Undoing a Weak Quantum Measurement of a Solid-State Qubit

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(Received 6 July 2006; published 20 October 2006)

We propose an experiment which demonstrates the undoing of a weak continuous measurement of a solid-state qubit, so that any unknown initial state is fully restored. The undoing procedure has only a finite probability of success because of the nonunitary nature of quantum measurement, though it is accompanied by a clear experimental indication of whether or not the undoing has been successful. The probability of success decreases with increasing strength of the measurement, reaching zero for a traditional projective measurement. Measurement undoing (''quantum undemolition'') may be interpreted as a kind of quantum eraser, in which the information obtained from the first measurement is erased by the second measurement, which is an essential part of the undoing procedure. The experiment can be realized using quantum dot (charge) or superconducting (phase) qubits.

DOI: 10.1103/PhysRevLett.97.166805 PACS numbers: 73.23.−b, 03.65.Ta, 03.67.Lx, 85.25.Cp

A careless scientist accidently turns on a quantum detector, disturbing a precious, unknown, quantum state. Dismayed by this event, he desperately asks if there is a way to get the state back. Is it possible to *undo* a quantum measurement? According to the traditional theory of projective quantum measurement [1], the answer is no: Wave function collapse is irreversible; the original state is gone forever and is impossible to resuscitate. However, as will be presently discussed, the situation is different for weak quantum measurements [2–5]. It is possible to fully restore any unknown premeasured state, though with a probability less than unity. Such undoing of the measurement disturbance [which we will also refer to as a quantum undemolition (QUD) measurement] can be accomplished by making an additional weak measurement, which ''erases'' the information obtained from the first measurement (somewhat similar to the quantum eraser of Scully and Drühl [6]). The success of the undoing procedure is indicated by observing a certain result of the second measurement, in which case the unknown premeasured state is fully restored. The probability of successfully undoing the quantum measurement decreases with increasing strength of the measurement, tending to zero for a projective measurement.

The possibility of physically undoing (or reversing) a quantum measurement has been previously discussed by Koashi and Ueda [7], who have proposed a quantum optics photon-counting implementation of the idea, using the Kerr effect. Reversible measurement has also been discussed by others [8,9] (see also the closely related articles [10]), though mainly from a more formal perspective. In this Letter, we investigate the undoing of *continuous* weak measurements, particularly applied to solid-state qubits. We first consider a quantum double dot qubit, measured by a quantum point contact, a system of extensive experimental investigation [11]. For this system, we discuss how to practically undo the measurement and calculate the undoing probability, as well as the mean undoing time. The second system we consider is a superconducting ''phase'' qubit [12], measured by a nearby SQUID. Coherent nonunitary evolution due to measurement has recently been experimentally verified in this system [13]. We describe the undoing procedure for the phase qubit and calculate the undoing probability, obtaining a result similar to the quantum dot system. The undoing procedure described for the phase qubit is only slightly more complicated than the experiment already done, providing a promising candidate for experimental verification. We briefly discuss the general theory of QUD measurement and show that our specific results maximize the general undoing probability, thus constituting ideal measurement reversal.

Charge qubit.—A double-quantum-dot (DQD) qubit, measured continuously by a symmetric quantum point contact (QPC) [14] [Fig. 1(a)], has been extensively studied in earlier papers. The measurement is characterized by the average currents I_1 and I_2 corresponding to the qubit

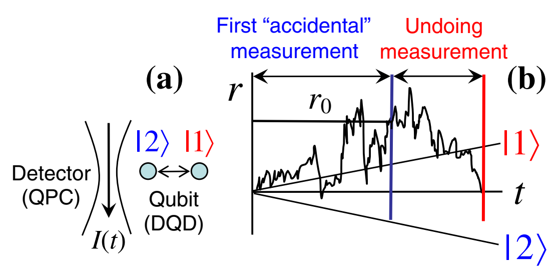

FIG. 1 (color online). (a) Schematic of a DQD charge qubit continuously measured by a QPC. (b) Illustration of the measurement undoing procedure for the charge qubit. The slanted lines indicate the detector output in the absence of noise, if the qubit is in state $|1\rangle$ or $|2\rangle$. The initial accidental measurement can be undone by waiting until the stochastic measurement result $r(t) = (\Delta I/S_I) \int_0^t [I(t') - I_0]dt'$ crosses the origin.

states $|1\rangle$ and $|2\rangle$ and by the shot noise spectral density S_I [15]. We treat the additive detector shot noise as a Gaussian, white, stochastic process and assume the detector is in the weakly responding regime $|\Delta I| \ll I_0$, where $\Delta I = I_1 - I_2$ and $I_0 = (I_1 + I_2)/2$, with QPC voltage bias larger than all other energy scales, so that the measurement process can be described by the quantum Bayesian formalism [4].

Let us first assume that there is no qubit Hamiltonian evolution [this can also be effectively done using "kicked" quantum nondemolition (QND) measurements [16]]. As was shown in Ref. [4], the QPC is an ideal quantum detector (which does not decohere the measured qubit), so that the evolution of the qubit density matrix ρ due to continuous measurement preserves the quantity $\rho_{12}/\sqrt{\rho_{11}\rho_{22}}$, while the diagonal matrix elements are normalized at all times and evolve according to the classical Bayes rule:

$$\frac{\rho_{11}(t)}{\rho_{22}(t)} = \frac{\rho_{11}(0)\exp[-(\bar{I}(t) - I_1)^2 t/S_I]}{\rho_{22}(0)\exp[-(\bar{I}(t) - I_2)^2 t/S_I]} = \frac{\rho_{11}(0)}{\rho_{22}(0)} e^{2r(t)}, \tag{1}$$

where $\bar{I}(t) = [\int_0^t I(t')dt']/t$ and we define the *measurement result* as $r(t) = [\bar{I}(t) - I_0]t\Delta I/S_I$. For times much longer than the "measurement time" $T_m = 2S_I/(\Delta I)^2$ (the time scale required to obtain a signal-to-noise ratio of 1), the average current $\bar{I}$ tends to either I_1 or I_2 because the probability density $P(\bar{I})$ of a particular $\bar{I}$ is

$$P(\bar{I}) = \sum_{i=1,2} \rho_{ii}(0)\sqrt{t/\pi S_I}\exp[-(\bar{I} - I_i)^2 t/S_I]. \tag{2}$$

Therefore, $r(t)$ tends to $\pm\infty$, continuously collapsing the state to either $|1\rangle$ (for $r \to \infty$) or $|2\rangle$ (for $r \to -\infty$). Critical to what follows, notice that, if $r(t) = 0$ at some moment t, then the qubit state becomes exactly the same as it was initially, $\rho(t) = \rho(0)$. This curious fact corresponds to an equal likelihood of the states $|1\rangle$ and $|2\rangle$ and, therefore, provides no information about the qubit.

Measurement undoing for the charge qubit.—Let the outcome of the "accidental" first measurement be r_0. The previous "no information" observation suggests the following strategy: Continue measuring, with the hope that after some time t the stochastic result of the second measurement $r_u(t)$ becomes equal to $-r_0$, so the total result $r(t) = r_0 + r_u(t)$ is zero, and therefore the initial qubit state is fully restored. If this happens, the measuring device is immediately switched off and the undoing procedure is successful [Fig. 1(b)]. However, $r(t)$ may never cross the origin, and then the undoing attempt fails.

The success probability P_s for this procedure may be calculated by noticing that the nondiagonal matrix elements of ρ do not enter the probability of the detector output (2) (this is true only in the case of a zero or QND-eliminated qubit Hamiltonian), and therefore the calculation is identical for a classical bit with probabilities $P_{1,2} = \rho_{11,22}(0)$ of being in state "1" or "2." These probabilities should be updated (using the classical Bayes formula) with the information obtained from result r_0: $\tilde{P}_1 = P_1 e^{r_0}/[P_1 e^{r_0} + P_2 e^{-r_0}]$, $\tilde{P}_2 = 1 - \tilde{P}_1$. Assume for definiteness $r_0 > 0$. We will now calculate the probability that the random variable $r(t)$ crosses the origin at least once, known in stochastic physics as a first passage process. It follows from (2) that both cases may be described by two different random walks with the initial condition $r = r_0$ at $t = 0$, described by the Green function solution of the two Fokker-Planck equations

$$\partial_t G_i(r, t) = -v_i \partial_r G_i + D\partial_r^2 G_i + \delta(r - r_0)\delta(t), \tag{3}$$

supplemented with an absorbing boundary condition at the origin, where $D = 1/(2T_m)$ is the diffusion coefficient, and $v_i = (-1)^{i+1}/T_m$ are the two different drift velocities, depending on the bit state. Equation (3) may be solved with standard methods [17], and, from the solution, the first passage time distribution $P_{\rm fpt}^{(i)}(t)$ is found from the probability current flux at the origin

$$P_{\rm fpt}^{(i)}(t) = \frac{r_0}{\sqrt{4\pi D t^3}} \exp[-(r_0 + v_i t)^2/(4Dt)]. \tag{4}$$

The probability that $r = 0$ is ever crossed is found by integrating (4) over all positive time to obtain $P_{c,1} = \exp(-v_1 r_0/D) = \exp(-2r_0)$ for the crossing probability if $i = 1$ and $P_{c,2} = 1$ if $i = 2$. This result is intuitive because, starting at $r_0 > 0$, a negative drift velocity must cause an eventual crossing, while a positive drift velocity can only occasionally be beaten by the noise term. The mean first passage time may also be calculated from (4) to obtain $t_{c,i} = r_0/|v_i|$, averaging only over successful attempts. Analogous results for $r_0 < 0$ are straightforward.

Combining these results, the probability $P_s = \tilde{P}_1 P_{c,1} + \tilde{P}_2 P_{c,2}$ for a successful quantum undemolition measurement is

$$P_s = e^{-|r_0|}/[e^{r_0}\rho_{11}(0) + e^{-r_0}\rho_{22}(0)], \tag{5}$$

and the mean waiting time $T_{\rm undo} = \tilde{P}_1 t_{c,1} + \tilde{P}_2 t_{c,2}$ until the measurement is undone is

$$T_{\rm undo} = T_m |r_0|. \tag{6}$$

Several comments are in order about the main results (5) and (6). (i) The probability of success P_s given by (5) becomes very small for $|r_0| \gg 1$ (when the measurement result indicates a particular qubit state with good confidence), eventually becoming $P_s = 0$ for a projective measurement, recovering the traditional statement of irreversibility in this limiting case. (ii) In the important special case when the initial state is pure, the state remains pure during the entire process. (iii) Although the starting point of the derivation (1) assumes that the initial state $\rho(0)$ is known, result (5) also applies to the situation where the initial state is *unknown*. (iv) Averaging (5) over different initial states $\rho^k(0)$ with corresponding probabilities $\mathcal{P}_k$ leads to the same result (5), just with $\rho(0)$ replaced by the averaged density matrix $\sum_k \mathcal{P}_k \rho^k(0)$. (v) If the qubit is entangled with other qubits, the QUD measurement restores the initial entangled state; the density matrix in (5) in this case can be obtained by tracing over all other qubits. (vi) After the initial measurement, the state evolution is given as a one-to-one map, whose inverse is known. However, the nonunitarity of the inverse makes its realization impossible via Hamiltonian evolution; therefore, a QUD measurement must be probabilistic. (vii) The undoing probability averaged over the result r_0, $P_{\rm av} = 1 - {\rm erf}[\sqrt{t/(2T_m)}]$, depends on the "strength" t/T_m of the first measurement but not on the initial state.

Phase qubit.—The second explicit example of erasing information and undoing a quantum measurement is for a superconducting phase qubit [12,13]. The system (similar to the "flux" qubit) is comprised of a superconducting loop interrupted by one Josephson junction [Fig. 2(a)], which is controlled by an external flux ϕ_e. Qubit states $|1\rangle$ and $|2\rangle$ [Fig. 2(b)] correspond to the two lowest states in a quantum well with potential energy $V(\phi)$, where ϕ is the superconducting phase difference across the junction (for consistency with the previous example, we do not use the more traditional notation $|0\rangle$ and $|1\rangle$). The qubit is measured by lowering the barrier (which is controlled by ϕ_e), so that the upper state $|2\rangle$ tunnels into the continuum with rate Γ, while state $|1\rangle$ does not tunnel out. The tunneling event is sensed by a two-junction detector SQUID inductively coupled to the qubit [Fig. 2(a)].

For sufficiently long tunneling time t, $\Gamma t \gg 1$, the measurement corresponds to the usual collapse: The qubit state is either projected onto the lower state $|1\rangle$ (if no tunneling is recorded) or destroyed (if tunneling happens). However, if the barrier is raised after a finite time $t \sim \Gamma^{-1}$, the measurement is weak: The qubit state is still destroyed if tunneling happens, while in the case of no tunneling (a null-result measurement) the qubit density matrix evolves in the rotating frame as [13,18]

$$\frac{\rho_{11}(t)}{\rho_{22}(t)} = \frac{\rho_{11}(0)}{\rho_{22}(0)e^{-\Gamma t}}, \qquad \frac{\rho_{12}(t)}{\sqrt{\rho_{11}(t)\rho_{22}(t)}} = \frac{\rho_{12}(0)e^{-i\varphi(t)}}{\sqrt{\rho_{11}(0)\rho_{22}(0)}}, \tag{7}$$

where the phase $\varphi(t)$ accumulates because of the change of energy difference between states $|1\rangle$ and $|2\rangle$ when the barrier is lowered by changing ϕ_e. Notice that, except for the effect of the extra phase $\varphi(t)$, the qubit evolution (7) is similar to the qubit evolution in the previous example; in particular, it also represents an ideal measurement which does not decohere the qubit and has a clear Bayesian interpretation. Formally, the evolution (7) corresponds to the measurement result $r = \Gamma t/2$ in Eq. (1). The coherent nonunitary evolution (7) has been experimentally verified in Ref. [13] using tomography of the postmeasurement state (in Ref. [13], the product Γt was actually varied by changing the tunneling rate Γ, while keeping the duration t constant).

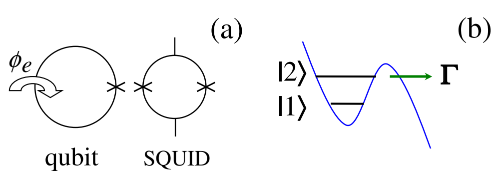

FIG. 2 (color online). (a) Schematic of a phase qubit controlled by an external flux ϕ_e and inductively coupled to the detector SQUID. (b) Energy profile $V(\phi)$ with quantized levels representing the qubit states. The tunneling event is sensed by the SQUID.

Measurement undoing for the phase qubit.—A slight modification of the experiment [13] can be used to demonstrate measurement undoing. Suppose the tunneling event did not happen during the first weak measurement, so the evolution (7) has occurred. The undoing of this measurement consists of three steps: (i) Exchange the amplitudes of states $|1\rangle$ and $|2\rangle$ by the application of a π pulse, (ii) perform another weak measurement, identical to the first measurement, and (iii) apply a second π pulse. If the tunneling event did not occur during the second measurement, then the information about the initial qubit state is erased (both basis states have equal likelihood for two null-result measurements). Correspondingly, according to Eq. (7) (which is applied for the second time with exchanged indices $1 \leftrightarrow 2$), any initial qubit state is fully restored (notice that the phase φ is also canceled).

The success probability P_s for the undoing procedure is just the probability that the tunneling does not happen during the second measurement. If we start with the qubit state $\rho(0)$, the state after the first measurement is given by Eq. (7). After the π pulse, the occupation of the upper state is $\rho'_{22} = \rho_{11}(0)/[\rho_{11}(0) + \rho_{22}(0)e^{-\Gamma t}]$, so the success probability $P_s = 1 - \rho'_{22}(1 - e^{-\Gamma t})$ can be expressed as

$$P_s = e^{-\Gamma t}/[\rho_{11}(0) + e^{-\Gamma t}\rho_{22}(0)], \tag{8}$$

which formally coincides with Eq. (5) for $r = \Gamma t/2$. While measurement undoing is most important for an unknown state, in the demonstration experiment the initial state can be known, and tomography of the final state can be used to check that it is identical to the initial state.

General theory of measurement undoing.—Applying positive operator-valued measure formalism [2], we can describe a general quantum measurement with result r by a linear operator M_r, so that for an initial state ρ the probability of result r is $P_r(\rho) = {\rm Tr}(E_r\rho)$, where $E_r = M_r^\dagger M_r$, and the state after measurement is $\tilde\rho = M_r\rho M_r^\dagger/{\rm Tr}(E_r\rho)$. Here E_r is a positive Hermitian operator, obeying the completeness relation $\sum_r E_r = 1$. In order to undo this measurement, we should apply the inverse operation characterized by $L_r = CM_r^{-1}$, where C is a complex number. Such an operation is physical (i.e., can be realized by a second measurement yielding a "lucky" result) only if all eigenvalues of $L_r^\dagger L_r$ are not larger than 1 (otherwise, completeness cannot be satisfied), which leads to the upper bound $|C|^2 \le \min_i p_i$, where $\{p_i\}$ is the set of eigenvalues of E_r. Therefore, the probability $P_s = {\rm Tr}(L_r^\dagger L_r\tilde\rho)$ of the lucky result corresponding to L_r is bounded by $(\min_i p_i)/P_r(\rho)$. Finally, recalling that $\{p_i\}$ are probabilities of the result r for eigenvectors of E_r, we find the upper bound for the probability of successful undoing (similar to the result of Ref. [7]):

$$P_s \leq (\min P_r)/P_r(\rho), \tag{9}$$

where ρ is the initial state and $\min P_r$ is the probability of the result r minimized over all possible initial states. Notice that averaging of P_s over the result r makes it independent of the initial state: $P_{\rm av} \leq \sum_r (\min P_r)$.

Let us compare the general upper bound (9) for P_s with the results (5) and (8) of the two previous examples. For the QPC measurement of the DQD qubit described by Eq. (1), the operator E_r is diagonal in the measurement basis $|1\rangle$, $|2\rangle$ and has matrix elements $p_i = (\pi S_I/t)^{-1/2} \exp[-(\bar{I} - I_i)^2 t/S_I] d\bar{I}$, $i = 1, 2$, related to the probability densities of the continuous variable $\bar{I}$. Then the upper bound (9) becomes $\min(p_1, p_2)/(p_1\rho_{11} + p_2\rho_{22})$, which coincides with Eq. (5) because $p_1/p_2 = e^{2r}$. We conclude that our undoing strategy is optimal, since the upper bound (9) is reached. For the example of phase qubit measurement, the operator E_r corresponding to the null-result measurement (no tunneling) is also diagonal in the measurement basis $|1\rangle$, $|2\rangle$ and has matrix elements 1 and $e^{-\Gamma t}$, respectively. Again, P_s given by (8) reaches the upper bound (9), thus confirming the optimality of the analyzed undoing procedure.

Explicit general procedure of measurement undoing.—We briefly discuss a procedure to undo (in principle) an arbitrary one-to-one measurement M_r for any number N of entangled charge qubits, using unitary rotations and measurement by a QPC with an extremely strong nonlinearity, so that tunneling (with rate γ) occurs in the QPC only when all qubits are in the state $|1\rangle$. For simplicity, assume $M_r = \sqrt{E_r}$ (the generalization is trivial). In the basis of 2^N vectors $|i\rangle$ diagonalizing E_r, the desired undoing operator $L_r = \sqrt{\min_j p_j} M_r^{-1}$ (see above) is also diagonal: $L_{r,ii} = \sqrt{(\min_j p_j)/p_i}$. It can be realized in 2^N steps. The ith step consists of a unitary rotation of the vector $|i\rangle$ into the state $|11\ldots1\rangle$, measurement by the QPC for duration $\tau_i = -\gamma^{-1} \ln L_{r,ii}^2$, and the reverse unitary rotation. In each step, the no-tunneling result corresponds to the measurement operator, which is almost unity, except for diagonal matrix element $e^{-\gamma\tau_i/2} = L_{r,ii}$ for the vector $|i\rangle$. The measurement undoing procedure is successful if no tunneling occurred in all steps. The corresponding success probability reaches the upper bound (9).

Undoing continuous measurement of an evolving charge qubit.—In our first example, we have assumed for simplicity that the qubit is not undergoing Hamiltonian evolution during the measurement process to be undone. We briefly note that a QUD measurement is also possible when Hamiltonian evolution is included. In this case, the qubit evolution is described by the Bayesian equations [4], which can be used to find the operator M_r from the measurement record $I(t)$ (the non-normalized version of the Bayesian equations is more appropriate for this purpose). Then the desired undoing operator L_r can be realized in three steps, corresponding to the singular value decomposition of L_r: unitary rotation, continuous measurement by QPC (with internal qubit dynamics turned off), and one more unitary rotation. (The details of this calculation and the explicit undoing procedure will be presented elsewhere.)

The authors thank Alexei Kitaev for useful discussions and Jonathan Dowling for suggesting the term quantum undemolition measurement. This work was supported by NSA and DTO under ARO Grant No. W911NF-04-1-0204 (A. N. K.), AFRL Grant No. F30602-01-1-0594, AFOSR Grant No. FA9550-04-1-0206, and TITF Grant No. 2001-055 (A. N. J.).

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