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Quantum feedback control of a solid-state qubit

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We have studied theoretically the basic operation of a quantum feedback loop designed to maintain a desired phase of quantum coherent oscillations in a single solid-state qubit. The degree of oscillations synchronization with external harmonic signal is calculated as a function of feedback strength, taking into account available bandwidth and coupling to environment. The feedback can efficiently suppress the dephasing of oscillations if the qubit coupling to the detector is stronger than the coupling to the environment.

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The principle of feedback control is used in a wide variety of physical and engineering problems. In particular, it can be applied in a straightforward way to tune the oscillation phase of a harmonic oscillator in order to achieve a desired synchronization with some reference oscillator. An intriguing and fundamental question is whether continuous feedback can be used to control quantum systems; for instance, whether or not it is possible to tune the phase of quantum coherent (Rabi) oscillations in a qubit (two-level system)?

At first sight the quantum feedback seems to be impossible because according to the ''orthodox'' collapse postulate[1] the quantum state is abruptly destroyed by the act of measurement. However, as was shown 2 decades ago, in particular by Leggett,[2] in a typical solid-state setup the collapse of a qubit state should be considered as a continuous process rather than as an instantaneous event.

While the Leggett theory as well as a majority of similar approaches can describe only *ensembles* of quantum systems, the theory describing the gradual collapse of a *single* solid-state qubit was developed only recently.[3–5] (A similar problem in quantum optics was solved much earlier—see, e.g., Refs. 6 and 7 and references in Ref. 4.) Basically, the theory says that the evolution of a single quantum system due to continuous measurement is governed by the information continuously acquired from the detector. Similarly to classical probability, the Bayes formula[8] that naturally takes into account incomplete information from the detector, can still be applied to the density matrix of the measured quantum system; thus the formalism is called Bayesian.[3]

In case of a poor detector the extra noise acting back onto the input disturbs the measured system more than the limit determined by the uncertainty principle; this leads to gradual decoherence of the measured system. In contrast, when measured with a good (quantum-limited) detector, the quantum system does not loose the coherence (even though the quantum state evolves randomly); moreover, its density matrix can be gradually purified[3] which basically means acquiring as much information about the system as permitted by quantum mechanics.

Since the Bayesian formalism allows us to monitor the continuous evolution of a quantum system in a process of measurement, this naturally gives rise to a possibility of continuous feedback control of a quantum system. In this paper, we will study the operation of a feedback loop proposed in Ref. 4 and designed to maintain a desired phase of quantum coherent oscillations in a solid-state qubit. (Quantum feedback in optics has been proposed and studied earlier—see, e.g., Refs. 7, 9–15.) In particular, we will study dependence of the loop operation on the feedback strength, available bandwidth, and dephasing due to environment.

As an example of the measurement setup (Fig. 1) we consider a qubit represented by a single electron in a double quantum dot (DQD), the location of which is measured by a quantum point contact (QPC) nearby in a way used in Ref. 16. If the electron is in the dot 2 (state $|2\rangle$) which is closer to QPC than dot 1, then the QPC tunnel barrier is higher and so the average current I_2 through QPC is smaller than the average current I_1 corresponding to the electron in the dot 1 (state $|1\rangle$). Consequently, from the QPC current one gets information about the electron location. We consider a realistic case of weak response, $\Delta I \equiv I_1 - I_2 \ll I_0 \equiv (I_1+I_2)/2$. In this case the measurement time $S_I/2(\Delta I)^2$, which is necessary to achieve signal-to-noise ratio equal to 1 (here S_I is the QPC shot noise), is much larger than e/I_0, so the QPC current $I(t)$ is continuous on the measurement time scale.

The evolution of the qubit density matrix ρ during the measurement process is described within the Bayesian formalism by equations[3,4]

$$\dot{\rho}_{11} = -\dot{\rho}_{22} = -2\frac{H}{\hbar}\ \mathrm{Im}\rho_{12} + \rho_{11}\rho_{22}\frac{2\Delta I}{S_I}[I(t)-I_0], \quad (1)$$

$$\dot{\rho}_{12} = i\frac{\varepsilon}{\hbar}\rho_{12} + i\frac{H}{\hbar}(\rho_{11}-\rho_{22}) - (\rho_{11}-\rho_{22})\frac{\Delta I}{S_I}[I(t)-I_0]\rho_{12} - \gamma\rho_{12}, \quad (2)$$

where ε and H are, respectively, the energy asymmetry and tunneling strength of the qubit [the qubit Hamiltonian is $\mathcal{H}_{qb} = (\varepsilon/2)(c_2^\dagger c_2 - c_1^\dagger c_1) + H(c_1^\dagger c_2 + c_2^\dagger c_1)$], and $\gamma = \gamma_d + \gamma_e$ is the dephasing rate due to the detector nonideality (γ_d) and coupling with the environment (γ_e).[17] Theoretically, $\gamma_d = 0$ when qubit is measured by a QPC; however, if instead of QPC we use a single-electron transistor (SET), then dephasing γ_d is usually quite significant[4,18] (except the case when the SET operates in a cotunneling regime[19,20]).

Notice that the ensemble dephasing rate $\Gamma = \gamma + (\Delta I)^2/4S_I$ is larger than γ because of differing evolution of the ensemble members due to random $I(t)$. Individual realizations can be simulated using the formula[4]

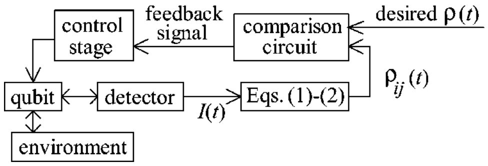

FIG. 1. Schematic of the quantum feedback loop maintaining the quantum oscillations in a qubit.

$$I(t)-I_0=(\rho_{11}-\rho_{22})\Delta I/2+\xi(t), \tag{3}$$

where $\xi(t)$ is the pure white noise with spectral density $S_\xi = S_I$. If Eqs. (1) and (2) are averaged over $\xi(t)$ (we use Stratonovich definition for stochastic differential equations), then we get usual ensemble-averaged equations (terms proportional to ΔI disappear and γ is replaced by Γ).

It is natural to characterize the effect of extra dephasing γ_d by the detector ideality (efficiency) $\eta \equiv 1/[1+\gamma_d 4S_I/(\Delta I)^2]$. One can show[4,21] that $\eta=(\hbar/2\epsilon_d)^2$ where ϵ_d is the total energy sensitivity of the detector $[\epsilon_d \equiv(\epsilon_i\epsilon_o)^{1/2}$, where ϵ_o is the usual (output) energy sensitivity and ϵ_i is a similar quantity characterizing backaction to the input]. So, an ideal case $\eta=1$ corresponds to a detector with quantum-limited sensitivity.

To realize a feedback loop (Fig. 1), we can monitor the qubit evolution using the detector current $I(t)$ plugged into Eqs. (1) and (2). Then the qubit state is compared with the desired state, and the difference signal is used to control the qubit parameters H and/or ε. In our example the feedback loop is designed to stabilize the quantum oscillations of the state of a symmetric qubit ($\varepsilon=0$), so the desired evolution is $\rho_{11}(t)=1-\rho_{22}(t)=[1+\cos(\Omega t)]/2$, $\rho_{12}(t)=\rho_{21}^*(t) = i\sin(\Omega t)/2$, where the frequency is $\Omega=(4H^2+\varepsilon^2)^{1/2}/\hbar = 2H/\hbar$. As a difference ("error") signal we use the phase difference $\Delta\phi$ ($|\Delta\phi|<\pi$) between the desired value $\phi_0(t) = \Omega t (\mathrm{mod}\, 2\pi)$ and the monitored value $\phi(t) \equiv \arctan(2\,\mathrm{Im}\rho_{12}(t)/[\rho_{11}(t)-\rho_{22}(t)])$. This difference is used to control the qubit parameter H (changing the barrier height of DQD); here we study a linear control: $H_{fb}=(1-F\times\Delta\phi)H$, where F is the dimensionless feedback factor.[22]

In this paper we neglect additional time delay[4] in the feedback network, however, we take into account the finite bandwidth of a line carrying detector current (which is a critical parameter for a possible experiment). More specifically, we average the current $I(t)$ with a rectangular window of duration τ_a, $I_a(t)\equiv\tau_a^{-1}\int_{t-\tau_a}^{t} I(t')dt'$, before plugging it into Eqs. (1) and (2), so that the "available" density matrix $\rho_a(t)$ differs from the "true" density matrix $\rho(t)$. Also, to compensate for the corresponding implicit time delay, we use $\Delta\phi = \phi_a - \Omega(t-\kappa\tau_a)$ with $\kappa=1/2$ (we found that $\kappa=1/2$ provides the best operation of the feedback loop).

Let us start with the case of ideal detector, $\eta=1$, absence of extra environment, $\gamma_e=0$, and infinite bandwidth, $\tau_a=0$. Figure 2 shows numerically calculated correlation function $K_z(\tau)\equiv\langle z(t+\tau)z(t)\rangle$ where $z\equiv\rho_{11}-\rho_{22}$, for several feedback factors: $F=0$, 0.05, and 0.5. The curves are obtained using Monte Carlo simulation[3,4] of the measurement process

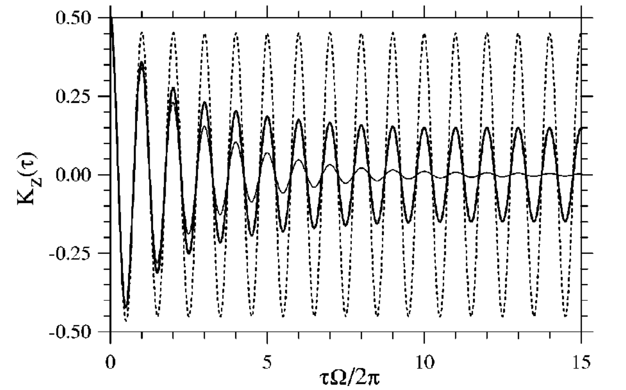

FIG. 2. Correlation function $K_z(\tau)$ of the qubit quantum oscillations for $\mathcal{C}=1$ and feedback factors $F=0$ (thin solid line), 0.05 (thick solid line), and 0.5 (dashed line). Nondecaying oscillations are due to synchronization by the feedback.

for moderately weak coupling between the qubit and detector: $\mathcal{C}\equiv\hbar(\Delta I)^2/S_I H=1$ (notice that the Q factor of oscillations[23] is equal to $8/\mathcal{C}$, so $\mathcal{C}=1$ is still a weak coupling). In the absence of feedback ($F=0$) the correlation function decays to zero, while for finite feedback factor the correlations remain for indefinitely long time (assuming perfect reference oscillator). The nondecaying correlations show that the quantum feedback loop really provides the synchronization of quantum oscillations. The degree of synchronization depends on the feedback factor F. One can see that for a moderate value of $F=0.5$ the synchronization is already very good [the ideal case would be $K_z(\tau)=\cos(\Omega\tau)/2$].

For analytical analysis we take into account that in the ideal case $\gamma_d=\gamma_e=0$ the qubit state is pure,[4] and using Eqs. (1)–(3) start with the equation

$$\frac{d}{dt}\Delta\phi = -\sin\phi\,\frac{\Delta I}{S_I}\left(\frac{\Delta I}{2}\cos\phi+\xi\right)-\frac{2FH}{\hbar}\Delta\phi, \tag{4}$$

which assumes the absence of 2π phase slips (good or moderate synchronization). For weak coupling ($\mathcal{C}/8\ll 1$) we can neglect the first term in parentheses and average the random term over $\sin\phi$ assuming almost harmonic evolution that leads to the simplified equation

$$\frac{d}{dt}\Delta\phi = \tilde{\xi} - \frac{2FH}{\hbar}\Delta\phi, \tag{5}$$

where $\tilde{\xi}(t)$ is the white noise with spectral density $S_{\tilde{\xi}} = (\Delta I)^2/2S$. This equation describes a particle diffusion in the parabolic potential (we again assume $|\Delta\phi|<\pi$). The corresponding Fokker-Planck equation has an exact solution that is used to calculate the correlation function $K_z(\tau) \approx \langle\cos[\Delta\phi(t)-\Delta\phi(t+\tau)]\rangle\cos\Omega\tau/2$. In this way we obtain the analytical expression

$$K_z(\tau)=\frac{\cos\Omega\tau}{2}\exp\left[\frac{\mathcal{C}}{16F}\left(e^{-2FH\tau/\hbar}-1\right)\right], \tag{6}$$

which fits well the Monte Carlo results when $\mathcal{C}/8\ll 1$ and $\mathcal{C}/16F\lesssim 1$ (weak coupling and moderate or good synchronization). As an example, the dots in Fig. 3 show the numeri-

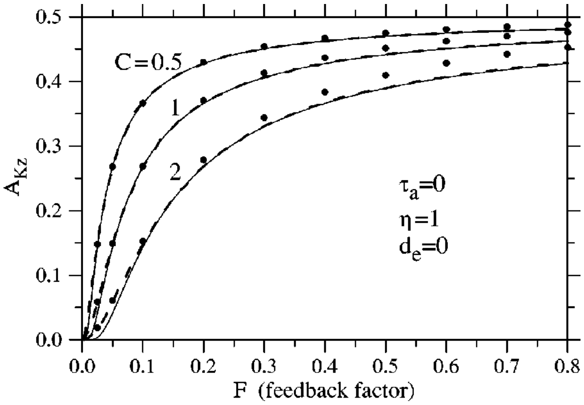

FIG. 3. Dots: asymptotic amplitude A_{Kz} of $K_z(\tau)$ oscillations as a function of feedback factor F for several couplings with the detector, $\mathcal{C}=0.5$, 1, and 2. Solid lines: analytical approximation $A_{Kz}=\exp(-\mathcal{C}/16F)/2$. Dashed lines: corresponding numerical results for $D^2/2$.

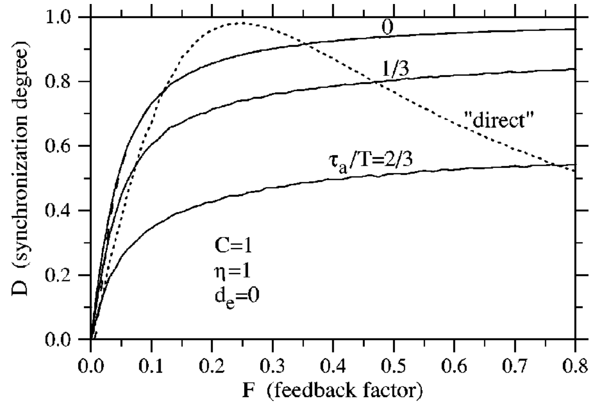

FIG. 4. Synchronization degree D as a function of feedback factor F for several values τ_a of detector signal averaging: $\tau_a/T=0$, 1/3, and 2/3, where $T=2\pi/\Omega$. Dashed line $D=\exp(-\mathcal{C}/32F)$ almost coincides with the upper curve. Dotted line corresponds to "direct" feedback with $\tau_a=T/10$.

cally calculated (using the least-mean-square fit) asymptotic amplitude A_{Kz} of $K_z(\tau)$ oscillations (at $\tau\rightarrow\infty$) as a function of the feedback factor F for three values of the coupling $\mathcal{C}$, while solid lines show the corresponding analytical curves $A_{Kz}=\exp(-\mathcal{C}/16F)/2$.

The correlation function $K_I(\tau)\equiv\langle I(t+\tau)I(t)\rangle$ of the detector current $I(t)$ is somewhat similar to $K_z(\tau)$, however, it also has the decaying contribution[23] due to correlation $K_{z\xi}$ and a δ-function contribution due to the detector noise. The analytical result for the same regime as above,

$$K_I(\tau)=\frac{S_I}{2}\delta(\tau)+\frac{(\Delta I)^2}{4}(1+e^{-2FH\tau/\hbar})K_z(\tau), \qquad (7)$$

also agrees well with the Monte Carlo results. The spectral density $S_I(\omega)$ of the detector current can be obtained as a Fourier transform of $K_I(\tau)$. While in the absence of feedback, the quantum oscillations in the qubit can provide only a moderate peak of $S_I(\omega)$ around frequency Ω (the peak height cannot be larger than four times the noise pedestal[23]) the feedback synchronization leads to the appearance of a δ function at the frequency of desired oscillations.

Besides the correlation function and spectral density, we have studied one more characteristic, D, of the synchronization degree. We define D as the average scalar product of the unit-length vector on the Bloch sphere corresponding to the desired state and the vector corresponding to the actual state of the qubit. The equivalent definition is $D\equiv 2\langle \mathrm{Tr}\rho\rho_d\rangle-1$, where ρ_d is the density matrix of the desired pure state. [The so-called fidelity is equal to either $(D+1)/2$ or $\sqrt{(D+1)/2}$, depending on the definition.[15]] Perfect synchronization corresponds to $D=1$. It is simple to show that in the limit of weak coupling and for symmetric distribution of $\Delta\phi$ (unshifted desired frequency), A_{Kz} coincides with $D^2/2$. Notice, however, that at moderate coupling, $D^2/2$ (dashed lines in Fig. 3) is much closer to the analytical result than A_{Kz}.

Upper solid line in Fig. 4 shows the dependence of D on the feedback factor F for $\mathcal{C}=1$ and $\tau_a=0$. One can see that D is proportional to F for small F ("soft" onset of the synchronization) and D is asymptotically approaching 1 at large F. The analytical result $D=\exp(-\mathcal{C}/32F)$ (dashed line in Fig. 4) is very close to the numerical results at moderate and good synchronization.

Finite available bandwidth of the detector current $I(t)$ (finite averaging time τ_a in our formalism) worsens the performance of the quantum feedback loop. The solid lines in Fig. 4 show the dependence of the synchronization degree $D(F)$ for $\tau_a/T=0$, 1/3, and 2/3, where $T=2\pi/\Omega$ is the oscillation period. Obviously, a significant information loss occurs when τ_a becomes comparable to T, leading to a decrease of D. The curves $D(F)$ saturate at large F allowing us to introduce the dependence $D_{max}(\tau)$. Calculations for the parameters of Fig. 4 show pretty good synchronization, $D_{max}=0.993$, for $\tau_a=T/30$, while $D_{max}=0.98$, 0.92, and 0.57 for $\tau_a=T/10$, $T/3$, and $2T/3$, respectively.

The main potential practical importance of the quantum feedback is the ability to suppress the effect of the qubit dephasing caused by interaction with the environment (see Fig. 1). This can be used, for example, for qubit initialization in a solid-state quantum computer. Solid lines in Fig. 5 show the dependence $D(F)$ for several magnitudes of the dephasing due to environment, $d_e=0$, 0.1, and 0.5, where $d_e\equiv\gamma_e/[(\Delta I)^2/4S_I]$ is the ratio between the qubit coupling to the environment and to the detector (we still assume an ideal detector). First of all, we see that the feedback still maintains the qubit phase synchronization for infinitely long time. However, for finite d_e the degree of synchronization D saturates at a level less than unity. We have studied numerically the dependence $D_{max}(d_e)$ for $\mathcal{C}=1/2$, 1, and 2 (while $\tau_a=0$ and $\eta=1$) and found a linear dependence at small d_e: $D_{max}\simeq 1-0.5d_e$. [A little better formula $D_{max}\simeq 1-0.5d_e/(1+d_e)$ works reasonably well up to $d_e\lesssim 1$.] This means that the feedback loop can efficiently suppress the qubit dephasing due to the coupling to the environment if this coupling is much weaker than the qubit coupling to a nearly ideal detector.

Notice that the solid lines shown in Figs. 4 and 5 are calculated assuming the feedback control of the tunnel ma-

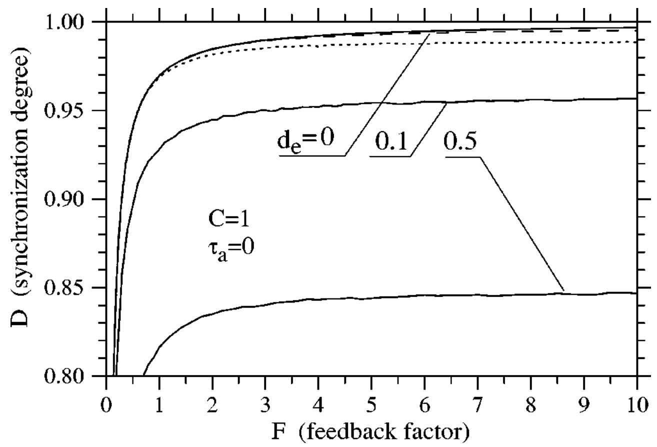

FIG. 5. Dependence $D(F)$ for $\mathcal{C}=1$, $\tau_a=0$, and several magnitudes of dephasing due to environment: $d_e=0$, 0.1, and 0.5. Dashed and dotted lines correspond to $d_e=0$ and limitation of H_{fb} by 0 and $H/2$, respectively.

trix element $H_{fb}=H(1-F\times\Delta\phi)$ even when H_{fb} becomes negative (this is also an assumption for the analytical results). To eliminate this unphysical assumption we have also performed numerical calculations with restrictions $H_{fb}>0$ and $H_{fb}>H/2$. This leads to rather minor modifications of the presented curves (dashed and dotted lines in Fig. 5 show the results for $d_e=0$ and $\tau_a=0$). However, important difference is that $D(F)$ goes down at large F, so the optimum D_{max} is achieved at some finite value of F.

Besides the discussed feedback based on $\Delta\phi$ calculation, we have also studied a ‘‘direct’’ feedback loop in which $H_{fb}(t)/H-1=F(2[I_a(t)-I_0]/\Delta I-\cos[\Omega(t-\tau_a/2)])\sin[\Omega(t-\tau_a/2)]$ (we call it also a ‘‘naive’’ feedback because this control formula is easily designed from the naive assumption that the detector current directly follows the evolution of ρ_{11}). Direct feedback is much simpler for experimental realization since it does not require real-time solution of the Bayesian equations (direct feedback in quantum optics has been studied in Refs. 7, 10–13). Surprisingly, the direct feedback can also provide a good phase synchronization of quantum oscillations if $F/\mathcal{C}$ is close to 1/4 (see dotted line in Fig. 4). However, it requires more careful choice of F and τ_a than for the Bayesian feedback, and also suffers more significantly from the restriction on H_{fb} variation.

Experimentally, besides the realization of quantum feedback control of a DQD continuously measured by a QPC, one can also think about the qubit based on a single-Cooper-pair box measured by a single-electron transistor (see discussion in Ref. 4). This realization can be preferable because of a rapid progress of metallic single-electronics technology. However, the problems are high output impedance of the single-electron transistor and its nonideality as a quantum detector. The third potential realization can be based on superconducting quantum interference devices. For any realization the major problem is the bandwidth: the feedback should be at least faster than the qubit dephasing. Because of that, the quantum feedback of a solid-state qubit should probably be attempted only after the realization of recently proposed Bell-type two-detector correlation experiment,[24] which would show the possibility of quantum monitoring, the first step to quantum feedback control.

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