/2}$. Surprisingly, this inequality is sufficient for exact calculation of $\rho_{12}(t)$ in the case of a QPC as a detector (we still assume $H = 0$). Averaging the inequality over all possible detector outputs $\overline{I}$ using distribution (26) we get the inequality

$$|\langle \rho_{12}(t+\tau)\rangle| \leq \langle |\rho_{12}(t+\tau)|\rangle \leq \langle [\rho_{11}(t+\tau)\rho_{22}(t+\tau)]^{1/2}\rangle = [\rho_{11}(t)\rho_{22}(t)]^{1/2} \exp[-(\Delta I)^2\tau/4S_0] \tag{27}$$

[here the decaying exponent is a consequence of changing ρ_{11} and ρ_{22} that reduces their average product]. On the other hand, from the conventional approach we know[15,17,26,27] that the ensemble-averaged qubit decoherence rate caused by a QPC is equal to $\Gamma_d = (\Delta I)^2/4S_0$, where $S_0 = 2eI_0(1-\mathcal{T})$ and $\mathcal{T}$ is the QPC transparency. This means that inequality (27) actually *reaches its upper bound.* This is possible *only if* in each realization of the measurement process an initially pure density matrix $\rho_{ij}(t)$ stays pure all the time, $|\rho_{12}|^2 = \rho_{11}\rho_{22}$. Moreover, since the phase of $\rho_{12}(t+\tau)$ should be the same for all realizations (to ensure $|\langle\rho_{12}(t+\tau)\rangle| = \langle|\rho_{12}(t+\tau)|\rangle$), the only possibility in absence of a detector-induced shift of ε is

$$\frac{\rho_{12}(t+\tau)}{[\rho_{11}(t+\tau)\rho_{22}(t+\tau)]^{1/2}} = \frac{\rho_{12}(t)}{[\rho_{11}(t)\rho_{22}(t)]^{1/2}} \exp(i\varepsilon\tau/\hbar) \tag{28}$$

(if the coupling with detector shifts ε, we just have to use the shifted value).

As the next step of the derivation, let us allow an arbitrary mixed initial state of the qubit (but still $H = 0$). It can always be represented as a mixture of two states with the same diagonal matrix elements, one of which is pure, while the other state does not have nondiagonal matrix elements. Since nondiagonal matrix elements for the latter state cannot appear in the process of measurement and since the evolution of the diagonal matrix elements is equal for both states, one can conclude that Eq. (28) remains valid, i.e. for mixed states the degree of purity is conserved (gradual purification does not occur at $H = 0$). The final step of the formalism derivation is differentiating Eqs. (17) and (28) over time and adding (in a simple way) the evolution due to H.

In this way we reproduce Eqs. (19)–(20). However, as seen from the derivation, now they are applicable to a *broader class of detectors* (which includes the finite-transparency QPC) for which $\Gamma_d = (\Delta I)^2/4S_0$. This relation can also be expressed as $\Gamma_d\tau_m = 1/2$ since $\tau_m = 2S_0/(\Delta I)^2$ for a weakly responding detector. As will be discussed in the next section, this is a condition of an ideal quantum detector.

5. NONIDEAL DETECTORS

The relation $\Gamma_d \tau_m = 1/2$ which is valid for the models of a tunnel junction and QPC as detectors, basically says that the *larger output noise S_0 of a detector leads to a smaller backaction* characterized by ensemble decoherence Γ_d. This is quite expected from quantum mechanical point of view (the faster we get information, the faster we should collapse the measured state). However, it is obviously not necessarily the case for an arbitrary solid-state detector; for example, the increase of output noise S_0 can be due to later stages of signal amplification, which do not affect Γ_d. In other words, it is easy to imagine a bad detector which produces a lot of both output and backaction noises.

To take into account an extra detector noise, we can phenomenologically add a dephasing rate γ_d into the Bayesian equations:

$$\dot{\rho}_{11} = -\dot{\rho}_{22} = -2\,\frac{H}{\hbar}\,\mathrm{Im}\,\rho_{12} + \rho_{11}\rho_{22}\,\frac{2\Delta I}{S_0}\,[I(t) - I_0], \tag{29}$$

$$\dot{\rho}_{12} = \mathrm{i}\,\frac{\varepsilon}{\hbar}\,\rho_{12} + \mathrm{i}\,\frac{H}{\hbar}\,(\rho_{11} - \rho_{22}) - (\rho_{11} - \rho_{22})\frac{\Delta I}{S_0}\,[I(t) - I_0]\,\rho_{12} - \gamma_d\rho_{12}. \tag{30}$$

This obviously increases the ensemble decoherence rate:

$$\Gamma_d = \frac{(\Delta I)^2}{4S_0} + \gamma_d. \tag{31}$$

A natural definition of a detector ideality (quantum efficiency) in this case is

$$\eta \equiv 1 - \frac{\gamma_d}{\Gamma_d} = \frac{1}{2\Gamma_d\tau_m}. \tag{32}$$

An upper limit for η is 100% because of a fundamental limitation

$$\Gamma_d\tau_m \geq 1/2, \tag{33}$$

which is a by-product of the Bayesian derivation for the case of quasicontinuous detector current – see inequality (27). The extra dephasing γ_d in Eq. (30) can be interpreted[7] as an effect of extra environment or (mathematically) as due to a second detector "in parallel", the output of which is not read out (then we have to average over possible outputs). It can be also interpreted[28] as an effect of extra noise S_{add} at the detector output, $S_0 = (\Delta I)^2/4\Gamma_d + S_{add}$.

Introduction of the detector ideality η allows us to consider a continuous transition from the conventional ensemble-averaged result (4)–(5) to the "pure" Bayesian result (19)–(20). The effect of a pure environment can be considered as a measurement with an extremely bad detector, $\eta = 0$. Technically it corresponds to $\Delta I = 0$ in Eqs. (29)–(30), transforming them into conventional equations. The case of a detector with very small efficiency, $\eta \ll 1$, can be treated in two steps: first, we analyze the effect of the decoherence term ($\gamma_d \approx \Gamma_d$), and then we use the classical (still Bayesian) analysis to relate the qubit density matrix and the measurement outcome. So, only for good detectors with the efficiency η comparable to unity, the quantum Bayesian approach discussed in this paper is really necessary. With the progress of technology, such detectors are becoming available. For example, the analysis of experimental data of the recent "which path" experiment[29] shows that their QPC had a quantum efficiency quite close to unity (very crudely, it can be estimated as 80%).

The account of the detector nonideality by introducing extra decoherence rate γ_d into Eq. (30) implicitly assumes the absence of a direct correlation between the output detector noise and the backaction noise affecting the qubit energy asymmetry ε. However, such correlation is a typical situation, for example, for a single-electron transistor as a detector.[30] In this case the knowledge of the noisy detector output $I(t)$ gives some information about the probable backaction noise "trajectory" $\varepsilon(t)$ which can be used to improve our knowledge of the qubit state. Compensation for the most probable trajectory $\varepsilon(t)$ leads to the improved Bayesian equations for the SET in which Eq. (30) is replaced with[28]

$$\dot{\rho}_{12} = \mathrm{i}\,\frac{\varepsilon}{\hbar}\,\rho_{12} + \mathrm{i}\,\frac{H}{\hbar}\,(\rho_{11} - \rho_{22}) - (\rho_{11} - \rho_{22})\frac{\Delta I}{S_0}\,[I(t) - I_0]\,\rho_{12} + \mathrm{i}\,K\,[I(t) - I_0]\,\rho_{12} - \tilde{\gamma}_d\rho_{12}, \tag{34}$$

where $K = (d\varepsilon/d\varphi)S_{I\varphi}/S_0\hbar$, φ is the electric potential of the SET central electrode, and $S_{I\varphi}$ is the mutual low-frequency spectral density between the current noise and φ noise. (The energy asymmetry induced by the measurement is included in ε.)

The dephasing rate $\tilde{\gamma}_d$ should now satisfy equation

$$\tilde{\gamma}_d = \Gamma_d - \frac{(\Delta I)^2}{4S_0} - \frac{K^2 S_0}{4} \tag{35}$$

to correspond to the the ensemble-averaged dynamics still described by Eqs. (4)–(5). The obvious inequality $\tilde{\gamma}_d \geq 0$ (in the opposite case the relation $|\rho_{12}|^2 \leq \rho_{11}\rho_{22}$ would be violated in a single realization of measurement) imposes a lower bound for the ensemble decoherence rate Γ_d:

$$\Gamma_d \geq \frac{(\Delta I)^2}{4S_0} + \frac{K^2 S_0}{4}\,, \tag{36}$$

which is stronger than the inequality $\Gamma_d \tau_m \geq 1/2$.

Inequality (36) can be easily expressed in terms of the energy sensitivity of an SET. Let us define the output energy sensitivity as $\epsilon_I \equiv (dI/dq)^{-2}S_I/2C$ where C is the total SET island capacitance, dI/dq is the response to the externally induced charge q, and we have changed the notation $S_I \equiv S_0$ for a more symmetric look of the formulas. Notice that ϵ_I has the same dimension as $\hbar$. Similarly, let us characterize the backaction noise intensity by $\epsilon_\varphi \equiv CS_\varphi/2$ and the correlation between two noises by the magnitude $\epsilon_{I\varphi} \equiv (dI/dq)^{-1}S_{I\varphi}/2$. Since in absence of other decoherence sources $\Gamma_d = S_\varphi(C\Delta E/2e\hbar)^2$, where ΔE is the energy coupling between qubit and single-electron transistor,[4,16] and using also the reciprocity property $\Delta q = C\Delta E/e = d\varepsilon/d\varphi$, we can rewrite Eq. (36) as

$$\epsilon \equiv (\epsilon_I \epsilon_\varphi - \epsilon_{I\varphi}^2)^{1/2} \geq \hbar/2. \tag{37}$$

This is a result known for 20 years[31] for SQUIDs (see also[15,32–35]).

When the limit $\epsilon = \hbar/2$ is achieved, the decoherence rate

$$\tilde{\gamma}_d = \frac{(\Delta I)^2}{4S_I}\left[\frac{\epsilon_I\epsilon_\varphi - \epsilon_{I\varphi}^2}{(\hbar/2)^2} - 1\right] \tag{38}$$

in Eq. (34) vanishes, $\tilde{\gamma}_d = 0$. In this sense the detector is ideal, $\tilde{\eta} = \tilde{\eta}_2 = 1$, where

$$\tilde{\eta} \equiv 1 - \frac{\tilde{\gamma}_d}{\Gamma_d} = \frac{\hbar^2(dI/dq)^2}{S_I S_\varphi} + \frac{(S_{I\varphi})^2}{S_I S_\varphi}\,, \quad \tilde{\eta}_2 \equiv \frac{(\hbar/2)^2}{\epsilon_I\epsilon_\varphi - \epsilon_{I\varphi}^2} = \frac{\hbar^2(dI/dq)^2}{S_I S_\varphi - S_{I\varphi}^2}, \tag{39}$$

even though it can be a nonideal detector ($\eta < 1$) by the previous definition, $\eta = \hbar^2(dI/dq)^2/S_I S_\varphi$. Notice a simple relation between different definitions of ideality,

$$\eta = \tilde{\eta} = \tilde{\eta}_2 = \frac{(\hbar/2)^2}{\epsilon_I\epsilon_\varphi} = \frac{1}{2\Gamma_d\tau_m}\,, \tag{40}$$

in absence of correlation between noises of $\varphi(t)$ and $I(t)$, $(S_{I\varphi})^2 \ll S_I S_\varphi$.

Even though Eqs. (34)–(40) were derived for the SET as a detector, it is rather obvious that they are applicable to virtually any solid-state detector with continuous output (for a dc SQUID the current output should obviously be replaced by the voltage output). In particular, the conclusion that reaching the quantum-limited total energy sensitivity $\epsilon = \hbar/2$ is equivalent to the detector ideality, is quite general.

As we already discussed, the tunnel junction and QPC at zero temperature (actually, for small temperatures $\beta^{-1} \ll eV$) are theoretically ideal quantum detectors. The fact that a SQUID can reach the limit of an ideal detector follows from the results of Ref..[31] A normal state SET is not a good quantum detector ($\eta \ll 1$) at usual operating points above the Coulomb Blockade threshold.[4,16] However, its quantum efficiency improves when we go closer to the threshold[4,32] and becomes much better when the operating point is in the cotunneling range (below the threshold), in which case the limit of an ideal detector can be achieved.[33] Superconducting SET is generally better than normal SET as a quantum-limited detector and can approach 100% ideality in the supercurrent regime[34] as well as in the double Josephson-plus-quasiparticle regime.[35] Finally, the resonant-tunneling SET[36] can reach ideality factor $\tilde{\eta}_2 = 3/4$ at large bias and complete ideality, $\tilde{\eta} = \tilde{\eta}_2 = 1$, in the small-bias limit.

6. SOME EXPERIMENTAL PREDICTIONS

6.1. Direct experiments

The Bayesian equations tell us that we can monitor the qubit evolution in a single realization of the measurement process using the record of the noisy detector output. In particular, for an ideal detector *we can monitor the qubit wavefunction* (except the overall phase) if the initial qubit state is pure or, for a mixed initial state, after monitoring for a sufficiently long time so that the gradual purification has enough time to produce a practically pure state.

This prediction (and hence, the validity of the Bayesian equations) can in principle be tested experimentally. For example,[7] let us first prepare the double-dot in the symmetric coherent state, $\rho_{11} = \rho_{22} = |\rho_{12}| = 1/2$, make $H = 0$ (raise the barrier), and begin measurement with a QPC [Fig. 1(b)]. According to our formalism, after some time τ (the most interesting case is $\tau \sim \tau_m$) the qubit state remains pure but becomes asymmetric ($\rho_{11} \neq \rho_{22}$) and can be calculated with Eqs. (19) and (20). To prove this, an experimentalist can use the knowledge of the wavefunction to move the electron "coherently" into the first dot with 100% probability. (Notice that if the qubit is in a mixed state, no unitary transformation can end up in the state $|1\rangle$ with certainty.) For instance, experimentalist switches off the detector at $t = \tau$, reduces the barrier (to get finite H), and creates the energy difference $\varepsilon = [(1-4|\rho_{12}(\tau)|^2)^{1/2} - 1]H\mathrm{Re}\rho_{12}(\tau)/|\rho_{12}(\tau)|^2$; then after the time period $\Delta t = [\pi - \arcsin(\mathrm{Im}\rho_{12}(\tau)\,\hbar\Omega/H)]/\Omega$ the "whole" electron will be moved into the first dot, that can be checked by the detector switched on again. [Here $\Omega \equiv (4H^2+\varepsilon^2)^{1/2}/\hbar$ is the frequency of unperturbed coherent ("Rabi") oscillations of the qubit.]

Another experimental idea[7] is to demonstrate the gradual purification of the double-dot density matrix. Let us start with a completely mixed (unknown) state ($\rho_{11} = \rho_{22} = 1/2$, $\rho_{12} = 0$) of the double-dot qubit with finite H. Then using the detector output $I(t)$ and Eqs. (19)–(20) an experimentalist gradually gets more and more knowledge about the randomly evolving qubit state (gradual purification), eventually ending up with almost pure wavefunction with precisely known *phase* of Rabi oscillations (we are not talking about the wavefunction phase, but about the phase of diagonal matrix elements oscillations). The final check of the wavefunction can be similar to that described in the previous paragraph. It can be even simpler, since with the knowledge of the phase of oscillations it is easy to stop the evolution by raising the barrier when the electron is in the first dot with certainty. Notice that if fast real-time calculations are not available, the moment of raising the barrier can be random, while lucky cases can be selected later from the record of $I(t)$.

Direct experiments of this kind as well as experiments on quantum feedback control and on Bayesian measurement of entangled qubits (discussed later), are still too difficult for realization with the present-day technology. In the next two subsections we discuss experiments which seem to be realizable (though very hard) at present.

6.2. Spectral density of the detector current

Naively thinking, a qubit with $H \neq 0$ should perform coherent (Rabi) oscillations with frequency Ω and these oscillations should lead to an oscillating contribution of the detector current $I(t)$. On the other hand, conventional Eqs. (4)–(5) seem naively to imply that the qubit eventually reaches a stationary state and no oscillations should be present in $I(t)$ after a sufficiently long observation. So, it is interesting to find what is the actual spectral density of the detector current $S_I(\omega)$ [it is easier to measure this quantity experimentally, than to record $I(t)$].

The Bayesian formalism predicts[23,18,24] the presence of the spectral peak at the Rabi frequency Ω, however, the height of this peak cannot be larger than 4 times the noise pedestal. In particular, in the case $\varepsilon = 0$

$$\frac{S_I(\omega)}{S_0} = 1 + \frac{4\eta}{(\omega/\Omega)^2 + (\omega^2-\Omega^2)^2/\Omega^2\Gamma_d^2}\,. \tag{41}$$

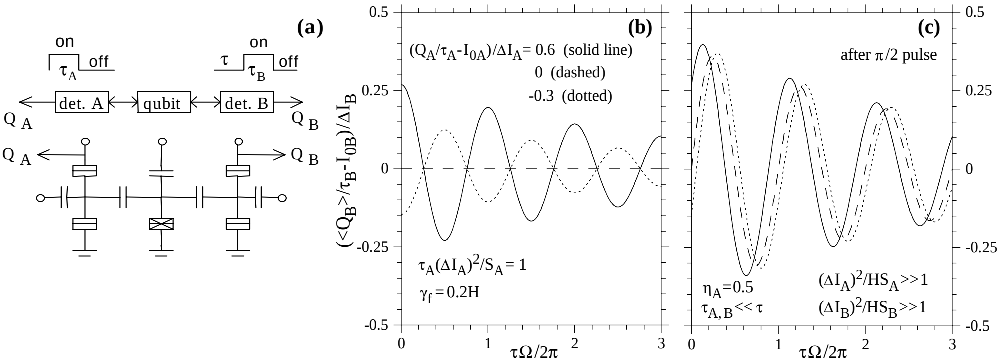

Figure 3. (a) Schematic of the proposed Bell-type correlation experiment,[37] in which a SCPB qubit is measured by two SETs during short time periods τ_A and τ_B shifted in time by τ. The first measurement leads to an incomplete collapse of the qubit initial state $(|1\rangle + |2\rangle)/\sqrt{2}$ and affects the result of the second measurement. (b) The average result $\langle Q_B\rangle$ of the second measurement for a selected result Q_A of the first measurement. The sign and amplitude of Rabi oscillations depend on Q_A, reflecting the change of the diagonal matrix elements of the qubit density matrix. (c) same as (b) if $\pi/2$ pulse is applied immediately after the first measurement. Now the phase of oscillations depends on Q_A. The full-swing oscillations (with amplitude of 0.5 in the ideal case) indicate a pure qubit state after the first measurement.

Actually, an experimental confirmation of this formula would not be a direct verification of the Bayesian formalism, since Eq. (41) can be also obtained by other methods, including the master equation method[15, 23, 27] and the method based on the Bloch equations[24] (see also Refs.[20–22]).

6.3. Bell-type experiment

Another experiment which also seems to be much easier than the direct experiments but can unambiguously test the Bayesian formalism, is a Bell-type experiment in which one qubit is measured by two detectors.[37] An idea (Fig. 3) is to prepare the qubit in a coherent state $(|1\rangle + |2\rangle)/\sqrt{2}$, then to switch on the first detector (A) for a relatively short time τ_A (so that the measurement is only partially completed), and to switch on the second detector (B) a little later. If the first measurement changes the qubit state according to the Bayesian formalism, then the second measurement can check this change. An output from a single run of the measurement are two charges Q_A and Q_B passed through two detectors. Performing the experiment many times and analyzing the correlation between Q_A and Q_B, one can recover the effect of the first measurement on the qubit state[37] (to check the change of the nondiagonal matrix element it is necessary to apply a $\pi/2$ pulse right after the first measurement). The main advantage of this Bell-type experiment in comparison with the direct Bayesian experiments is that the wide bandwidth for the output signal is not necessary; instead, it is traded for the wide bandwidth of two input lines (switching detectors on and off), which is much easier to realize experimentally.

7. QUANTUM FEEDBACK CONTROL OF A QUBIT

The Bayesian formalism can be used as a basis for the design and analysis of a quantum feedback control of a solid-state qubit. As an example, such feedback control can maintain for arbitrary long time the desired phase of a qubit Rabi oscillations, synchronizing them with a classical reference oscillator, even in presence of dephasing environment.[4, 19] The overall idea is very close to a classical feedback loop [Fig. 4(a)]. The oscillating qubit evolution is monitored by a weakly coupled detector ($\mathcal{C} \equiv \hbar(\Delta I)^2/S_0H < 1$), the phase $\phi(t)$ of actual Rabi oscillations is compared with the desired phase $\phi_0(t)$, and the difference signal $\Delta\phi$ is used to control the qubit barrier height. If qubit is slightly behind the desired phase, then H is decreased, so the oscillations run faster to catch up; if the qubit is ahead of proper phase, H is increased. It is natural to use a linear control: $H_{fb} = H(1 - F \times \Delta\phi)$, where F is a dimensionless feedback factor.

The only difference of this loop from a classical feedback is that even weakly coupled detector disturbs the qubit oscillations. However, the induced fluctuations of the oscillation phase are slow, and the information obtained from the detector happens to be enough to monitor the phase fluctuations and compensate them. The

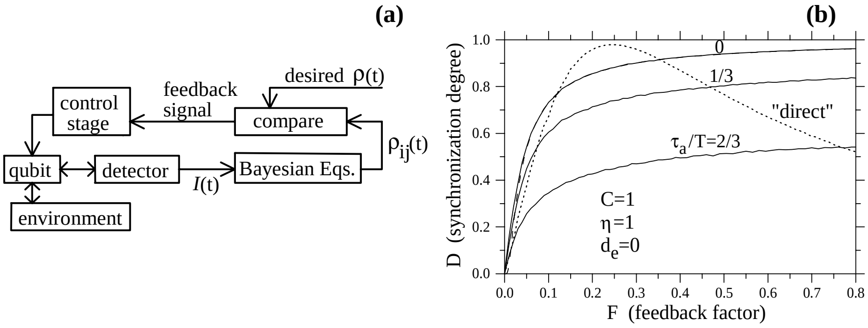

Figure 4. (a) Schematic of the quantum feedback loop maintaining the Rabi oscillations of a qubit by synchronizing them with a classical harmonic signal. (b) Solid lines: the synchronization degree D as a function of the feedback factor F for several values of available bandwidth τ_a^{-1}. While synchronization can approach 100% for wide bandwidth, it worsens when τ_a becomes comparable to the oscillation period $T = 2\pi/\Omega$. Dashed line (alsmost coinciding with the upper solid line): analytical result $D = \exp(-\mathcal{C}/32F)$. Dotted line: synchronization degree for a direct feedback with $\tau_a = T/10$. (From Ref..[19])

quantitative analysis[19] shows that in a limit of good synchronization and absence of extra environment the qubit correlation function $K_z(\tau) \equiv \langle z(t)z(t+\tau)\rangle$ (here $z \equiv \rho_{11} - \rho_{22}$) is given by

$$K_z(\tau) = \frac{\cos \Omega\tau}{2} \exp\left[\frac{\mathcal{C}}{16F}\left(e^{-2FH\tau/\hbar} - 1\right)\right], \tag{42}$$

and does not decay to zero at $\tau \to \infty$. Correspondingly, the degree of the qubit synchronization, $D \equiv 2\langle \mathrm{Tr}\rho\rho_d\rangle - 1$ (here ρ_d is the desired density matrix corresponding to ideal oscillations) is found to be $D = \exp(-\mathcal{C}/32F)$ and approaches 100% at $F \gg \mathcal{C}$.

The quality of the qubit oscillations synchronization decreases with the decrease of available feedback bandwidth τ_a^{-1} [Fig. 4(b)]. It also decreases when the qubit is dephased by an extra environment. For a weak dephasing rate γ_e we found numerically[19] a dependence $D_{max} \simeq 1 - 0.5d_e$ where $d_e \equiv \gamma_e/[(\Delta I)^2/4S_0]$. This means that the feedback loop can efficiently suppress the qubit dephasing due to coupling to the environment if this coupling is much weaker than the coupling to a nearly ideal detector.

Besides the linear feedback $H_{fb} = H(1 - F \times \Delta\phi)$, we have also studied the "direct" feedback $H_{fb}(t)/H - 1 = F\{2[I(t) - I_0]/\Delta I - \cos \Omega t\} \sin \Omega t$ and found that it can also provide a good phase synchronization if $F/\mathcal{C}$ is close to 1/4 [Fig. 4(b)]. The direct feedback is much easier for an experimental realization because it does not require solving the Bayesian equations (29)–(30) in real time.

8. BAYESIAN MEASUREMENT OF ENTANGLED QUBITS

The Bayesian formalism has been generalized to a continuous quantum measurement of entangled qubits in Ref..[38] Suppose a detector is coupled to N entangled qubits. In the "measurement" basis there are 2^N states $|i\rangle$ of the qubits which correspond to up to 2^N different dc current levels I_i of the detector (some of these currents can coincide, for example, if two or more qubits are coupled equally strong to the detector). It has been shown that the generalization of Eqs. (29)–(30) for this case is[28, 38]

$$\dot{\rho}_{ij} = \frac{-\mathrm{i}}{\hbar}[H_{qb}, \rho]_{ij} + \rho_{ij}\,\frac{1}{S_0}\sum_k \rho_{kk}\left[\left(I(t) - \frac{I_k + I_i}{2}\right)(I_i - I_k) + \left(I(t) - \frac{I_k + I_j}{2}\right)(I_j - I_k)\right] - \gamma_{ij}\rho_{ij}, \tag{43}$$

where the first term describes the unitary evolution due to the Hamiltonian of qubits H_{qb} and

$$\gamma_{ij} = (\eta^{-1} - 1)(I_i - I_j)^2/4S_0, \tag{44}$$

while Eq. (21) is replaced by

$$I(t) = \sum_i \rho_{ii}(t)I_i + \xi(t). \tag{45}$$

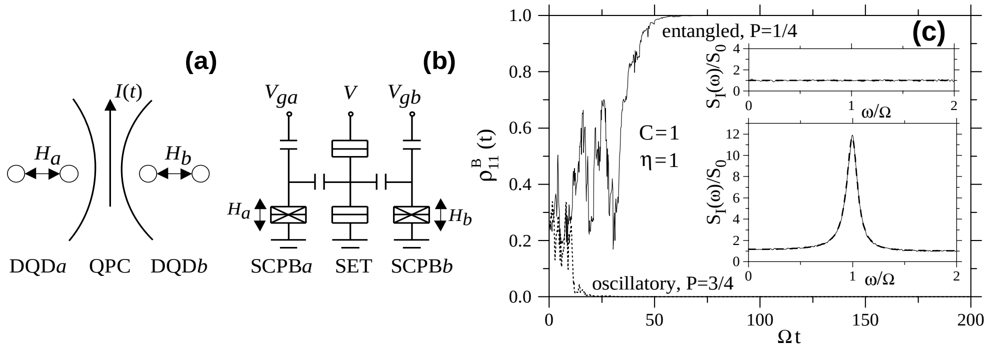

Figure 5. (a) Two qubits made of double quantum dots measured by an equally coupled quantum point contact. (b) Similar setup made of single-Cooper-pair boxes measured by a single-electron transistor. (c) Two Monte Carlo realizations of the two-qubit state evolution starting from the fully mixed state for a symmetric setup (ρ_{11}^B is the diagonal component of the two-qubit density matrix, corresponding to the Bell state $(|\uparrow_a\downarrow_b\rangle - |\downarrow_a\uparrow_b\rangle)/\sqrt{2}$). With probability 1/4 the qubits become fully entangled, $\rho_{11}^B \to 1$ ("spontaneous entanglement"); then the detector output is a pure noise (upper inset). With probability 3/4 the state is gradually collapsed into the orthogonal subspace, $\rho_{11}^B \to 0$; then the detector signal shows a spectral peak at the Rabi frequency Ω with the peak-to-pedestal ratio of 32/3. (From Ref..[39])

Notice that there is no mutual decoherence ($\gamma_{ij} = 0$) between states $|i\rangle$ and $|j\rangle$ even for a nonideal detector if the corresponding classical currents coincide, $I_i = I_j$. This is because the detector noise cannot destroy the coherence between states which are equally coupled to the detector.

Translating Eq. (43) from Stratonovich form into Itô form, we get

$$\dot{\rho}_{ij} = \frac{-i}{\hbar}[H_{qb}, \rho]_{ij} + \rho_{ij}\,\frac{1}{S_0}\left(I(t) - \sum_k \rho_{kk} I_k\right)\left(I_i + I_j - 2\sum_k \rho_{kk} I_k\right) - \left(\gamma_{ij} + \frac{(I_i - I_j)^2}{4S_0}\right)\rho_{ij}, \tag{46}$$

while in the ensemble-averaged equations the second term of Eq. (46) [containing $I(t)$] is averaged to zero.

These Bayesian equations have been applied in Ref.[39] to the analysis of a simple setup (Fig. 5) in which a detector is equally coupled to two similar qubits (both qubits are symmetric, $\varepsilon_a = \varepsilon_b = 0$, and do not interact directly with each other). An interesting effect has been found in the case when the Rabi frequencies $\Omega_a = 2H_a/\hbar$ and $\Omega_b = 2H_b/\hbar$ exactly coincide. Then there are two possible scenarios of the two-qubit evolution due to measurement, starting from a general mixed state. Either qubits become *fully entangled* collapsing into the Bell ("spin-0") state $(|\uparrow_a\downarrow_b\rangle - |\downarrow_a\uparrow_b\rangle)/\sqrt{2}$ (we call this process spontaneous entanglement), or the state falls into the orthogonal subspace of the two-qubit Hilbert space. Experimentally these two scenarios can be distinguished by different spectral density $S_I(\omega)$ of the detector current [Fig. 5(c)]. In the case of Bell state, $S_I(\omega)$ is just the flat noise S_0 of the detector because the signals from two qubits compensate each other, while in the other scenario $S_I(\omega)$ has a spectral peak at the Rabi frequency, which height is equal to $32\eta/3$ for a weakly coupled detector ($\mathcal{C}_a = \mathcal{C}_b < 1$).[39]

The probabilities of two scenarios are equal to the contributions of two subspaces in the initial state $\rho(0)$; for the case of fully mixed initial state they are equal to 1/4 and 3/4, respectively. The considered setup can obviously be used for a *preparation of the Bell state* without direct interaction between two qubits. Notice that if the state has collapsed into the orthogonal subspace, we can apply some noise which affects ε_a and/or ε_b and therefore mixes the two-qubit density matrix, and try the measurement again. In this way the probability $1 - (3/4)^M$ to obtain the Bell state can be made arbitrary close to 100% by allowing sufficiently large number M of attempts.

In actual experiment the symmetry of the setup cannot be made exact. In this case the Bell state and the oscillating state are not infinitely stable and there will be switching between them. The calculations show[39] that the switching rate $\Gamma_{B\to O}$ from the Bell state into the oscillating state is equal to $\Gamma_{B\to O} = (\Delta\Omega)^2/2\Gamma_d$ due to slightly different Rabi frequencies [$\Gamma_d = \eta^{-1}(\Delta I)^2/4S_0$], $\Gamma_{B\to O} = (\Delta\mathcal{C}/\mathcal{C})^2\Gamma_d/8$ due to slightly different coupling, and $\Gamma_{B\to O} = (\gamma_a + \gamma_b)/2$ due to an extra environment acting on two qubits separately. The rate of the return switching is 3 times smaller, $\Gamma_{O\to B} = \Gamma_{B\to O}/3$. Notice that in this case the averaged height of the Rabi spectral peak is equal to $8\eta S_0$, which is exactly twice as much as for a single qubit.

Even though such experiment on spontaneous entanglement is still extremely difficult for a realization, it should be noted that for the observation of the phenomenon the detector quantum efficiency η should not necessarily be close to 100%; it should only be large enough to allow distinguishing the Rabi spectral peak with the peak-to-pedestal ratio of $32\eta/3$.

9. CONCLUSION

We have discussed the basic derivation and some applications of the Bayesian approach to continuous quantum measurement of solid-state qubits. Even though this is a new subject for the solid-state community, many similar formalisms have been developed in other fields of quantum physics. Generally, this type of approach which takes into account the measurement outcome, is usually called selective or conditional quantum measurement. However, there is a rather broad variety of formalisms and their interpretations within the approach (for reviews see, e.g. Refs.[3, 40]). Some of key words related to this subject are: quantum trajectories, quantum state diffusion, quantum jumps, weak measurements, stochastic evolution of the wavefunction, stochastic Schrödinger equation, complex Hamiltonian, restricted path integral, quantum Bayes theorem, etc. The approach of conditional quantum measurements is relatively well developed in quantum optics. In particular, the optical quantum feedback control has been well studied theoretically (see, e.g.[41–45]) and was recently realized experimentally.[46] In the subject of continuous quantum measurement of single systems the quantum optics is well ahead of the solid-state physics. However, the interest to this problem in the solid-state community is rapidly growing, and the technology is coming to the level when experimental studies become possible.

We have discussed in this paper two solid-state experiments which seem to be realizable today. First, it is the measurement of the spectral peak of the detector current at the Rabi frequency of qubit oscillations and comparison with the theoretical prediction that the peak-to-pedestal ratio is equal to 4 in the best case (actually, this behavior is consistent with a recent experiment[47] on a single-spin precession; however, the problem is still not completely clear). Second, the Bell-type correlation experiment with one qubit measured by two detectors would be able to verify that the qubit state remains pure during the whole measurement process and show the possibility of monitoring the qubit evolution precisely. This would be the first step towards realization of the quantum feedback control of solid-state qubits. A continuous monitoring of entangled qubits would be another very interesting experiment. While in quantum optics it took about 10 years from the development of the theory of continuous conditional quantum measurements to the first successful experiments, we hope that this period will be significantly shorter in the solid-state physics and engineering.

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