

Nonideal quantum detectors in Bayesian formalism

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The Bayesian formalism for a continuous measurement of solid-state qubits is extended to a model which takes into account several factors of the detector nonideality. In particular, we consider additional classical output and backaction noises (with finite correlation), together with quantum-limited output and backaction noises, and take into account possible asymmetry of the detector coupling. The formalism is first derived for a single qubit and then generalized to the measurement of entangled qubits.

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I. INTRODUCTION

The problem of continuous qubit measurement is of a significant importance for solid-state quantum computing¹ because the measurement of a solid-state qubit typically requires a significant time and thus can interplay nontrivially²⁻⁹ with the intrinsic evolution of the qubit system. The evolution of a single solid-state qubit (without ensemble averaging) due to continuous measurement can be described by the Bayesian formalism (for a review see Ref. 10) which takes into account the noisy measurement output of the detector. Apart from different notations, the Bayesian formalism practically coincides with the version of the quantum trajectory formalism⁶ adapted to solid-state setups from the theory developed for quantum optics.¹¹

The interaction between the measured quantum system and continuously operating detector leads to a random gradual change of the measured system, which is often called quantum backaction. However, the noisy component of the detector output is correlated with this backaction; therefore the evolution of the measured system during the measurement process can be inferred from the noisy detector output. In the quantum Bayesian approach the detector output is taken into account using the classical Bayes theorem,¹² which is applied in a straightforward way to the diagonal elements of the density matrix of the measured system, while a special care is taken of nondiagonal matrix elements.

Actually, there are many approaches with a similar methodology (though the formalisms may look very different) and similar results, which have been developed in various areas of physics. These approaches have been pioneered by the theories developed more than two decades ago by Davies, Kraus, and Holevo,¹³ which have generalized the “orthodox” quantum measurement and lead to the theories describing continuous quantum measurement. An important contribution has been the approach of the restricted path integral.¹⁴ Among all areas of physics, the theory of continuous quantum measurement has been best developed in quantum optics (see, e.g., Ref. 15), where it is often called as the quantum trajectory approach (for more references see Ref. 10).

The Bayesian approach is developed mostly for the continuous measurement of solid-state qubits and necessarily requires several simplifying assumptions. Even though the detector gets entangled with the qubit in a course of measurement, we assume that the detector cannot be in a

coherent state for a significantly long time. Therefore the entanglement is neglected and we discuss the random evolution of the qubit, correlated with the random evolution of the detector (in some sense this is a quasiclassical approach). Moreover, we assume that intrinsic quantum evolution of the detector is much faster than qubit evolution, so that the so-called Markov approximation can be used and the detector noise spectral density can be assumed to be frequency independent. This condition in typical situations can be expressed as $eV/\hbar \gg \Omega$, where V is the detector bias voltage and Ω is a typical (Rabi) frequency of the qubit evolution. In other words, the energy involved in the detector operation should be much larger than the typical energy of the measured system (actually, without this condition the interaction between the qubit and detector cannot be meaningfully considered as a measurement). The purpose of these assumptions is to ensure that the detector output is a classical quantity and is not involved in further quantum interactions (in the quantum theory of linear amplifiers developed by Caves¹⁶ this condition corresponds to large amplifier gain in units of number of quanta—see discussion in Ref. 16). One more usual assumption of the Bayesian approach (which can be easily removed) is that the detector output is continuous. While a discontinuous detector output (“quantum jumps”) is a more typical situation in the quantum trajectory approach applied to solid-state qubits,⁶ both formalisms basically coincide (as has been shown in Ref. 6), in spite of some technical differences.

One of the main predictions of the Bayesian formalism is the absence of the single qubit decoherence during the measurement by a good (ideal) detector,⁴ in contrast to decoherence of an ensemble of qubits^{17,18} (even though each measurement path involves experimentally monitorable pure qubit states, the paths are random and therefore different, leading to the decoherence as a result of ensemble averaging or averaging over the measurement result). Moreover, the state of a solid-state qubit can be gradually purified in a course of continuous measurement. In particular, this makes possible to monitor the phase of quantum coherent (Rabi) oscillations of the qubit. Such monitoring can be naturally used in the quantum feedback control^{19,20} of the Rabi oscillations which suppresses the qubit decoherence due to environment (for quantum feedback in quantum optics see, e.g., Refs. 21–24). Another potentially useful application of the Bayesian formalism is a recent prediction that two qubits can be made fully entangled by their continuous measurement with an equally coupled detector.²⁵

The efficiency of the quantum feedback loop operation crucially depends on the ideality (quantum efficiency) of the detector. For example, 100% synchronization between the qubit Rabi oscillations and desired pure oscillations is possible only for 100% ideal detector.²⁰ Many other effects related to continuous measurement of solid-state qubits, which have been predicted using the Bayesian formalism (see, e.g., Refs. 4 and 25–28) also depend significantly on the detector ideality. The ideality η of a continuously operating solid-state detector can be generally defined as a ratio between the detector performance and the performance of a quantum-limited detector, in which the output and backaction noises are strictly related, reaching the lower bound of an inequality similar to the Heisenberg uncertainty relation. More exact definition will be discussed later.

A quantum point contact (QPC) at low temperature is theoretically an ideal quantum detector⁴ that follows from the results of Refs. 2 and 29. A nearly ideal operation of the QPC has been demonstrated experimentally.^{30,31} The fact that a superconducting quantum interference device (SQUID) can theoretically reach the limit of an ideal detector follows³² from the results of Ref. 33. A normal state single-electron transistor (SET) is not a good quantum detector at usual operating points above the Coulomb blockade threshold.^{3,19} However, its quantum efficiency improves when we go closer to the threshold^{19,34} and becomes much better when the operating point is in the cotunneling range (below the threshold), in which case the limit of an ideal detector can be achieved.^{35,36} Superconducting SET is generally better than normal SET as a quantum-limited detector and can approach 100% ideality in the supercurrent regime³⁷ as well as in the double Josephson-plus-quasiparticle regime.³⁸ Finally, the resonant-tunneling SET (Ref. 5) can reach complete ideality in the small-bias limit.

In the simplest version of the Bayesian formalism⁴ a nonideal solid-state detector is modeled as an ideal symmetrically coupled detector³⁹ and a “pure dephaser” in parallel (environment or just extra backaction noise). In this case the nonideality leads to an extra term in the Bayesian equations, which introduces the gradual decay of the nondiagonal elements of the density matrix of the measured qubit. It was implied that such backaction dephasing is also equivalent to the extra noise at the detector output. However, the equivalence has never been proven explicitly, and this is one of the goals of the present paper.

In a more advanced version of the Bayesian formalism,^{40,19} a possible correlation between the output noise of a nonideal detector and the backaction noise is taken into account. However, the formulas for the evolution of the qubit density matrix in this case have been presented without any derivation, just from physical intuition. Moreover, comparison of these formulas with the results obtained by Goan and Milburn²⁸ for an ideal but asymmetrically coupled detector (which shifts the energy levels of the measured qubit) reveals some difference. Even though the difference is minor (second order in the detector response, which is assumed to be small), it points to some incorrectness of the initial formulas of Ref. 40 (corrected formulas can be found in Ref. 10). The main goal of this paper is to present a mathematical

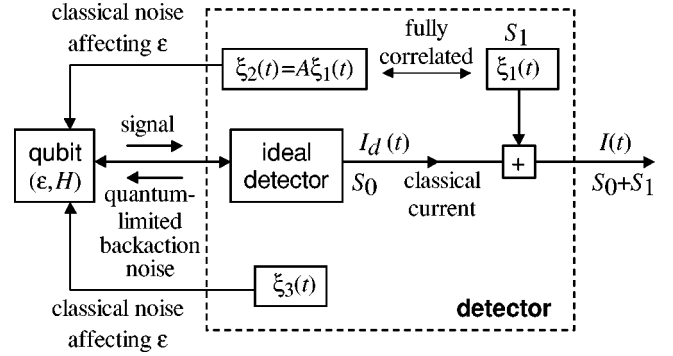


FIG. 1. Schematic of a nonideal solid-state detector measuring a qubit state. The detector is modeled as an ideal quantum detector and three sources of additional classical noise: output noise $\xi_1(t)$ with the white spectral density S_1 , backaction noise $\xi_2(t)$ fully correlated with $\xi_1(t)$, and uncorrelated backaction noise $\xi_3(t)$. The total noise density $S_0 + S_1$ of the output detector current $I(t)$ includes the contribution S_0 from the noise of ideal detector current $I_d(t)$.

derivation of the Bayesian formalism for a nonideal detector with correlated output and backaction noises, using a *phenomenological* model which adds correlated classical noise to the quantum-limited noise of an ideal detector. In spite of introducing the quantum-limited and classical noises in the model on different footing, there is no distinction between them in the final equations for the measured qubit evolution, and the detector is characterized by six real numbers: the dc current (operating point), the response, the total output noise, the total backaction noise, their correlation, and the induced shift of the qubit energy asymmetry. After the derivation of formalism for the measurement of one qubit, we generalize it to the continuous measurement of an arbitrary number of entangled qubits.

Notice that the issue of the asymmetric detector coupling to qubit has been recently discussed for a QPC in terms of the tunneling phase dependence on the qubit state.^{31,32,41,42} For a small-transparency QPC the formalism is significantly simplified²⁸ and is a direct generalization of the model of Ref. 2. We will use the results of Ref. 28 to model an ideal asymmetric detector.

While we model the detector nonideality by an additional classical noise, let us mention a different approach to the nonideality in Ref. 28, in which a random fraction of electrons tunneled through the detector is assumed to be missing. In our opinion, such model is not well applicable to solid-state detectors, even though it perfectly describes the inefficiency of a photodetector in a similar problem in quantum optics.

II. MODEL

We will use the phenomenological model of a nonideal solid-state detector of a qubit state shown in Fig. 1. It consists of an ideal detector and three sources of additional classical noise. We assume that the detector output is the noisy current $I(t)$ (we have in mind a QPC or a SET as a detector). The ideal detector is characterized by the output noise spec-

tral density S_0 [we assume flat (“white”) noise spectrum] and its backaction onto the measured qubit which will be called quantum-limited noise or just quantum noise. (Actually, because of the quantum relation between the output noise and the backaction, the output noise could also be called quantum; however, we will avoid such terminology, emphasizing the assumption that the quantum behavior does not propagate beyond the ideal detector.)

The first source of an additional classical noise adds the noisy component $\xi_1(t)$ with the white spectral density S_1 to the output $I_d(t)$ of the ideal detector, so that the final output is $I(t) = I_d(t) + \xi_1(t)$. The second noise source is the classical noise $\xi_2(t)$ which is 100% correlated with (proportional to) the noise $\xi_1(t)$ and affects the qubit energy asymmetry ε . [The qubit Hamiltonian is

$$\mathcal{H}_{qb} = \frac{\varepsilon}{2}(c_2^\dagger c_2 - c_1^\dagger c_1) + H(c_1^\dagger c_2 + c_2^\dagger c_1), \quad (1)$$

where the tunneling strength H is assumed to be real without loss of generality.] The relative magnitude of the noise $\xi_2(t) = A\xi_1(t)$ is characterized by the parameter A . Finally, the third classical noise source is the white noise $\xi_3(t)$ which also affects the qubit energy asymmetry ε [so that $\varepsilon \rightarrow \varepsilon + \xi_2(t) + \xi_3(t)$]. The second and third noise sources together are obviously equivalent to one white noise source, partially correlated with $\xi_1(t)$. However, we prefer to split it into the fully correlated and uncorrelated parts for clarity. Obviously, the qubit parameter H can also be affected by the detector noise; however, we do not take this effect into account, because the qubit dephasing is more naturally caused by the noise of its energy asymmetry ε (which corresponds to the measured degree of freedom) and also because the induced noise of H is negligible, for example, for a single-Cooper-pair qubit measured by an SET.

Let us start with the symmetric ideal detector, neglect all classical noises $\xi_{1,2,3}(t)$ and use the basic Bayesian formalism to describe the measurement process (i.e., the result of quantum backaction onto qubit); then the evolution of the qubit density matrix $\rho_{ij}(t)$ is^{4,10,19}

$$\dot{\rho}_{11} = -\dot{\rho}_{22} = -2\frac{H}{\hbar} \text{Im} \rho_{12} + \rho_{11}\rho_{22} \frac{2\Delta I}{S_0} [I_d(t) - I_0], \quad (2)$$

$$\dot{\rho}_{12} = i\frac{\varepsilon}{\hbar} \rho_{12} + i\frac{H}{\hbar} (\rho_{11} - \rho_{22}) - (\rho_{11} - \rho_{22}) \frac{\Delta I}{S_0} [I_d(t) - I_0] \rho_{12}. \quad (3)$$

Here $\Delta I \equiv I_1 - I_2$ is the detector response, I_1 is the average detector current for the qubit state $|1\rangle$, I_2 is the average current for the state $|2\rangle$, and $I_0 \equiv (I_1 + I_2)/2$. For the validity of Eqs. (2) and (3) we have to assume^{4,10,19} the weakly responding detector, $|\Delta I| \ll I_0$, sufficiently large detector voltage (much larger than the qubit energies), and assume that the passage of individual electrons in the detector is much faster than the qubit evolution, $I_0/e \gg (4H^2 + \varepsilon^2)/\hbar$, so that the current can be considered as continuous (see the brief discussion of the conditions in the Introduction).

For simulations, Eqs. (2) and (3) should be complemented by the equation

$$I_d(t) - I_0 = \frac{\Delta I}{2} (\rho_{11} - \rho_{22}) + \xi_0(t), \quad (4)$$

where $\xi_0(t)$ is the pure output noise of the ideal detector with flat spectral density S_0 .

Equations (2) and (3) are written in the so-called Stratonovich form,⁴³ which assumes symmetric definition of the derivative, $\dot{\rho}(t) = \lim_{\tau \rightarrow 0} [\rho(t + \tau/2) - \rho(t - \tau/2)]/\tau$. For the forward definition of derivative, $\dot{\rho}(t) = \lim_{\tau \rightarrow 0} [\rho(t + \tau) - \rho(t)]/\tau$ (Itô form), Eqs. (2) and (3) transform into¹⁹

$$\dot{\rho}_{11} = -\dot{\rho}_{22} = -2\frac{H}{\hbar} \text{Im} \rho_{12} + \rho_{11}\rho_{22} \frac{2\Delta I}{S_0} \xi_0(t), \quad (5)$$

$$\begin{aligned} \dot{\rho}_{12} = & i\frac{\varepsilon}{\hbar} \rho_{12} + i\frac{H}{\hbar} (\rho_{11} - \rho_{22}) - (\rho_{11} - \rho_{22}) \frac{\Delta I}{S_0} \rho_{12} \xi_0(t) \\ & - \frac{(\Delta I)^2}{4S_0} \rho_{12}, \end{aligned} \quad (6)$$

while the relation (4) remains unchanged. [The general rule of transformation is the following:⁴³ for an arbitrary system of equations $\dot{x}_i(t) = G_i(\mathbf{x}, t) + F_i(\mathbf{x}, t)\xi(t)$ in the Stratonovich form, the corresponding equations in the Itô form are $\dot{x}_i(t) = G_i(\mathbf{x}, t) + F_i(\mathbf{x}, t)\xi(t) + (S_\xi/4)\sum_k F_k(\mathbf{x}, t)\partial F_i(\mathbf{x}, t)/\partial x_k$.] The advantage of the Itô form is that the averaging over the noise $\xi_0(t)$ is straightforward (we just need to eliminate terms with ξ_0), so it is easy to obtain the ensemble averaged evolution:

$$\dot{\rho}_{11} = -\dot{\rho}_{22} = -2\frac{H}{\hbar} \text{Im} \rho_{12}, \quad (7)$$

$$\dot{\rho}_{12} = i\frac{\varepsilon}{\hbar} \rho_{12} + i\frac{H}{\hbar} (\rho_{11} - \rho_{22}) - \frac{(\Delta I)^2}{4S_0} \rho_{12}. \quad (8)$$

On the other hand, the advantage of the Stratonovich form is the validity of usual calculus rules (which do not work in the Itô form) and therefore easier physical interpretation of equations.

Let us emphasize that the single qubit in this model (which assumes ideal detector) does not decohere during the measurement process, as easier to see from Eqs. (2) and (3). However, because of the probabilistic nature of quantum measurement, the ensemble of qubits does decohere since different qubits will go along different “trajectories.” The ensemble decoherence rate $(\Delta I)^2/4S_0$ is determined by this quantum randomness and therefore can be naturally called “quantum-limited” decoherence rate. (It can also be called “information-limited” ensemble decoherence, since its origin is the tendency of qubit state to evolve either into state $|1\rangle$ or $|2\rangle$, corresponding to the information acquired from the measurement.¹⁹)

While it is not trivial to take into account additional classical noises ξ_1 and ξ_2 (this will be done in the following sections), the account of the noise ξ_3 is very simple. It obvi-

ously leads to the additional dephasing term $-\gamma_3\rho_{12}$ in Eqs. (3), (6), and (8), where $\gamma_3=S_3/4\hbar^2$ is proportional to the spectral density S_3 of $\xi_3(t)$. We will characterize this noise by the dephasing rate γ_3 instead of characterizing it by S_3 . (The relation $\gamma_3=S_3/4\hbar^2$ can be easily derived adding $\xi_3(t)$ to ε in the Stratonovich form, then translating the equation into Itô form, and averaging over ξ_3 .)

The natural definition of the detector ideality factor η in this case is^{4,10,19}

$$\eta \equiv \frac{\Gamma_0}{\Gamma_\Sigma}, \quad (9)$$

where $\Gamma_0 \equiv (\Delta I)^2/4S_0$ is the quantum-limited contribution and $\Gamma_\Sigma = \Gamma_0 + \gamma_3$ is the total ensemble dephasing rate. Simply speaking, this definition of ideality is the ratio between quantum contribution and total backaction noise.

III. IDEAL SYMMETRIC DETECTOR AND ADDITIONAL OUTPUT NOISE

Let us now take into account additional output noise $\xi_1(t)$, while $\xi_2(t)$ is still zero. We also switch off $\xi_3(t)$, since it is trivial to add its effect later. In order to derive Bayesian equations in this case, let us also assume $H = \varepsilon = 0$ (“frozen” qubit) and add the effects of H and ε later. For $H = \varepsilon = 0$, Eqs. (2) and (3) have a simple solution^{4,19} which can be interpreted as a consequence of the “quantum Bayes theorem”:⁴⁴

$$\begin{aligned} \rho_{11}(\tau) &= \left[1 + \frac{\rho_{22}(\tau)}{\rho_{11}(\tau)} \right]^{-1} \\ &= \left[1 + \frac{\rho_{22}(0) \exp[-(\bar{I}_d - I_2)^2 \tau / S_0]}{\rho_{11}(0) \exp[-(\bar{I}_d - I_1)^2 \tau / S_0]} \right]^{-1}, \quad (10) \end{aligned}$$

$$\rho_{22}(\tau) = 1 - \rho_{11}(\tau), \quad (11)$$

$$\rho_{12}(\tau) = \rho_{12}(0) \frac{[\rho_{11}(\tau)\rho_{22}(\tau)]^{1/2}}{[\rho_{11}(0)\rho_{22}(0)]^{1/2}} \quad (12)$$

$$\begin{aligned} &= \frac{\rho_{12}(0) \exp\left[-\frac{(\Delta I)^2 \tau}{4S_0}\right] \exp\left[-\frac{(\bar{I}_d - I_0)^2 \tau}{S_0}\right]}{\rho_{11}(0) \exp\left[-\frac{(\bar{I}_d - I_1)^2 \tau}{S_0}\right] + \rho_{22}(0) \exp\left[-\frac{(\bar{I}_d - I_2)^2 \tau}{S_0}\right]}, \quad (13) \end{aligned}$$

where \bar{I}_d is the average of the detector current during the time interval between 0 and τ :

$$\bar{I}_d = \frac{1}{\tau} \int_0^\tau I_d(t) dt. \quad (14)$$

Here Eq. (10) is the consequence of the classical Bayes theorem¹² and Eq. (12) says that the degree of the qubit purity is conserved. [It is easy to include the effect of finite

ε , which just leads to an extra factor $\exp(i\varepsilon\tau/\hbar)$ in Eqs. (12) and (13); however, we will not do that in order to keep the formulas shorter.]

Since the detector output is now $I(t) = I_d(t) + \xi_1(t)$, we have to express $\rho_{ij}(\tau)$ in terms of $I(t)$ and average it over the noise $\xi_1(t)$. Naively thinking, we have to use the substitution

$$\bar{I}_d = \bar{I} - x, \quad \bar{I} = \frac{1}{\tau} \int_0^\tau I(t) dt, \quad x = \frac{1}{\tau} \int_0^\tau \xi_1(t) dt, \quad (15)$$

and average Eqs. (10) and (13) over the noise contribution x using the weight factor $p(x) = (2\pi D_1)^{-1/2} \exp(-x^2/2D_1)$ where $D_1 = S_1/2\tau$ is the variance of x . However, this is not a correct procedure because the probability distribution of x is correlated with \bar{I} (though it is not correlated with \bar{I}_d). So instead, we have to use the conditional distribution of x for a given \bar{I} :

$$p(x) = P(x) \left/ \int P(x') dx' \right., \quad (16)$$

$$\begin{aligned} P(x) &= \frac{\exp(-x^2/2D_1)}{\sqrt{2\pi D_1}} \left[\rho_{11}(0) \frac{\exp[-(\bar{I} - x - I_1)^2/2D_0]}{\sqrt{2\pi D_0}} \right. \\ &\quad \left. + \rho_{22}(0) \frac{\exp[-(\bar{I} - x - I_2)^2/2D_0]}{\sqrt{2\pi D_0}} \right], \quad (17) \end{aligned}$$

where $D_0 = S_0/2\tau$. (Let us stress again that both \bar{I}_d and x are assumed to be classical quantities.) Introducing the weight factor $p(x)$ into Eq. (10), substituting $\bar{I}_d = \bar{I} - x$, and integrating over x , we get the averaged equation

$$\rho_{11}(\tau) = \left[1 + \frac{\rho_{22}(0) \exp[-(\bar{I} - I_2)^2 \tau / S_\Sigma]}{\rho_{11}(0) \exp[-(\bar{I} - I_1)^2 \tau / S_\Sigma]} \right]^{-1}, \quad (18)$$

where $S_\Sigma \equiv S_0 + S_1$ is the total output noise. The only difference compared with Eq. (10) is the change of \bar{I}_d into \bar{I} and change of S_0 into S_Σ (this is quite expected since ρ_{11} behaves as a classical probability and the classical Bayes formula still works).

To calculate $\rho_{12}(\tau)$ averaged over the noise $\xi_1(t)$, we have to do a similar procedure. We multiple Eq. (13) by the weight factor $p(x)$, use substitution $\bar{I}_d = \bar{I} - x$, and integrate over x . In this way we obtain

$$\begin{aligned} &\rho_{12}(\tau) \\ &= \frac{\rho_{12}(0) \exp\left[-\frac{(\Delta I)^2 \tau}{4S_0}\right] \exp\left[-\frac{(\bar{I} - I_0)^2 \tau}{S_\Sigma}\right]}{\rho_{11}(0) \exp\left[-\frac{(\bar{I} - I_1)^2 \tau}{S_\Sigma}\right] + \rho_{22}(0) \exp\left[-\frac{(\bar{I} - I_2)^2 \tau}{S_\Sigma}\right]}. \quad (19) \end{aligned}$$

Comparing Eqs. (13) and (19), we see that \bar{I}_d changes into \bar{I} and S_0 changes into S_Σ , except in the second factor of the

numerator, where S_0 remains unchanged. Let us represent this factor as $\exp[-(\Delta I)^2\tau/4S_\Sigma]\exp(-\gamma_1\tau)$, where

$$\gamma_1 = \frac{(\Delta I)^2 S_1}{4S_0 S_\Sigma}. \quad (20)$$

So, the effect of additional output noise ξ_1 on the Eqs. (10)–(13) is the following: the output current I_d from the ideal part of the detector changes into the output current I , the spectral density S_0 corresponding to the ideal part of the detector changes into the total output noise S_Σ , and the non-diagonal matrix element acquires the dephasing factor $\exp(-\gamma_1\tau)$. Differentiating the new equations over time (if we do it in a simple first-order way, we automatically get equations in the Stratonovich form) and adding terms due to H , ε , and noise ξ_3 , we obtain

$$\dot{\rho}_{11} = -\dot{\rho}_{22} = -2\frac{H}{\hbar} \text{Im} \rho_{12} + \rho_{11}\rho_{22} \frac{2\Delta I}{S_\Sigma} [I(t) - I_0], \quad (21)$$

$$\begin{aligned} \dot{\rho}_{12} = & i\frac{\varepsilon}{\hbar} \rho_{12} + i\frac{H}{\hbar} (\rho_{11} - \rho_{22}) - (\rho_{11} - \rho_{22}) \frac{\Delta I}{S_\Sigma} [I(t) - I_0] \rho_{12} \\ & - (\gamma_1 + \gamma_3) \rho_{12}. \end{aligned} \quad (22)$$

We see that the effect of the extra output noise $\xi_1(t)$ is similar to the effect of the extra backaction noise $\xi_3(t)$ and both lead to the qubit dephasing.

From physical reasoning, the way of separation of the detector into the ideal part and additional noise sources $\xi_1(t)$ and $\xi_3(t)$ is arbitrary, as long as the total output noise S_Σ and total ensemble qubit dephasing rate Γ_Σ are fixed (in other words, S_Σ and Γ_Σ are the only physically relevant quantities). It is easy to check that Eqs. (21) and (22) satisfy this requirement because

$$\gamma_1 + \gamma_3 = \Gamma_\Sigma - \frac{(\Delta I)^2}{4S_\Sigma}, \quad \Gamma_\Sigma = \frac{(\Delta I)^2}{4S_0} + \gamma_3. \quad (23)$$

[The total ensemble dephasing rate Γ_Σ can be formally found from Eqs. (21) and (22) by translating them into Itô form that adds ensemble dephasing rate $(\Delta I)^2/4S_\Sigma$.]

Comparing Eqs. (21) and (22) with Eqs. (2) and (3), we naturally introduce a more general definition of the detector ideality:

$$\eta = \frac{(\Delta I)^2/4S_\Sigma}{\Gamma_\Sigma}, \quad (24)$$

which is again the ratio of the quantum-limited part of the ensemble qubit dephasing and its total dephasing rate. (Notice that the numerator is not the “real” quantum backaction determined by S_0 , but the information-limited backaction determined by S_Σ .) In particular, for our model in the case $\xi_3(t)=0$ (no classical backaction) we obtain $\eta = S_0/(S_0 + S_1)$.

IV. CORRELATED OUTPUT AND BACKACTION NOISES

Now let us add the classical backaction noise $\xi_2(t)$ which affects the qubit energy ε [so that $\varepsilon \rightarrow \varepsilon + \xi_2(t)$], and which is 100% correlated with the output noise source, $\xi_2(t) = A\xi_1(t)$. We again start with Eqs. (10)–(13) for the “fro-

zen” qubit. Addition of $\xi_2(t)$ does not affect diagonal matrix elements, so Eq. (18) remains unchanged. In order to calculate $\rho_{12}(\tau)$ averaged over the noises ξ_1 and ξ_2 , we multiply Eq. (13) by the factor $\exp[i\hbar^{-1}\int_0^\tau \xi_2(t)dt] = \exp(iAx\tau/\hbar)$ and average it over x with the weight $p(x)$ given by Eqs. (16) and (17). This leads to the equation

$$\begin{aligned} \rho_{12}(\tau) &= \frac{\rho_{12}(0) \exp\left[-\frac{(\Delta I)^2\tau}{4S_0}\right] \exp\left[-\frac{(\bar{I}-I_0)^2\tau}{S_\Sigma}\right]}{\rho_{11}(0) \exp\left[-\frac{(\bar{I}-I_1)^2\tau}{S_\Sigma}\right] + \rho_{22}(0) \exp\left[-\frac{(\bar{I}-I_2)^2\tau}{S_\Sigma}\right]} \\ &\times \exp\left[\frac{i}{\hbar}(\bar{I}-I_0)A\frac{S_1}{S_\Sigma}\tau\right] \exp\left[-\frac{S_0S_1}{S_\Sigma}\frac{A^2}{4\hbar^2}\tau\right], \end{aligned} \quad (25)$$

in which only the last line is different from Eq. (19).

Differentiating this equation over time (again, we automatically obtain the Stratonovich form) and adding terms due to H , ε , and ξ_3 , we get

$$\begin{aligned} \dot{\rho}_{12} = & i\frac{\varepsilon}{\hbar} \rho_{12} + i\frac{H}{\hbar} (\rho_{11} - \rho_{22}) + \frac{i}{\hbar} [I(t) - I_0] A \frac{S_1}{S_\Sigma} \rho_{12} \\ & - (\rho_{11} - \rho_{22}) \frac{\Delta I}{S_\Sigma} [I(t) - I_0] \rho_{12} - (\gamma_1 + \gamma_2 + \gamma_3) \rho_{12}, \end{aligned} \quad (26)$$

where γ_1 is given by Eq. (20) and γ_2 is

$$\gamma_2 = \frac{S_0S_1}{S_\Sigma} \frac{A^2}{4\hbar^2}. \quad (27)$$

Physically relevant parameters of the detector are the total output noise $S_\Sigma = S_0 + S_1$, total ensemble dephasing rate Γ_Σ , and the correlation between output and backaction noises. Since three sources of the backaction noise in Fig. 1 are uncorrelated, the ensemble dephasing rate is

$$\Gamma_\Sigma = (\Delta I)^2/4S_0 + A^2S_1/4\hbar^2 + \gamma_3 \quad (28)$$

[the same result can be obtained by translating Eq. (26) into Itô form and performing ensemble averaging]. Following Refs. 40 and 19, let us characterize the noise correlation by the magnitude (real number)

$$K = \frac{S_{\varepsilon I}}{\hbar S_\Sigma}, \quad (29)$$

where $S_{\varepsilon I}$ is the mutual spectral density of the detector output noise and induced fluctuations of ε (strictly speaking, $S_{\varepsilon I}$ in our notation is only the real part of the mutual spectral density, while the imaginary part can formally describe the detector response³²). In our case $K = AS_1/\hbar S_\Sigma$. Expressing qubit evolution only in terms of physically relevant quantities, we obtain

$$\dot{\rho}_{11} = -\dot{\rho}_{22} = -2\frac{H}{\hbar} \text{Im} \rho_{12} + \rho_{11}\rho_{22} \frac{2\Delta I}{S_\Sigma} [I(t) - I_0], \quad (30)$$

$$\begin{aligned} \dot{\rho}_{12} = & i \frac{\tilde{\varepsilon}}{\hbar} \rho_{12} + i \frac{H}{\hbar} (\rho_{11} - \rho_{22}) + iK[I(t) - I_0] \rho_{12} - (\rho_{11} - \rho_{22}) \\ & \times \frac{\Delta I}{S_\Sigma} [I(t) - I_0] \rho_{12} - \tilde{\gamma} \rho_{12}, \end{aligned} \quad (31)$$

where

$$\tilde{\gamma} = \Gamma_\Sigma - \frac{(\Delta I)^2}{4S_\Sigma} - \frac{K^2 S_\Sigma}{4} \quad (32)$$

and the new notation $\tilde{\varepsilon} = \varepsilon$ is introduced to accommodate the measurement-induced shift of ε discussed in the next section.

Equations (30)–(32) are the main result of this paper for the measurement of one qubit (the results of the next section will lead to the same equations). Comparing Eqs. (30)–(32) with similar equations presented (but not derived) in Refs. 40 and 19, we notice a difference: the term $iK[I(t) - I_0] \rho_{12}$ was incorrectly replaced in Refs. 40 and 19 by the term $iK[I(t) - (\rho_{11}I_1 + \rho_{22}I_2)] \rho_{12}$. Notice though that the effect of their difference $iK(\Delta I/2)(\rho_{11} - \rho_{22}) \rho_{12}$ is minor since $\rho_{11} - \rho_{22}$ is the oscillating magnitude and averages to zero. As will be mentioned later, Eqs. (30)–(31) in the case $\tilde{\gamma} = 0$ (this is possible for asymmetric ideal detector) coincide with the corresponding equations of Ref. 28 if $\tilde{\varepsilon}$ includes the detector-induced shift.

To translate Eqs. (30)–(31) from Stratonovich to Itô form, notice that

$$I(t) - I_0 = \frac{\Delta I}{2} (\rho_{11} - \rho_{22}) + \xi_0(t) + \xi_1(t) \quad (33)$$

and the sum of two output white noises $\xi_{0+1}(t) \equiv \xi_0(t) + \xi_1(t)$ is the white noise with the spectral density S_Σ . Then using the standard rule of translation, we obtain Itô equations

$$\dot{\rho}_{11} = -\dot{\rho}_{22} = -2 \frac{H}{\hbar} \text{Im} \rho_{12} + \rho_{11} \rho_{22} \frac{2\Delta I}{S_\Sigma} \xi_{0+1}(t), \quad (34)$$

$$\begin{aligned} \dot{\rho}_{12} = & i \frac{\tilde{\varepsilon}}{\hbar} \rho_{12} + i \frac{H}{\hbar} (\rho_{11} - \rho_{22}) + iK \xi_{0+1}(t) \rho_{12} - (\rho_{11} - \rho_{22}) \\ & \times \frac{\Delta I}{S_\Sigma} \xi_{0+1}(t) \rho_{12} - \left(\tilde{\gamma} + \frac{(\Delta I)^2}{4S_\Sigma} + \frac{K^2 S_\Sigma}{4} \right) \rho_{12}, \end{aligned} \quad (35)$$

while the relation between output current $I(t)$ and pure noise $\xi_{0+1}(t)$ is still given by Eq. (33). (Notice that the above-mentioned incorrect term is correct in the Itô form of the equation, so the mistake was due to mixing the Stratonovich and Itô forms.) The corresponding ensemble-averaged equations can be obtained by erasing terms containing $\xi_{0+1}(t)$ in Eqs. (34) and (35):

$$\dot{\rho}_{11} = -\dot{\rho}_{22} = -2 \frac{H}{\hbar} \text{Im} \rho_{12}, \quad (36)$$

$$\dot{\rho}_{12} = i \frac{\tilde{\varepsilon}}{\hbar} \rho_{12} + i \frac{H}{\hbar} (\rho_{11} - \rho_{22}) - \Gamma_\Sigma \rho_{12}, \quad (37)$$

where the total ensemble decoherence rate Γ_Σ is given by Eqs. (28) and/or (32).

Since $\tilde{\gamma} > 0$ [otherwise solution of Eqs. (34) and (35) would violate inequality $|\rho_{12}|^2 \leq \rho_{11}\rho_{22}$], we obtain the fundamental limitation for the ensemble dephasing:

$$\Gamma_\Sigma \geq \frac{(\Delta I)^2}{4S_\Sigma} + \frac{K^2 S_\Sigma}{4}. \quad (38)$$

Besides the definition of the detector ideality η given by Eq. (24) which would give $\eta = 1 - (\tilde{\gamma} + K^2 S_\Sigma/4)/\Gamma_\Sigma$, it is natural to introduce another definition of the ideality¹⁹

$$\tilde{\eta} = 1 - \frac{\tilde{\gamma}}{\Gamma_\Sigma} = \frac{(\Delta I)^2/4S_\Sigma + K^2 S_\Sigma/4}{\Gamma_\Sigma}, \quad (39)$$

since the term $K^2 S_\Sigma/4$ does not correspond to the dephasing of a single qubit. One more possible definition of ideality (which also gives 100% if $\tilde{\gamma} = 0$) is

$$\tilde{\eta}_2 = \frac{1}{1 + \tilde{\gamma}/[(\Delta I)^2/4S_\Sigma]} = \frac{(\Delta I)^2/4S_\Sigma}{\Gamma_\Sigma - K^2 S_\Sigma/4}, \quad (40)$$

so that $(\tilde{\eta}_2)^{-1/2}$ directly corresponds¹⁰ to the total energy sensitivity of the detector in units of $\hbar/2$. In the case $K = 0$ all the definitions coincide, $\eta = \tilde{\eta} = \tilde{\eta}_2$.

V. ACCOUNT OF ASYMMETRIC IDEAL DETECTOR

So far we have assumed that the ideal part of the detector in Fig. 1 does not induce the shift of the qubit energy asymmetry ε (i.e., in our terminology assumed symmetrically coupled detector). However, in general the coupling with detector changes ε , so it should be treated self-consistently.¹⁹ As an example, the operating point of an SET as a detector is slightly shifted by different charge states of the measured qubit. This generally leads to the change of the average potential v of the SET island, which affects back the qubit energy asymmetry ε . Notice that v can also be temporary shifted by a fluctuation of the current through SET, leading to the correlation between the output and backaction noises. So, in this example the shift of ε and noise correlation are closely related. Similar situation occurs when as a detector we use a QPC, which location relative to the qubit is geometrically asymmetric.^{31,42} Then the qubit state affects the phase of the QPC current,⁴⁵ and in return each electron passing through the QPC affects the phase difference between qubit states $|1\rangle$ and $|2\rangle$, thus leading to effective shift of ε . Correspondingly, the noise of the QPC current causes the correlated noise of ε , so again these effects are closely related.

The asymmetric coupling can be relatively easy taken into account for a small-transparency QPC using the model analyzed in Ref. 28. The detector and its interaction with the qubit are described by Hamiltonians

$$\mathcal{H}_{det} = \sum_l E_l a_l^\dagger a_l + \sum_r E_r a_r^\dagger a_r + \sum_{l,r} (T a_l^\dagger a_l + T^* a_l^\dagger a_r),$$

$$\mathcal{H}_{int} = \sum_{l,r} (c_1^\dagger c_1 - c_2^\dagger c_2) \left(\frac{\Delta T}{2} a_r^\dagger a_l + \frac{\Delta T^*}{2} a_l^\dagger a_r \right), \quad (41)$$

in which we neglect the dependence of tunnel matrix elements T and ΔT on the electron states (l, r) in electrodes. The only difference of this model from the model of Ref. 2 [which leads to Eqs. (2) and (3)] is the possibility of complex T and ΔT (actually, T can be assumed real without loss of generality). Following the procedure developed in Ref. 2, it is possible to show⁴⁶ that in the corresponding Bloch equation for $\dot{\rho}_{12}^n$ (where ρ_{ij}^n is the density matrix with account of the number n of electrons passed through the detector) the term² $\sqrt{I_1 I_2} e^{-1} \rho_{12}^{n-1}$ should be replaced by $\exp(i\varphi) \sqrt{I_1 I_2} e^{-1} \rho_{12}^{n-1}$, where $\varphi = \arg[(T + \Delta T/2)(T^* - \Delta T^*/2)]$. Therefore each electron tunneling through the detector adds the phase φ to ρ_{12} , thus affecting the qubit energy asymmetry ε .

The assumption of weak detector response implies $|\Delta T| \ll |T|$, so that $|\varphi| \ll 1$.²⁸ The extra phase leads to the extra term $i(\theta/\hbar)I_d \rho_{12}$ in Eq. (3) where $\theta \equiv \hbar \varphi/e$. Separating it into the average and fluctuating parts, we obtain the following equations for the asymmetric ideal detector in the Stratonovich form:

$$\dot{\rho}_{11} = -\dot{\rho}_{22} = -2\frac{H}{\hbar} \text{Im} \rho_{12} + \rho_{11} \rho_{22} \frac{2\Delta I}{S_0} [I_d(t) - I_0], \quad (42)$$

$$\begin{aligned} \dot{\rho}_{12} = & i\frac{\tilde{\varepsilon}}{\hbar} \rho_{12} + i\frac{H}{\hbar} (\rho_{11} - \rho_{22}) + i\frac{\theta}{\hbar} [I_d(t) - I_0] \rho_{12} \\ & - (\rho_{11} - \rho_{22}) \frac{\Delta I}{S_0} [I_d(t) - I_0] \rho_{12}, \end{aligned} \quad (43)$$

where $\tilde{\varepsilon} = \varepsilon + \Delta\varepsilon$ and $\Delta\varepsilon = \theta I_0 = \hbar \varphi I_0/e$. Equations (42) and (43) in Itô form have been obtained in Ref. 28 (notice a different definition of θ).⁴⁷ Let us mention that even though Eqs. (2) and (43) have been derived for a particular model of the detector [Eq. (41)], we expect them to be applicable to a significantly broader class of asymmetric ideal detectors (i.e., finite-transparency QPC's, quantum-limited dc SQUIDS, etc.). Notice that Eqs. (42) and (43) are formally similar to Eqs. (30) and (31) except $\tilde{\gamma} = 0$ (ideal detector) and the self-consistent value $\tilde{\varepsilon}$ is now different from ε .

The solution of Eqs. (42) and (43) in the simple case $H = 0$ is still given by Eqs. (10) and (11) for ρ_{11} and ρ_{22} , while Eqs. (12) and (13) for ρ_{12} should be replaced by

$$\rho_{12}(\tau) = \rho_{12}(0) \frac{[\rho_{11}(\tau) \rho_{22}(\tau)]^{1/2}}{[\rho_{11}(0) \rho_{22}(0)]^{1/2}} e^{i\tilde{\varepsilon}\tau/\hbar} e^{i\theta(\bar{I}_d - I_0)\tau/\hbar} \quad (44)$$

$$\begin{aligned} & \rho_{12}(0) \exp\left[-\frac{(\Delta I)^2 \tau}{4S_0} - \frac{(\bar{I}_d - I_0)^2 \tau}{S_0}\right] \exp\left[i\frac{\tilde{\varepsilon}\tau + \theta(\bar{I}_d - I_0)\tau}{\hbar}\right] \\ & = \frac{\rho_{12}(0) \exp\left[-\frac{(\bar{I}_d - I_1)^2 \tau}{S_0}\right] + \rho_{22}(0) \exp\left[-\frac{(\bar{I}_d - I_2)^2 \tau}{S_0}\right]}{\rho_{11}(0) \exp\left[-\frac{(\bar{I}_d - I_1)^2 \tau}{S_0}\right] + \rho_{22}(0) \exp\left[-\frac{(\bar{I}_d - I_2)^2 \tau}{S_0}\right]}. \end{aligned} \quad (45)$$

Let us now add the classical noises ξ_1 , ξ_2 , and ξ_3 (see Fig. 1) to our model of asymmetric ideal detector.⁴⁸ Using the procedure explained in two previous sections, we multiply Eq. (45) by the factor $\exp(iAx\tau/\hbar)$ where $x = \tau^{-1} \int_0^\tau \xi_1(t) dt$, average the resulting equation and Eq. (10) over the distribution $p(x)$ given by Eqs. (16) and (17), differentiate the result over time, and add the terms due to H and ξ_3 . In this way we obtain the following Stratonovich equations:

$$\dot{\rho}_{11} = -\dot{\rho}_{22} = -2\frac{H}{\hbar} \text{Im} \rho_{12} + \rho_{11} \rho_{22} \frac{2\Delta I}{S_\Sigma} [I(t) - I_0], \quad (46)$$

$$\begin{aligned} \dot{\rho}_{12} = & i\frac{\tilde{\varepsilon}}{\hbar} \rho_{12} + i\frac{H}{\hbar} (\rho_{11} - \rho_{22}) + \frac{i}{\hbar} \frac{AS_1 + \theta S_0}{S_\Sigma} [I(t) - I_0] \rho_{12} \\ & - (\rho_{11} - \rho_{22}) \frac{\Delta I}{S_\Sigma} [I(t) - I_0] \rho_{12} - \tilde{\gamma} \rho_{12}, \end{aligned} \quad (47)$$

where

$$\tilde{\gamma} = \frac{(\Delta I)^2 S_1}{4S_0 S_\Sigma} + \frac{S_0 S_1}{S_\Sigma} \frac{(A - \theta)^2}{4\hbar^2} + \gamma_3. \quad (48)$$

It is interesting to notice that the single-qubit decoherence rate $\tilde{\gamma}$ can be decreased by adding the backaction noise $\xi_2(t) = A\xi_1(t)$ with $A = \theta$ (i.e., having the same correlation with the output noise as the quantum backaction), even though it increases the ensemble dephasing rate

$$\Gamma_\Sigma = (\Delta I)^2/4S_0 + \theta^2 S_0/4\hbar^2 + A^2 S_1/4\hbar^2 + \gamma_3. \quad (49)$$

One can easily see that introducing the total correlation between the output noise and the backaction (including quantum) noise,

$$K = \frac{AS_1 + \theta S_0}{\hbar S_\Sigma}, \quad (50)$$

we reduce Eqs. (46) and (47) to Eqs. (30) and (31) discussed in the previous section. (The only new feature is the measurement-induced energy shift $\varepsilon \rightarrow \tilde{\varepsilon}$, which can also

have a simple classical contribution due to the shift of the detector operating point.) Therefore the corresponding Itô equations are still given by Eqs. (34) and (35) and the ensemble-averaged equations are still given by Eqs. (36) and (37), while the relation between the single-qubit decoherence $\tilde{\gamma}$ and ensemble decoherence Γ_Σ still satisfies Eq. (32). Thus we conclude that the Bayesian description of the measurement process given by Eqs. (30)–(37), which is expressed in terms of the physically relevant quantities K , S_Σ , and Γ_Σ , remains valid in the case when the ideal part of the detector is assumed asymmetrically coupled to the qubit. Similarly, the limitation (38) for Γ_Σ remains valid and the definitions (39) and (40) of the detector ideality can still be used.

Let us emphasize again that the phenomenologically introduced separation between the quantum and classical noise contributions does not show up in the Bayesian description of the continuous single qubit measurement [Eqs. (30)–(32)], and the detector in our model is completely described by six quantities: dc output current (“operating point”) I_0 , response ΔI , output noise S_Σ , ensemble dephasing rate Γ_Σ , correlation magnitude K , and the induced qubit energy shift $\Delta\varepsilon$. The quantities S_Σ , Γ_Σ , and K are analogous to the output, input, and cross-correlation noise terms, which are usually used for the description of a classical amplifier (and similarly for the description of a quantum amplifier—see, e.g., Ref. 32 and references therein). The induced energy shift $\Delta\varepsilon$ is somewhat similar to the effect of a finite amplifier input impedance onto the previous stage parameters.

VI. DETECTOR MEASURING ENTANGLED QUBITS

Finally, let us consider the case when the detector is coupled to N arbitrarily entangled and arbitrary interacting qubits. Following Ref. 49 we introduce the measurement basis consisting of 2^N states $|i\rangle$ and up to 2^N different levels I_i of the detector average current (some average currents can coincide). For an ideal symmetric detector and “frozen” qubits, $H_{qbs}=0$, where H_{qbs} is the Hamiltonian of intrinsic evolution of the qubits, the evolution of the density matrix ρ of qubits due to measurement is described by simple “quantum Bayes” equation,^{44,49}

$$\rho_{ij}(\tau) = \rho_{ij}(0) \frac{\sqrt{P_i(\tau)P_j(\tau)}}{\sum_k \rho_{kk}(0)P_k(\tau)}, \quad (51)$$

where $P_i(\tau)$ is the probability of obtaining particular measurement result (\bar{I}_d in this case) assuming state $|i\rangle$ of the measured system:

$$P_i(\tau) = \frac{1}{\sqrt{2\pi D_0}} \exp\left[-\frac{(\bar{I}_d - I_i)^2}{2D_0}\right], \quad D_0 = \frac{S_0}{2\tau} \quad (52)$$

(we again assume that the currents I_i do not differ much, and therefore the detector noise S_0 is state independent).

To take into account additional output noise $\xi_1(t)$ with spectral density S_1 , we have to perform the averaging of the density matrix ρ_{ij} over $x = \bar{I} - \bar{I}_d$ with the weight factor

$$p(x) = P(x) \int P(x') dx', \quad (53)$$

$$P(x) = \frac{\exp\left(-\frac{x^2}{2D_1}\right)}{\sqrt{2\pi D_1}} \sum_k \rho_{kk}(0) \frac{\exp\left[-\frac{(\bar{I} - x - I_k)^2}{2D_0}\right]}{\sqrt{2\pi D_0}}. \quad (54)$$

This procedure (without account of other noise sources) will lead to the equation presented in Ref. 49 and corresponds to the detector ideality $\eta = S_0/(S_0 + S_1)$, similar to the one-qubit case.

For the classical backaction noise which is 100% correlated with $\xi_1(t)$, we should take into account that it can be coupled differently to different qubits. Let us assume that the energy of each state $|i\rangle$ is affected by the classical backaction noise, proportional to ξ_1 , so that $\varepsilon_i \rightarrow \varepsilon_i + A_i \xi_1(t)$, where A_i are arbitrary constants. Then Eq. (51) should be multiplied by the factor $\exp[i(A_j - A_i)x\tau/\hbar]$ (in the one-qubit case the previously defined A would correspond to $A_2 - A_1$).

Similarly, to take into account the possible asymmetry of the quantum backaction noise, let us assume that each electron tunneling through the ideal part of the detector shifts the phases corresponding to states $|i\rangle$ (differently for different states), that leads to the extra factor $\exp[i(\theta_i - \theta_j)\bar{I}_d\tau/\hbar]$ in Eq. (51).

The uncorrelated classical noise $\xi_3(t)$ is also assumed to be coupled differently to the states $|i\rangle$, so that $\varepsilon_i \rightarrow \varepsilon_i + g_i \xi_3(t)$, where g_i are some constants. The averaging over noise ξ_3 is simple and leads to the extra factor $\exp[-(g_i - g_j)^2 S_3 \tau / 4\hbar^2]$ in Eq. (51).

Taking into account the effect of θ_i , averaging over the noise $\xi_1(t)$ (and fully correlated backaction noise) and $\xi_3(t)$, differentiating equations over time, and adding the intrinsic evolution of qubits, we finally obtain the following equation in the Stratonovich form:

$$\begin{aligned} \dot{\rho}_{ij} = & -\frac{i}{\hbar} [H_{qbs}, \rho]_{ij} + i \frac{\Delta\varepsilon_{ij}}{\hbar} \rho_{ij} + i K_{ij} \left[I(t) - \frac{I_i + I_j}{2} \right] \rho_{ij} \\ & + \rho_{ij} \frac{1}{S_\Sigma} \sum_k \rho_{kk} \left[\left(I(t) - \frac{I_i + I_k}{2} \right) (I_i - I_k) \right. \\ & \left. + \left(I(t) - \frac{I_j + I_k}{2} \right) (I_j - I_k) \right] - \tilde{\gamma}_{ij} \rho_{ij}, \end{aligned} \quad (55)$$

where the first term describes the intrinsic evolution of qubits due to H_{qbs} , $\Delta\varepsilon_{ij} = (\theta_i - \theta_j)(I_i + I_j)/2$ is the effective energy shift due to detector asymmetry, $S_\Sigma = S_0 + S_1$ is the total output noise,

$$K_{ij} = \frac{(A_j - A_i)S_1 + (\theta_i - \theta_j)S_0}{\hbar S_\Sigma} \quad (56)$$

is the correlation factor between output and backaction noises, and the dephasing rate is

$$\begin{aligned} \tilde{\gamma}_{ij} = & \frac{(I_i - I_j)^2 S_1}{4S_0 S_\Sigma} + \frac{(g_i - g_j)^2 S_3}{4\hbar^2} \\ & + \frac{S_0 S_1}{4\hbar^2 S_\Sigma} [(A_j - A_i) - (\theta_i - \theta_j)]^2. \end{aligned} \quad (57)$$

Notice that there are obviously no dephasing terms for diagonal matrix elements. Also notice that Eq. (55) is applicable to both linear and nonlinear detectors, including purely quadratic detectors,⁵⁰ as long as the condition of weak response is satisfied.

Translating Eq. (55) into Itô form, we obtain

$$\begin{aligned} \dot{\rho}_{ij} = & -\frac{i}{\hbar} [H_{qbs}, \rho]_{ij} + i \frac{\Delta \varepsilon_{ij}}{\hbar} \rho_{ij} + i K_{ij} \left(I(t) - \sum_k \rho_{kk} I_k \right) \rho_{ij} \\ & + \rho_{ij} \frac{1}{S_\Sigma} \left(I(t) - \sum_k \rho_{kk} I_k \right) \left(I_i + I_j - 2 \sum_k \rho_{kk} I_k \right) - \Gamma_{ij} \rho_{ij}, \end{aligned} \quad (58)$$

where the ensemble dephasing Γ_{ij} is related to the single system dephasing $\tilde{\gamma}_{ij}$ as

$$\Gamma_{ij} = \frac{(I_i - I_j)^2}{4S_\Sigma} + \frac{K_{ij}^2 S_\Sigma}{4} + \tilde{\gamma}_{ij} \quad (59)$$

and in our particular case is equal to

$$\Gamma_{ij} = \frac{(I_i - I_j)^2}{4S_0} + \frac{(A_j - A_i)^2 S_1}{4\hbar^2} + \frac{(\theta_i - \theta_j)^2 S_0}{4\hbar^2} + \frac{(g_i - g_j)^2 S_3}{4\hbar^2}. \quad (60)$$

Since the combination $I(t) - \sum_k \rho_{kk} I_k$ in Eq. (58) is a pure noise because of the relation

$$I(t) = \sum_k \rho_{kk} I_k + \xi_0(t) + \xi_1(t), \quad (61)$$

the ensemble averaged evolution is described by the reduced equation

$$\dot{\rho}_{ij} = -\frac{i}{\hbar} [H_{qbs}, \rho]_{ij} + i \frac{\Delta \varepsilon_{ij}}{\hbar} \rho_{ij} - \Gamma_{ij} \rho_{ij}. \quad (62)$$

Because of the reciprocity, it is natural to assume that the backaction couplings $A_j - A_i$, $\theta_i - \theta_j$, and $g_i - g_j$ are proportional to the signal coupling $I_i - I_j$, so that $A_j - A_i = (I_i - I_j)a$, $\theta_i - \theta_j = (I_i - I_j)\Theta$, and $g_i - g_j = (I_i - I_j)g$. (Actually, this assumption implies detector linearity and also that all interactions with qubits occur via one ‘‘port of entry.’’ It is not valid, for example, when several geometrical parts of the detector interact with qubits in different ways.) With this assumption the parameters $\Delta \varepsilon_{ij}$, K_{ij} , $\tilde{\gamma}_{ij}$, and Γ_{ij} used in evolution equations (55) and (58) become

$$\Delta \varepsilon_{ij} = \Theta(I_i^2 - I_j^2)/2, \quad (63)$$

$$K_{ij} = \frac{aS_1 + \Theta S_0}{\hbar S_\Sigma} (I_i - I_j), \quad (64)$$

$$\tilde{\gamma}_{ij} = (I_i - I_j)^2 \left[\frac{S_1}{4S_0 S_\Sigma} + \frac{(a - \Theta)^2 S_0 S_1}{4\hbar^2 S_\Sigma} + \gamma_{3,n} \right], \quad (65)$$

$$\Gamma_{ij} = (I_i - I_j)^2 \left(\frac{1}{4S_0} + \frac{a^2 S_1}{4\hbar^2} + \frac{\Theta^2 S_0}{4\hbar^2} + \gamma_{3,n} \right), \quad (66)$$

where $\gamma_{3,n} = gS_3/4\hbar^2$. Notice that there will be no dephasing between states $|i\rangle$ and $|j\rangle$ if the detector is equally coupled to these states, $I_i = I_j$.

The detector ideality in this case can be characterized by a single number (or few numbers for different definitions), which does not depend on the state of the measured system. Extending the definitions (24), (39), and (40) discussed in previous sections, the detector ideality can be characterized by the parameter combinations

$$\eta = \frac{1/4S_\Sigma}{\Gamma_{\Sigma,n}}, \quad \tilde{\eta} = \frac{1/4S_\Sigma + K_n^2 S_\Sigma/4}{\Gamma_{\Sigma,n}},$$

$$\tilde{\eta}_2 = \frac{1/4S_\Sigma}{\Gamma_{\Sigma,n} - K_n^2 S_\Sigma/4}, \quad (67)$$

where $K_n \equiv (aS_1 + \Theta S_0)/\hbar S_\Sigma$ and $\Gamma_{\Sigma,n} \equiv 1/4S_0 + a^2 S_1/4\hbar^2 + \Theta^2 S_0/4\hbar^2 + \gamma_{3,n}$. In the case $a = \Theta = 0$ all definitions of ideality coincide and the evolution equation (55) reduces to the equation derived in Ref. 49. In the case of finite a and/or Θ , more natural definitions are $\tilde{\eta}$ and $\tilde{\eta}_2$ (again, $\tilde{\eta}_2^{-1/2}$ is the total energy sensitivity in units of $\hbar/2$). However, ideality η can also be a useful parameter, for example, if there is no way to control the degree of freedom affected by the backaction noise $\Theta \xi_0 + a \xi_1$, and therefore the corresponding dephasing cannot be reduced by a feedback procedure.

VII. CONCLUSION

In this paper we have analyzed the process of continuous measurement of a solid-state qubit by a nonideal solid state detector. We have considered the phenomenological model of the detector (Fig. 1) consisting of an ideal (quantum-limited) part and classical noise sources which contribute to the output (ξ_1) and backaction ($\xi_2 + \xi_3$) noises. The possible correlation between classical output and backaction noise sources is taken into account by separating the backaction noise into a contribution $\xi_2(t)$ fully correlated with output noise $\xi_1(t)$ and the uncorrelated contribution $\xi_3(t)$. For the description of the ideal part we have started with the Bayesian equations of Refs. 4, 10 and 19 and then used the model of an asymmetrically coupled ideal detector developed in Ref. 28. The asymmetric coupling changes the self-consistent energy difference between two qubit states. Also, this change fluctuates in time and the fluctuations are correlated with the output noise, thus producing an effect similar to the correlation of classical noises.

The main result of the paper for the one-qubit case is the

derivation of evolution equations (30)–(31) and (34)–(35) in Stratonovich and Itô form, respectively. In these equations the detector is characterized by the total output noise S_Σ , induced ensemble qubit decoherence rate Γ_Σ , and the total correlation K [see Eq. (50)] between output and backaction noises, so that the phenomenological detector separation into the quantum part and extra noises is irrelevant. (Notice that these three quantities are the counterparts of output, input, and cross-correlation noise terms used for the description of a classical amplifier.) The relation between ensemble and single qubit decoherence rates is given by Eq. (32), which leads to the fundamental limitation (38) for the ensemble decoherence rate. The discussed definitions of the detector ideality [see Eqs. (24), (39), and (40)] are various combinations of the single qubit decoherence rate, ensemble decoherence, and the “information acquisition” rate $(\Delta I)^2/4S_\Sigma$. A

100% ideal detector corresponds to the absence of single qubit decoherence.

The theory developed for a single qubit measurement is generalized to the case of entangled qubits in section VI. The evolution equation is given by Eqs. (55) and (58), while the relation between ensemble and single system decoherence rates is given by Eq. (59).

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⁴⁶R. Ruskov (unpublished).

⁴⁷In Ref. 28 it is stated that the effect of finite θ is always small in

the “quantum diffusion” (“weakly responding”) case, $|\Delta I| \ll I_0$. We think that actually its effect can be significant.

⁴⁸Notice that for the model of a nonideal detector which “sometimes misses the detection,” the quantum trajectory approach (Ref. 28) leads to an additional dephasing term in Eq. (43), similar to the effect of our classical noises ξ_1 and ξ_3 .

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