Entanglement-Assisted Weak Value Amplification

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Large weak values have been used to amplify the sensitivity of a linear response signal for detecting changes in a small parameter, which has also enabled a simple method for precise parameter estimation. However, producing a large weak value requires a low postselection probability for an ancilla degree of freedom, which limits the utility of the technique. We propose an improvement to this method that uses entanglement to increase the efficiency. We show that by entangling and postselecting n ancillas, the postselection probability can be increased by a factor of n while keeping the weak value fixed (compared to n uncorrelated attempts with one ancilla), which is the optimal scaling with n that is expected from quantum metrology. Furthermore, we show the surprising result that the quantum Fisher information about the detected parameter can be almost entirely preserved in the postselected state, which allows the sensitive estimation to approximately saturate the relevant quantum Cramér-Rao bound. To illustrate this protocol we provide simple quantum circuits that can be implemented using current experimental realizations of three entangled qubits.

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Weak value amplification (WVA) is an enhanced detection scheme that was first suggested by Aharonov, Albert, and Vaidman [1]. (See [2] and [3] for recent reviews.) The scheme exploits the fact that postselecting the weak measurement of an ancilla can produce a linear detector response with an anomalously high sensitivity to small changes in an interaction parameter. The sensitivity arises from coherent "superoscillatory" interference in the ancilla [4], which is controlled by the choice of preparation and postselection of the ancilla. The price that one pays for this increase in sensitivity is a reduction in the potential signal (and thus the potential precision of any estimation) due to the postselection process [5–10]. Nevertheless, by using this technique one can still consistently recover a large fraction of the maximum obtainable signal in a relatively simple way [11,12]. The relevant information is effectively concentrated into the small set of rarely postselected events [13].

A growing number of experiments have successfully used WVA to precisely estimate a diverse set of small physical parameters, including beam deflection (to picoradian resolution) [14–21], frequency shifts [22], phase shifts [23,24], angular shifts [25], temporal shifts [26], velocity shifts [27], and temperature shifts [28]. More experimental schemes have also been proposed [29–36]. These experimental results have shown remarkable resilience to the addition of temporally correlated noise, such as beam jitter [13]. Moreover, some of these experiments have reported precision near the standard quantum limit, which is surprising due to the intrinsic postselection loss. These observations have prompted the question of whether WVA can be improved further by combining it with other metrology techniques. One such improvement that has been proposed is to recycle the events that were discarded by the postselection back into the measurement [37]. Another investigation has shown that in certain cases it may be possible to achieve precision near the optimal Heisenberg limit with seemingly classical resources [38].

In this Letter, we supplement these efforts by asking whether adding quantum resources can also improve the efficiency of WVA. We find that using entangled ancilla preparations and postselections does indeed provide such an improvement. The postselection probability can be increased while preserving the amplification factor, which decreases the number of discarded events required to achieve the same sensitivity. Alternatively, the amplification can be enhanced directly while preserving the same postselection probability. These improvements scale optimally as the number of entangled ancillas increases; however, using even a small number of entangled ancillas provides a notable improvement. Moreover, we show the nontrivial result that despite the introduction of entanglement and a postselection that discards most of the collected data from the increased number of possible outcomes, nearly all the quantum Fisher information about the estimated parameter can still be concentrated into the rarely postselected state. As such, the simple linear estimator can nearly saturate the optimal quantum Cramér-Rao bound expected from the initial state.

As a concrete proposal that demonstrates this optimal scaling, we consider using n entangled ancilla qubits [39] to estimate a small controlled phase applied to a meter qubit. Since recent experiments with optical [40], solid-state [41,42] and NMR [43] systems have already verified

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the weak value (WV) effect using one or two qubits, we provide a simple set of similar quantum circuits that can be implemented experimentally using only three qubits.

Weak value amplification.—As a brief review, recall that for a typical WVA experiment one uses an interaction Hamiltonian of the form

$$\hat{H}_{\rm int} = \hbar g \hat{A} \otimes \hat{F} \delta(t - t_0), \tag{1}$$

where $\hat{A}$ is an ancilla observable, $\hat{F}$ is a meter observable, and g is the small coupling parameter that one would like to estimate. The time factor $\delta(t - t_0)$ indicates that the interaction between the ancilla and the meter is impulsive, i.e., happening on a much faster time scale than the natural evolution of both the ancilla and the meter. We leave the dimension of $\hat{A}$ arbitrary for our discussion.

An experimenter prepares the meter in a pure state $|\phi\rangle$ and the ancilla in a pure initial state $|\Psi_i\rangle$, then weakly couples them using the interaction Hamiltonian of Eq. (1), and then postselects the ancilla into a pure final state $|\Psi_f\rangle$, discarding the events where the postselection fails. This procedure effectively prepares an *enhanced* meter state that includes the effect of the ancilla $|\phi'\rangle = \hat{M}|\phi\rangle/||\hat{M}|\phi\rangle||$, which we write here in terms of a Kraus operator $\hat{M} = \langle\Psi_f|\exp(-ig\hat{A} \otimes \hat{F})|\Psi_i\rangle$. Averaging a meter observable $\hat{R}$ using this updated meter state yields $\langle\hat{R}\rangle_{|\phi'\rangle} = \langle\phi|\hat{M}^\dagger \hat{R}\, \hat{M}\,|\phi\rangle/\langle\phi|\hat{M}^\dagger\hat{M}|\phi\rangle$.

For small g, this observable average is well approximated by the following second-order expansion [2,44]:

$$\langle\hat{R}\rangle_{|\phi'\rangle} \approx \frac{2g{\rm Im}(\alpha A_w) + g^2\beta|A_w|^2}{1 + g^2\sigma^2|A_w|^2}, \tag{2}$$

where $\alpha = \langle\hat{R}\,\hat{F}\rangle_{|\phi\rangle}$, $\beta = \langle\hat{F}\,\hat{R}\,\hat{F}\rangle_{|\phi\rangle}$, and $\sigma^2 = \langle\hat{F}^2\rangle_{|\phi\rangle}$ are correlation parameters that are fixed by the choice of meter observables and the initial meter state, while

$$A_w = \frac{\langle\Psi_f|\hat{A}|\Psi_i\rangle}{\langle\Psi_f|\Psi_i\rangle} \tag{3}$$

is a complex WV controlled by the ancilla [1]. Note that we have assumed that the initial meter state is unbiased $\langle\hat{F}\rangle_{|\phi\rangle} = \langle\hat{R}\rangle_{|\phi\rangle} = 0$ to obtain the best response.

Most WVA experiments operate in the linear response regime where the second-order terms in Eq. (2) can be neglected, which produces [45]

$$\langle\hat{R}\rangle_{|\phi'\rangle} \approx 2g[{\rm Re}A_w{\rm Im}\alpha + {\rm Im}A_w{\rm Re}\alpha]. \tag{4}$$

This linear relation shows how a large ancilla WV can amplify the sensitivity of the meter for detecting small changes in g.

For concreteness, we consider a reference case when the meter is a qubit. State-of-the-art quantum computing technologies can already realize single qubit unitary gates and two-qubit CNOT and controlled rotation gates with high fidelity (e.g., [40–43,46–49]), so this example can be readily tested in the laboratory. The meter qubit is prepared in the state $|\phi\rangle = |+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$. The Pauli Z operator $\hat{\sigma}_z = \hat{F} = \hat{R}$ will serve as both meter observables. These choices fix the constants $\alpha = 1, \beta = 0$, and $\sigma^2 = 1$ in Eq. (2), yielding the meter response

$$\langle\hat{\sigma}_z\rangle_{|+'\rangle} \approx \frac{2g{\rm Im}A_w}{1 + g^2|A_w|^2}. \tag{5}$$

The nonlinearity in the denominator regularizes the detector response, placing a strict upper bound of $g|A_w| < 1$ on the magnitudes that are useful for WVA. The meter has a linear response in a more restricted range of roughly $g|A_w| < 1/10$. In practice, one typically assumes that $g|A_w| \ll 1$.

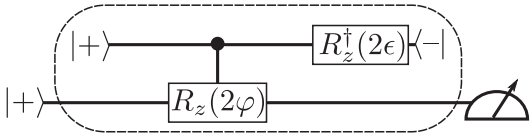

FIG. 1. Quantum circuit that simulates the WVA of a small parameter φ. A meter qubit is prepared in the state $|+\rangle = R_y(\pi/2)|0\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$. An ancilla qubit is prepared in the same state $|\Psi_i\rangle = |+\rangle$. The ancilla is used as a control for a Z rotation $R_z(2\varphi)$ of the meter, which simulates the unitary $\hat{U} = \exp(-i\varphi\hat{A} \otimes \hat{\sigma}_z/2)$ with $\hat{A} = \hat{\sigma}_z$. The ancilla is then postselected in the nearly orthogonal state $\langle\Psi_f| = \langle -|R_z^\dagger(2\epsilon) = \langle 0|R_y^\dagger(-\pi/2)R_z^\dagger(2\epsilon) = (\langle 0|e^{i\epsilon} - \langle 1|e^{-i\epsilon})/\sqrt{2}$ with probability $P_s \approx \epsilon^2$ by performing two rotations, measuring in the Z basis, and keeping only the $\langle 0|$ events. Finally, the meter qubit is measured in the Z basis, which yields the linear response $\langle\hat{\sigma}_z\rangle_{+'} \approx \varphi{\rm Im}A_w$ amplified by the weak value $A_w \approx i/\epsilon$. The probability for a single success after n attempts, $P_s^{(n)} = 1 - (1 - P_s)^n \approx n\epsilon^2$, is approximately linear in n.

As detailed in Fig. 1, we couple a single ancilla qubit to the meter using a controlled-Z rotation by a small angle 2φ, which sets $g = \varphi/2$ and $\hat{A} = \hat{\sigma}_z$. The ancilla is initialized in the state $|\Psi_i\rangle = |+\rangle$ and postselected in the nearly orthogonal state $|\Psi_f\rangle = R_z(2\epsilon)|-\rangle = (e^{-i\epsilon}|0\rangle - e^{i\epsilon}|1\rangle)/\sqrt{2}$ with a probability $P_s = \sin^2(\epsilon) \approx \epsilon^2$, which produces the WV $A_w = i\cot(\epsilon) \approx i/\epsilon$. The offset angle ϵ of the postselection must satisfy $\varphi/2 < \epsilon < \pi/4$ for amplification, and $5\varphi < \epsilon < \pi/4$ for linear response.

Postselection probability.—While a large WV can effectively amplify the small parameter g in the linear response, it also has a shortcoming of low efficiency. When A_w is large, Eq. (3) indicates that $\langle\Psi_f|\Psi_i\rangle$ must be small. This implies that the ancilla postselection probability is also small, since it approximates

$$P_s \approx |\langle\Psi_f|\Psi_i\rangle|^2 \tag{6}$$

for small g. Thus, the larger A_w is, the less likely it becomes to prepare the enhanced meter state $|\phi'\rangle$.

We now show that adding quantum resources to the ancilla can improve this efficiency while keeping the amplification factor of the weak value A_w the same. Specifically, we consider coupling n entangled ancillas to the meter simultaneously. To make a fair comparison with the uncorrelated case, the probability of successfully postselecting n entangled ancillas once should show an improvement over the probability of successfully postselecting a single ancilla once after n independent attempts. The latter probability has linear scaling in n when P_s is small

$$P_s^{(n)} = 1 - (1 - P_s)^n \approx nP_s. \tag{7}$$

We will see that entangled ancillas can achieve *quadratic* scaling with n, improving the efficiency by a factor of n.

To show this improvement, we couple the meter to n identical single-ancilla observables $\hat{a}$ using the interaction in Eq. (1), which effectively couples the meter to a single joint ancilla observable

$$\hat{A} = \hat{A}_1 + \cdots + \hat{A}_n, \tag{8}$$

where $\hat{A}_k = \hat{1} \otimes \cdots \hat{a} \cdots \otimes \hat{1}$ is shorthand for the observable $\hat{a}$ of the kth ancilla. Notably the minimum and maximum eigenvalues of this joint observable, $\Lambda_{\min(\max)} = n\lambda_{\min(\max)}$, are determined by the eigenvalues of $\hat{a}$. Similarly, the corresponding eigenstates are product states of the eigenstates of $\hat{a}$: $|\Lambda_{\min(\max)}\rangle = |\lambda_{\min(\max)}\rangle^{\otimes n}$. The n ancillas will be collectively prepared in a joint state $|\Psi_i\rangle$ and then postselected in a joint state $|\Psi_f\rangle$ to produce a joint WVA factor A_w, just as in Eq. (3). An example circuit that implements this procedure with qubits is illustrated in Fig. 2.

The ability to improve the postselection efficiency hinges upon the fact that there can be different choices of $|\Psi_i\rangle$ and $|\Psi_f\rangle$ that will produce the same WV A_w. However, these different choices will generally produce different postselection probabilities. Therefore, among these different choices of joint preparations and postselections there exists an optimal choice that maximizes the postselection probability.

We find this optimum in two steps. First, we maximize the postselection probability over all possible postselections $|\Psi_f\rangle$ while keeping the WV A_w and the preparation $|\Psi_i\rangle$ fixed. Second, we maximize this result over all preparations $|\Psi_i\rangle$.

To perform the first maximization, note that Eq. (3) implies $\langle\Psi_f|(\hat{A} - A_w)|\Psi_i\rangle = 0$, so $|\Psi_f\rangle$ must be orthogonal to $(\hat{A} - A_w)|\Psi_i\rangle$. This gives a constraint on the possible postselections $|\Psi_f\rangle$, so the maximization of P_S in Eq. (6) should be taken over the subspace $\mathcal{V}^\perp$ orthogonal to $(\hat{A} - A_w)|\Psi_i\rangle$. As shown in the Supplemental Material [50], the result of this maximization is

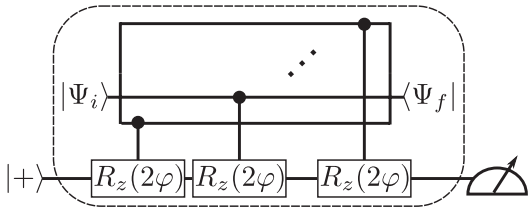

FIG. 2. Quantum circuit that simulates the entanglement-assisted WVA of a small parameter φ. As in Fig. 1, a meter qubit is prepared in the state $|+\rangle$, while n ancilla qubits are prepared in a entangled state $|\Psi_i\rangle$. Each ancilla is then used as a control for a Z rotation $R_z(2\varphi)$ of the meter, simulating the unitary $\hat{U} = \exp(-i\varphi\hat{A} \otimes \hat{\sigma}_z/2)$ with $\hat{A}$ being the sum of ancilla observables $\hat{\sigma}_z$. The ancillas are then postselected in an entangled state $|\Psi_f\rangle$, and the meter qubit is measured in the Z basis, yielding the linear response $\langle\hat{\sigma}_z\rangle_{+'} \approx \varphi \mathrm{Im} A_w$ amplified by a joint WV A_w.

$$\max_{|\Psi_f\rangle \in \mathcal{V}^\perp} P_s \approx \frac{\mathrm{Var}(\hat{A})_{|\Psi_i\rangle}}{|A_w|^2}, \tag{9}$$

where $\mathrm{Var}(\hat{A})_{|\Psi_i\rangle} = \langle\Psi_i|\hat{A}^2|\Psi_i\rangle - [\langle\Psi_i|\hat{A}|\Psi_i\rangle]^2$ is the variance of $\hat{A}$ in the initial state. This approximation applies when the WV is larger than any eigenvalue of $\hat{A}$: $|\Lambda| \ll |A_w| < 1/g$. However, since $\Lambda_{\min(\max)} = n\lambda_{\min(\max)}$, we must be careful to fix $|A_w|$ to be at least n times larger than the eigenvalues of $\hat{a}$.

Now we consider maximizing the variance over an arbitrary initial state $|\Psi_i\rangle$, which produces [51]

$$\max_{|\Psi_i\rangle} \mathrm{Var}(\hat{A})_{|\Psi_i\rangle} = \frac{n^2}{4}(\lambda_{\max} - \lambda_{\min})^2, \tag{10}$$

showing *quadratic* scaling with n. Therefore, according to Eq. (9) the maximum postselection probability also scales quadratically with n, showing a factor of n improvement over the linear scaling of the uncorrelated ancilla attempts in Eq. (7).

The preparation states that show this quadratic scaling of the variance have the maximally entangled form [51]

$$|\Psi_i\rangle = \frac{1}{\sqrt{2}}(|\lambda_{\max}\rangle^{\otimes n} + e^{i\theta}|\lambda_{\min}\rangle^{\otimes n}), \tag{11}$$

where $e^{i\theta}$ is an arbitrary relative phase. We provide a simple circuit to prepare such a state for n qubits in Fig. 3, choosing $\theta = 0$.

According to the derivation in the Supplemental Material [50], the corresponding postselection states that maximize the postselection probability are

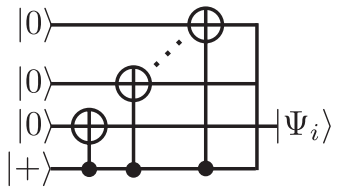

FIG. 3. Quantum circuit to prepare the optimal entangled preparation for n ancilla qubits. The state $|\Psi_i\rangle = (|0\rangle^{\otimes n} + |1\rangle^{\otimes n})/\sqrt{2}$ is prepared from a single $|+\rangle$ state by a sequence of CNOT gates. Because of this construction, we note that the ordering of the two-qubit gates in Figs. 2, 3, and 4 can be further optimized to pre- and postselect $(n-1)$ of the ancilla qubits sequentially, which allows the n-qubit entangled ancilla to be practically simulated using only three physical qubits.

$$|\Psi_f\rangle \propto -(n\lambda_{\min} - A_w^*)|\lambda_{\max}\rangle^{\otimes n} + e^{i\theta}(n\lambda_{\max} - A_w^*)|\lambda_{\min}\rangle^{\otimes n}, \quad (12)$$

which explicitly depend on the chosen value of A_w. We also provide a simple circuit to implement this postselection with n qubits in Fig. 4(a).

Weak value scaling.—So far we have shown that we can increase the postselection probability by a factor of n when the WV is kept fixed. Alternatively, we can hold the postselection probability fixed to increase the maximum WV by a factor of $\sqrt{n}$.

Given a specific postselection probability P_s, the postselected state $|\Psi_f\rangle$ must have the form

$$|\Psi_f\rangle = \sqrt{P_s}|\Psi_i\rangle + \sqrt{1-P_s}e^{i\theta}|\Psi_i^\perp\rangle, \quad (13)$$

where $|\Psi_i^\perp\rangle$ is an arbitrary state orthogonal to $|\Psi_i\rangle$. This implies that we can write the WV in Eq. (3) as

$$A_w = \langle\Psi_i|\hat{A}|\Psi_i\rangle + \sqrt{\frac{1-P_s}{P_s}}e^{-i\theta}\langle\Psi_i^\perp|\hat{A}|\Psi_i\rangle. \quad (14)$$

For large A_w and small P_s, then we can approximately neglect the first term. Since $e^{i\theta}$ is arbitrary, we can also assume that $\langle\Psi_i^\perp|\hat{A}|\Psi_i\rangle$ is positive. The maximum $\langle\Psi_i^\perp|\hat{A}|\Psi_i\rangle$ can be achieved when $|\Psi_i^\perp\rangle$ is parallel to the component of $\hat{A}|\Psi_i\rangle$ in the complementary subspace orthogonal to $|\Psi_i\rangle$. This choice produces $\langle\Psi_i^\perp|\hat{A}|\Psi_i\rangle = \|\hat{A}|\Psi_i\rangle - |\Psi_i\rangle\langle\Psi_i|\hat{A}|\Psi_i\rangle\| = [\mathrm{Var}(\hat{A})_{|\Psi_i\rangle}]^{1/2}$. Therefore, the largest WV that can be obtained from the initial state $|\Psi_i\rangle$ with a small P_s will approximate

$$\max|A_w| \approx \sqrt{\frac{\mathrm{Var}(\hat{A})_{|\Psi_i\rangle}}{P_s}}. \quad (15)$$

That is, the variance controls the scaling for the maxima of both P_s and A_w. Comparing Eqs. (9) and (15), it follows that if P_s can be improved by a factor of n, then it is also possible to improve A_w by a factor of $\sqrt{n}$. Furthermore, maximizing the variance produces the same initial state as Eq. (11), so the only difference between maximizing P_s and A_w is the choice of postselection state. We provide a simple circuit to implement this alternative postselection with n qubits in Fig. 4(b).

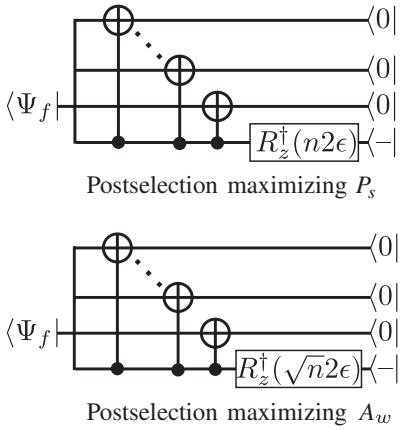

FIG. 4. Quantum circuits for attaining optimal postselections, using the preparation in Fig. 3. (a) Keeping $A_w \approx \mathrm{i}/\epsilon$ fixed and maximizing P_s produces the entangled postselection $\langle\Psi_f| = \langle 0|^{\otimes n}e^{in\epsilon} - \langle 1|^{\otimes n}e^{-in\epsilon}$ with $P_s \approx n^2\epsilon^2$, which is a factor of n larger than the single ancilla $P_s^{(n)}$ in Fig. 1. This postselection can be implemented as a sequence of CNOT gates and a rotation of the last qubit by $R_z^\dagger(n2\epsilon)$ and $R_y^\dagger(-\pi/2)$ before measuring all qubits in the Z basis and keeping only $\langle 0|$ events. For small ϵ this state is equivalent to Eq. (12). (b) Keeping $P_s = P_s^{(n)} \approx n\epsilon^2$ and maximizing A_w produces a similar state $\langle\Psi_f| = \langle 0|^{\otimes n}e^{i\sqrt{n}\epsilon} - \langle 1|^{\otimes n}e^{-i\sqrt{n}\epsilon}$ with $A_w \approx i\sqrt{n}/\epsilon$, which is a factor of $\sqrt{n}$ larger than A_w in Fig. 1.

Fisher information.—An improvement factor of $\sqrt{n}$ in the estimation precision is the best that we can expect from using entangled ancillas, according to well-known results from quantum metrology [51–55]. We are thus faced with the conundrum of how such a rare postselection can possibly show such optimal scaling with n. After all, most of the (potentially informative) data are being discarded by the postselection.

To understand this behavior, we compare the quantum Fisher information $I(g)$ about g contained in the post-interaction state $|\Phi_g\rangle = \exp(-ig\hat{A}\otimes\hat{F})|\Psi_i\rangle|\phi\rangle$ to the Fisher information $I'(g)$ that remains in the postselected state $\sqrt{P_s}|\phi'\rangle$. As detailed in the Supplemental Material [50], in the linear response regime $g|A_w|\mathrm{Var}(\hat{F})^{1/2} \ll 1$ with an initially unbiased meter $\langle\hat{F}\rangle_{|\phi\rangle} = 0$, and assuming a fixed $P_s \ll 1$ with maximal $|A_w|$, we obtain

$$I'(g) \approx \eta I(g)[1 - |gA_w|^2\mathrm{Var}(\hat{F})] \le I(g), \quad (16)$$

where $\eta = \mathrm{Var}(\hat{A})_{|\Psi_i\rangle}/\langle\hat{A}^2\rangle_{|\Psi_i\rangle}$ is an efficiency factor.

Remarkably, η can reach 1 when $\langle\hat{A}\rangle_{|\Psi_i\rangle} = 0$, implying that nearly all the original Fisher information $I(g)$ can be

concentrated into one rarely obtained $|\phi'\rangle$, up to a small reduction by $|gA_w|^2\mathrm{Var}(\hat{F}) \ll 1$. The remaining information is distributed among the discarded meter states, and could be retrieved in principle [8–10]. For the example with $\hat{F} = \hat{a} = \hat{\sigma}_z$, the initial state in Eq. (11) yields $\eta = 1$, $\mathrm{Var}(\hat{F}) = 1$, and a total Fisher information of $I(g) = 4\langle \hat{A}^2 \rangle_{|\Psi_i\rangle} = 4n^2$ (see the Supplemental Material [50]). The Cramér-Rao bound is thus $[I'(g)]^{-1/2} = (1/2n)[1 - |gA_w|^2]^{-1/2}$ for the precision of any unbiased estimation of $g = \varphi/2$ using $|\phi'\rangle$, confirming the optimal scaling with n.

Conclusion.—In summary, we have considered using entanglement to enhance the WVA of a small parameter. If the amplification factor is held fixed, then n entangled ancillas can improve the postselection probability by a factor of n compared to n attempts with uncorrelated ancillas. This improvement in postselection efficiency addresses a practical shortcoming of WVA, and achieves the optimal scaling with n that can be expected from quantum metrology. Indeed, we have shown that WVA can nearly saturate the expected quantum Cramér-Rao bound, despite the low efficiency of postselection. This result demonstrates that the practical benefits of WVA (e.g., amplified sensitivity and technical noise suppression) may be productively combined with existing quantum metrological techniques, encouraging further research and development. To this end, we have provided simple quantum circuits for the protocol that are readily implementable by existing quantum computing architectures that possess three or more qubits.

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