Autonomous Demand Response in Heterogeneous Smart Grid Topologies

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***Abstract*—Autonomous demand response (DR) is scalable and has minimal control overhead on utilities by encouraging users to minimize their own energy expenditure. While the prior results on autonomous DR are promising, they are all limited to *homogeneous* grid topologies, such as a microgrid or a small distribution feeder. In this paper, we take the first step to investigate autonomous DR in *heterogeneous* grid topologies, i.e., a macrogrid, where users who participate in autonomous DR programs are scattered across different buses. Our analysis requires expanding the existing autonomous DR to also include power flow analysis across the power grid. To gain insight, we perform two analytical case studies and show that the results can be very different from the results previously reported on homogeneous autonomous DR systems. We also provide recommendations to design efficient, fair, and practical autonomous demand response systems in heterogeneous smart grid topologies.**

***Keywords*: Autonomous demand response, heterogeneous grid, locational marginal price, game theory, Nash equilibrium.**

I. Introduction

Demand response (DR) programs are implemented by utilities to control the energy consumption at the consumer side of the meter in response to changes in grid operating conditions [1]. One approach in DR is direct load control (DLC), where the utility remotely controls energy consumption for certain high-load household appliances such as air-conditioners and water-heaters [2]. An alternative for DLC is smart pricing, where users are encouraged to *individually* and *voluntarily* manage their loads, e.g., by reducing their consumption at peak hours [3]. This can be done using automated Energy Consumption Scheduling (ECS) units that are embedded in users' smart meters, as suggested in [4]. For each user, the ECS unit finds the best load schedule to minimize the user's electricity bill while fulfilling the user's energy needs. This can lead to *autonomous* DR programs that are self-organizing and burden a minimal control overhead on utilities.

The literature on autonomous DR using smart pricing is extensive, c.f., [4]–[8]. In many cases the analytical tool that is used to study autonomous DR systems is Game Theory [9]. A common assumption in most prior game-theoretic studies of autonomous DR systems is that the users who participate in DR form a *homogeneous* grid topology, as shown in Fig. 1(a). That is, they are either part of a microgrid or connected to the same grid bus or a single generator. The cost function $C(\cdot)$ indicates the cost of power generation or purchase, which

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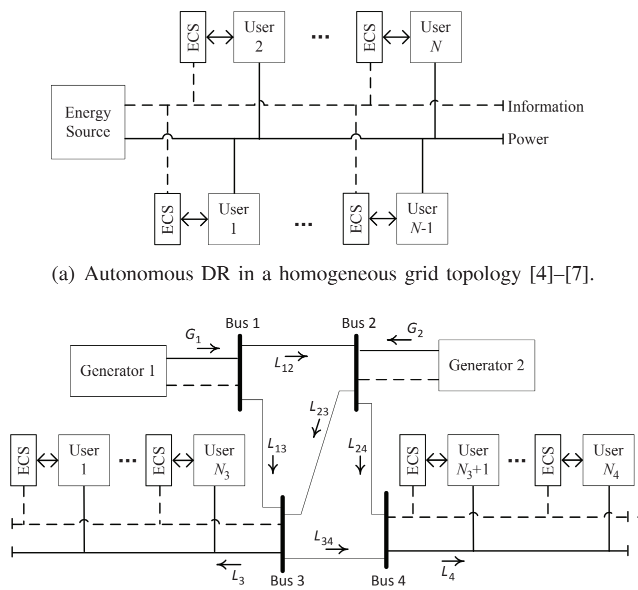

(a) Autonomous DR in a homogeneous grid topology [4]–[7].

(b) Autonomous DR in a heterogeneous grid topology.

Fig. 1. Two types of autonomous demand response systems in smart grids.

is usually a convex function of the aggregate load. Based on this cost function, the price of electricity is determined, and all users are charged with an *equal* price. However, in most practical cases, the DR system is *heterogeneous*, i.e., the generators and/or the users are distributed across different buses, as in Fig. 1(b). In that case, users may face different prices, making their interactions within the autonomous DR framework significantly more complicated. To the best of our knowledge, no prior work has addressed game-theoretic analysis of autonomous DR systems in heterogeneous topologies.

In general, the price of electricity at each bus in heterogeneous grid topologies is determined based on *locational marginal prices* (LMPs), which depend on parameters such as the line capacities and location, type, and generation cost for each generator. Most existing deregulated electricity markets in the United States currently use LMPs to settle various bulk sale and ancillary service transactions [10]. Although setting retail prices according to LMPs is still not a common practice in most U.S. regions, it is recently shown that by reflecting the prices in the wholesale market to the demand side, users will be better encouraged to consume electricity more efficiently [11]. Therefore, in this paper, we assume that the price of electricity that each user faces at each bus is the LMP at that bus. To facilitate a game-theoretic analysis of heterogeneous autonomous DR systems, we use closed-form models for LMPs at each bus based on optimal power flow (OPF) analysis to accordingly form the payoffs for each user. We show that, due to the complexity of such closed-form models, the game-theoretic analysis of autonomous DR systems in heterogeneous grid topologies is significantly more difficult compared to that in homogeneous grid topologies. Therefore, we limit our analysis to a case study based on the grid topology in Fig. 1(b). Also, in order to simplify the analysis, we use the classic one-stage game model with pure strategies. The results are insightful and quite different from the results reported in [4]–[8] for autonomous DR systems in *homogeneous* grid topologies. Based on these results, we make recommendations to design heterogeneous DR systems that more stable that benefit both the grid and consumers.

II. System Model and Problem Statement

Consider the heterogeneous autonomous DR system in Fig. 1(b). Let $\mathcal{N}_3 = \{1, \ldots, N_3\}$ and $\mathcal{N}_4 = \{N_3+1, \ldots, N_4\}$ denote the set of users that are connected to buses 3 and 4, respectively. We define $\mathcal{N} = \mathcal{N}_3 \cup \mathcal{N}_4$. For each user $n \in \mathcal{N}$, we denote l_n^h as the hourly load of user n at hour $h \in \mathcal{H}$, where $\mathcal{H} = \{1, \ldots, H\}$. For a daily DR analysis, we have $H = 24$. The daily electricity bill for user n at bus i is calculated as

$$B_n = \sum_{h=1}^{H} l_n^h \times \text{LMP}_i^h \tag{1}$$

where LMP_i^h is the locational marginal price at hour h on bus i. Under the autonomous DR paradigm [4], the ECS device corresponding to user n seeks to schedule energy consumption for user n such that user n's electricity bill is minimized. Without loss of generality, we assume that each user $n \in \mathcal{N}$ has exactly one load whose operation is time shiftable within a given time frame of $[\alpha_n, \beta_n]$, where $1 \leq \alpha_n < \beta_n \leq H$. Let E_n denote the total electric energy that has to be consumed in order to finish the operation of user n's load. Several examples for the choices of parameters α_n, β_n, and E_n are available in [4], [12]. User n's ECS device schedules his load profile $\boldsymbol{l}_n = (l_n^1, \ldots, l_n^H)$ to minimize user n's daily electricity bill by solving the following optimization problem:

$$\begin{aligned} \underset{\boldsymbol{l}_n}{\textbf{Minimize}} \quad & \sum_{h=1}^{H} l_n^h \times \text{LMP}_i^h \\ \textbf{Subject to} \quad & \sum_{h=\alpha_n}^{\beta_n} l_n^h = E_n. \end{aligned} \tag{2}$$

Since the LMP at every hour h depends on the load profile of *all* users at hour h, the bill for each user n can be affected not only by his own load profile $\boldsymbol{l}_n$, but also by the load profiles of other users $\boldsymbol{l}_{-n} = (\boldsymbol{l}_m, \forall m \in \mathcal{N} \backslash \{n\})$. This leads to forming the following autonomous DR game among users:

- **Players**: All users in set $\mathcal{N}$.
- **Actions**: For every user $n \in \mathcal{N}$, his load profile $\boldsymbol{l}_n$.
- **Payoffs**: For every user $n \in \mathcal{N}$, minus his bill: $-B_n$.

Autonomous DR game models similar to the one above have already been addressed, e.g., in [4]–[7]. However, all previous works have focused only on homogeneous topologies while here we focus on heterogeneous topologies. The latter is more challenging due to the complexity of the LMP models when it comes to game-theoretic analysis. We are particularly interested in investigating the solution concept of Nash equilibrium for the formulated game model. For the rest of this paper, we make the following assumptions: First, there are only two users in the system, one on bus 3 and the other one on bus 4. That is, $N_3 = 1$ and $N_4 = 2$. Second, demand response is limited to two hours. That is, $H = 2$. Third, the cost functions for generators 1 and 2 are linear: $C_1(G_1) = c_1 G_1$ and $C_2(G_2) = c_2 G_2$, where $c_1 < c_2$. Finally, we use the Continuous LMP model, as shown in Fig. 2, that was introduced in [13] to tackle the discontinuity of LMPs at critical loads [14]. We will see that even under these simplifying assumptions, the analysis of the autonomous DR game is still very challenging and the results are very different from those for homogeneous systems in [4]–[8]. The rest of the parameters in our case studies are as follows. We assume that $c_1 = 10$\$/MWh and $c_2 = 12$\$/MWh. The reactance of all transmission lines is 1 in a per unit system. The maximum generation capacity of each generator is 1000 MW. We have $E_3 = 300$ MWh and $E_4 = 200$ MWh. This setup simply serves as an example to gain insights. More general cases can be considered in the future, as we will explain in Section V.

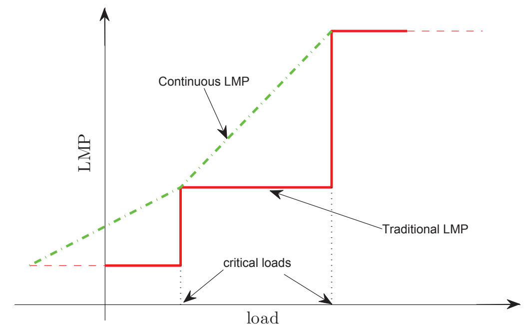

Fig. 2. Traditional LMP versus continuous LMP (CLMP) [13] curves.

III. The Scenario when Nash Equilibrium Exists

Consider the power system in Fig. 1(b). Further to the choice of system parameters explained in Section II, assume that the capacity of the transmission line between bus 1 and bus 2 is 80 MW. The capacity of the rest of the transmission lines in the system is 1000 MW. In other words, we assume that the transmission line between bus 1 and bus 2 is the only bottleneck and the rest of the lines have large enough capacities. For the user connected to bus 3, the load profile is denoted by $\boldsymbol{l}_3 = (l_3^1, l_3^2) = (l_3^1, E_3 - l_3^1)$, where the second equality is due to the constraint in (2). For the user connected to bus 4, the load profile is denoted by $\boldsymbol{l}_4 = (l_4^1, l_4^2) = (l_4^1, E_4 - l_4^1)$. Therefore, for the user connected to bus 3, the only action variable is l_3^1. Similarly, for the user connected to bus 4, the only action variable is l_4^1. For notational simplicity, we define

$$x \triangleq l_3^1, \qquad y \triangleq l_4^1. \tag{3}$$

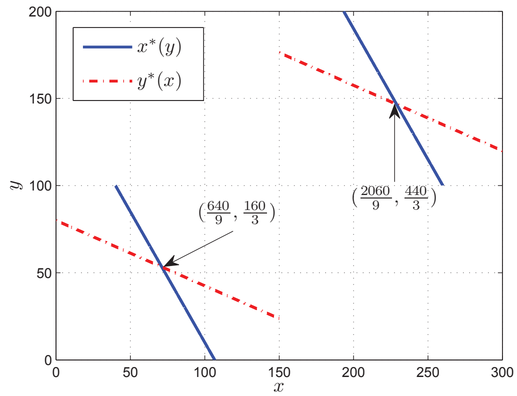

Fig. 3. Best response functions for the scenario studied in Section III.

From (3), the load profile for the user connected to bus 3 is denoted as $(x, E_3 - x)$. Similarly, the load profile for the user connected to bus 4 is denoted as $(y, E_4 - y)$. Clearly, we have

$$0 \le x \le E_3 = 300, \qquad 0 \le y \le E_4 = 200.$$

The following theorem provides the formulations for CLMPs.

Theorem 1: For the heterogeneous grid topology in Fig. 1(b) and given the choices of system parameters in this section, the continuous LMPs at load buses 3 and 4 are obtained as

$$\mathrm{CLMP}_3(x,y) = \begin{cases} \frac{3.6x - 40y + 6400}{640 - 4y} & \text{if } 3x + 4y \le 640, \\ 11.2 & \text{if } 3x + 4y > 640, \end{cases} \quad (4)$$

$$\mathrm{CLMP}_4(x,y) = \begin{cases} \frac{6.4y - 30x + 6400}{640 - 3x} & \text{if } 3x + 4y \le 640, \\ 11.6 & \text{if } 3x + 4y > 640. \end{cases} \quad (5)$$

The proof of Theorem 1 is given in Appendix A. In (4) and (5), constraint $3x + 4y > 640$ gives the region of congestion. From Theorem 1, the CLMPs (and also the LMPs) reach their maximum values when the grid is congested. From (1), for the user connected to bus 3, the electricity bill is equal to

$$B_3(x,y) = x \times \mathrm{CLMP}_3(x,y) + \\ (300 - x) \times \mathrm{CLMP}_3(300 - x, 200 - y). \quad (6)$$

From this, together with (4), and after rewording the terms, the bill for the user at bus 3 can be modeled as

- If $0 \le x < \frac{640-4y}{3}$, then
$$B_3(x,y) = 3360 - 1.2x + \frac{3.6x^2}{640 - 4y}.$$
- If $\frac{640-4y}{3} \le x \le \frac{1060-4y}{3}$, then
$$B_3(x,y) = 3360.$$
- If $\frac{1060-4y}{3} < x \le 300$, then
$$B_3(x,y) = 3000 + 1.2x + \frac{3.6(x-300)^2}{4y - 160}.$$

Similarly, for the user connected to bus 4, we have

$$B_4(x,y) = y \times \mathrm{CLMP}_4(x,y) + \\ (200 - y) \times \mathrm{CLMP}_4(300 - x, 200 - y). \quad (7)$$

From this, together with (5), the bill can be rephrased as:

- If $0 \le y < \frac{640-3x}{4}$, then
$$B_4(x,y) = 2320 - 1.6y + \frac{6.4y^2}{640 - 3x}.$$
- If $\frac{640-3x}{4} \le y \le \frac{1060-3x}{4}$, then
$$B_4(x,y) = 2320.$$
- If $\frac{1060-3x}{4} < y \le 200$, then
$$B_4(x,y) = 2000 + 1.6y + \frac{6.4(y-200)^2}{3x - 260}.$$

Finally, the best response, i.e., the solution of problem (2), for the users connected to buses 3 and 4 becomes:

$$x^*(y) = \begin{cases} \frac{640-4y}{6} & \text{if } 0 \le y < 100, \\ 40, 260 & \text{if } y = 100, \\ \frac{1960-4y}{6} & \text{if } 100 < y \le 200, \end{cases} \quad (8)$$

and

$$y^*(x) = \begin{cases} \frac{640-3x}{8} & \text{if } 0 \le x < 150, \\ \frac{95}{4}, \frac{705}{4} & \text{if } x = 150, \\ \frac{1860-3x}{8} & \text{if } 150 < x \le 300, \end{cases} \quad (9)$$

respectively. These results are illustrated in Fig. 3. From the results in this figure, we can see that there are two Nash equilibria for the autonomous DR game in this scenario:

$$(x^*, y^*) = \left(\tfrac{640}{9}, \tfrac{160}{3}\right), \quad \text{and} \quad (x^*, y^*) = \left(\tfrac{2060}{9}, \tfrac{440}{3}\right). \quad (10)$$

The bill amounts are the same in both cases:

$$B_3(x^*, y^*) = \frac{9952}{3}, \qquad B_4(x^*, y^*) = \frac{6832}{3}. \quad (11)$$

At Nash equilibria, the total power generation cost in the system is obtained as $\$10 \times \frac{3360}{9} + \$12 \times \frac{1140}{9} = \$\frac{15760}{3} \approx \$5253$. On the other hand, it can be shown that the optimal generation cost is equal to $\$10 \times 416 + \$12 \times 84 = \$5168$ and it is reached when $640 < 3x + 4y < 1060$. The calculated optimal cost can serve as a benchmark to assess the efficiency of the Nash equilibria. Therefore, we can conclude that

$$\text{Efficiency at Nash Equilibria} = 1 - \frac{5253 - 5168}{5168} \approx \%98.$$

That is, the heterogeneous autonomous DR system in this scenario can lead to a system performance which is very close to optimal. This observation matches the results in homogeneous autonomous DR studies in [4]–[8] and may suggest that as far as optimality at Nash equilibrium is concerned there is no major deference between homogeneous and heterogeneous autonomous DR systems. However, as we will see in the next section, these results can significantly change if we make some slight changes in the choices of system parameters.

IV. The Scenario when Nash Equilibrium Does Not Exist

Consider the power system in Fig. 1(b) and assume that the choices of system parameters are the same as those in Section III, except that this time we assume that the capacity of the transmission line between bus 3 and bus 4 is 30 MW. The capacity of the rest of the transmission lines in the system is 1000 MW. In other words, we assume that the transmission line between bus 3 and bus 4 is the only bottleneck and the rest of the lines have large enough capacities. Recall that for the results in Section III, the bottleneck was the transmission line between the generator buses 1 and 2. The following theorem provides the formulations for CLMPs in this new scenario.

Theorem 2: For the heterogeneous grid topology in Fig. 1(b) and given the choices of system parameters in this section, the Continuous LMPs at load buses 3 and 4 are obtained as

$$\mathrm{CLMP}_3(x,y) = \begin{cases} \frac{4x+34y-1440}{5y-240} & \text{if } 3y-2x \leq 240 < 4y-x, \\ 10 & \text{if } -240 < 4y-x \leq 240, \\ \text{not defined} & \text{otherwise,} \end{cases} \tag{12}$$

$$\mathrm{CLMP}_4(x,y) = \begin{cases} 18 & \text{if } 3y-2x < 240 \leq 4y-x, \\ \frac{4y-x+840}{60} & \text{if } -240 \leq 4y-x < 240, \\ \text{not defined} & \text{otherwise.} \end{cases} \tag{13}$$

The proof of Theorem 2 is given in Appendix B. For the cases where LMPs are not defined, these are the scenarios where the bottleneck transmission line between bus 3 and bus 4 would reach its maximum capacity and the grid can no longer support any additional load at certain buses.

It this example, constraint $3y-2x < 240 < 4y-x$ is the region of congestion (see the Appendix). From (12) and (13), the congestion causes a decrease in the CLMP of user 3, while it causes an increase in the CLMP of user 4. This is in sharp contrast to the scenario in Section III, where the congestion increases the CLMPs of *both* users 3 and 4.

Same as in Section IV, the users' bills can be stated as (6) and (7). Combining (6) and (12), and after rewording the terms, the bill for the user on bus 3 can be modeled as:

- If $\frac{3y-240}{2} \leq x < 4y-260$, then
$$B_3(x,y) = 3000 + 2x \times \frac{2x-8y+480}{5y-240}.$$
- If $4y-260 < x < 4y-240$, then
$$B_3(x,y) = \frac{-2080x^2 + 480x(29y-1600)}{(5y-240)(5y-760)} + 600 \times \frac{17y-3280}{5y-760}.$$
- If $4y-240 \leq x \leq \frac{3y+240}{2}$, then
$$B_3(x,y) = 3000 + 2(300-x)\frac{-2x+8y-520}{760-5y}.$$
- Elsewhere, $B_3(x,y)$ is not defined.

Similarly, using (7) and (13), $B_4(x,y)$ is modeled as:

- If $\frac{2x-240}{3} \leq y < \frac{x+240}{4}$, then
$$B_4(x,y) = \frac{4y^2 - y(x+240) + 216000}{60}.$$
- If $\frac{x+240}{4} < y < \frac{x+260}{4}$, then
$$B_4(x,y) = 3600.$$
- If $\frac{x+260}{4} \leq y \leq \frac{2x+240}{3}$, then
$$B_4(x,y) = \frac{4y^2 - y(x+1060) + 200x + 268000}{60}.$$

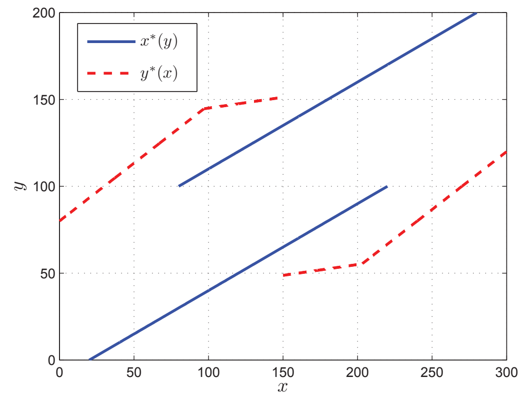

Fig. 4. Best response functions for the scenario studied in Section IV.

- Else, $B_4(x,y)$ is not defined.

Finally, the best responses are obtained as

$$x^*(y) = \begin{cases} 2y+20 & \text{if } 0 \leq y < 100, \\ 80, 220 & \text{if } y = 100, \\ 2y-120 & \text{if } 100 < y \leq 200, \end{cases} \tag{14}$$

and

$$y^*(x) = \begin{cases} \frac{2x+240}{3} & \text{if } 0 \leq x \leq \frac{1260}{13}, \\ \frac{x+1060}{8} & \text{if } \frac{1260}{13} < x < 150, \\ \frac{195}{4}, \frac{605}{4} & \text{if } x = 150, \\ \frac{x+240}{8} & \text{if } 150 < x < \frac{2640}{13}, \\ \frac{2x-240}{3} & \text{if } \frac{2640}{13} \leq x \leq 300. \end{cases} \tag{15}$$

These results are shown in Fig. 4(a). Since there is no crossing between the blue solid lines and the red dashed lines, we can conclude that Nash equilibrium does *not* exist in this scenario. This is in sharp contrast with the results in Section III and those on homogeneous autonomous DR systems in [4]–[8]. Next, we show that the reason for not have a Nash equilibrium is that the two users have conflicting interests:

- For the user on bus 3, it is preferred to *cause* congestion on the transmission line between buses 3 and 4. This is because congestion will *decrease* the price on bus 3.
- For the user on bus 4, it is preferred to *avoid* congestion on the transmission line between buses 3 and 4. This is because congestion will *increase* the price on bus 4.

To see this, consider the CLMP models in (12) and (13). The line between buses 3 and 4 becomes congested if we have

$$3y - 2x \leq 240 \leq 4y - x. \tag{16}$$

From (12), congestion can *decrease* the price of electricity for the user on bus 3. Note that, if (16) holds, then

$$\frac{4x+34y-1440}{5y-240} \leq 10. \tag{17}$$

On the other hand, from (13), congestion can *increase* the price for the user on bus 4. Note that, if (16) holds, then

$$18 \geq \frac{4y-x+840}{60}. \tag{18}$$

Therefore, while user 3 selects his load profit to cause congestion, user 4 selects his load profile to avoid congestion. Therefore, the two users can never settle on a fixed load profile as they can always improve their payoffs by unilateral changes.

V. Design Recommendations and Future Works

Based on the two cases that we analytically studied in Section III and IV, we can make the following remarks:

- *Remark 1*: Unlike the case for homogeneous autonomous DR games that always have Nash equilibrium, c.f. [4]–[8], the existence of Nash equilibrium in heterogeneous autonomous DR games depends on the system parameters. In particular, it seems that the location of the bottleneck transmission line is a key factor on this issue.
- *Remark 2*: From the Proofs of Theorems 1 and 2 in the Appendix, the sharp difference between the results in Sections III and IV is due to the different signs of the generation *shift factors* of the congested lines in the two scenarios. In Section III, the generation shift factors to the congested line from the two users/load have the *same* sign. However, in Section IV, these generation shift factors have *different* signs. This observation can lead to the following recommendation for designing autonomous DR systems in heterogeneous grids: We should avoid designing a power system, where there exists a congested transmission line such that the generation shift factors from different users/load (to that congested line) have different signs. In other words, the transmission lines in which generation shift factors from different users/load have different signs must have enough capacity to prevent them from being congested. Otherwise, autonomous DR systems at the buses of these users/load can be prone to instability as the Nash equilibrium may not exist.
- *Remark 3*: In this paper, we used a classic one-stage game model with pure strategies. However, it is interesting to also study the problem in a multi-stage game model, and/or to investigate mixed Nash equilibria [9]. Furthermore, our analysis was limited to the case with two users only. Having more users will make it possible to apply coalitional game theory [15] among users that are on the same bus since they all face the same LMPs. Such coalitions may still have conflicting interests with other coalitions on other buses. Finally, looking at daily scheduling mechanisms, i.e. setting $H = 24$ instead of $H = 2$, may reveal more interesting properties of the autonomous DR systems in heterogeneous grid topologies.
- *Remark 4*: The analysis in this paper was possible because we were able to obtain closed-form models for LMPs as shown in the appendix. Extending these closed-form models to more general cases can be challenging, but there are some interesting scenarios that may still have tractable LMP models. For example, changing the grid topology and adding more buses and transmission lines will not change our analysis. While increasing the number of bottleneck links can increase the number of critical loads in LMP models, which would make the expressions more rigorous, the analysis will not be that different compared to the case with a single bottleneck link.

VI. Conclusions

This paper represents the first steps towards devising autonomous DR systems in heterogeneous grid topologies. To gain insight, we formulated the interactions among users in an example 4-bus heterogeneous DR system as a two-person game. This required us to obtain closed-form models for LMPs at each load bus. By analyzing two different scenarios with respect to the location of the congested transmission line, we showed that a heterogeneous autonomous DR may or may *not* have a Nash equilibrium. This is in sharp contrast to the previous results on homogeneous autonomous DR systems, where Nash equilibrium always exists. These results helped us making recommendations for designing stable autonomous DR systems in heterogeneous power grids. We also discussed several potential future research directions.

Appendix

In this Appendix, we prove Theorems 1 and 2. But first, we go through some preliminaries and define some new notations. Let L_{ij} denote the power flow on transmission line between buses i and j. Also let $L_{ij}^{\max}$ denote the capacity of such line. Clearly, we have $|L_{ij}| \leq L_{ij}^{\max}$ for all transmission lines. We take bus 1 in Fig. 1(b) as the reference bus. Next, let SF_{ij}^{k} denote the *generation shift factor* [13], [14] to line ij from bus k. For each non-reference bus $k \in \{2, 3, 4\}$, we define

$$\mathbf{SF}^k := \left[\ \mathrm{SF}_{12}^k \quad \mathrm{SF}_{13}^k \quad \mathrm{SF}_{23}^k \quad \mathrm{SF}_{24}^k \quad \mathrm{SF}_{34}^k\ \right], \tag{19}$$

as the vector of all shift factors from bus k. From the definition of shift factor, it is easy to show that

$$\mathbf{SF}^2 = \frac{1}{8} \times \begin{bmatrix} -5 \\ -3 \\ +2 \\ +1 \\ -1 \end{bmatrix}, \quad \mathbf{SF}^3 = \frac{1}{8} \times \begin{bmatrix} -3 \\ -5 \\ -2 \\ -1 \\ +1 \end{bmatrix}, \quad \mathbf{SF}^4 = \frac{1}{8} \times \begin{bmatrix} -4 \\ -4 \\ 0 \\ -4 \\ -4 \end{bmatrix}.$$

Since power loss is negligible, at each instance, the total generated power is equal to the total load at buses 3 and 4. Thus, the total generation cost at each instance becomes

$$c_1 G_1 + c_2 G_2 = c_1 \times (x + y) + (c_2 - c_1) \times G_2, \tag{20}$$

where, at each instance, G_1 and G_2 denote the generated powers at buses 1 and 2, and x and y denote the load at buses 3 and 4. Since $c_2 - c_1 > 0$, the total cost is minimized if G_2 is minimized. That is, in the DC-OPF, the total load $x + y$ has to be solely supplied by the generator with the lower price, unless a transmission line congestion occurs. Thus, noting the fact that $G_2 \geq 0$ and $G_1 = x + y - G_2 \geq 0$, we have

$$G_2^{\text{opt}} = \min \left\{ G_2 \,\middle|\, 0 \leq G_2 \leq x + y, |L_{ij}| < L_{ij}^{\max}\ \forall\, i, j \right\}.$$

On the other side, using the definition of generation shift factor, the vector of power flows on power lines is equal to

$$\boldsymbol{L} = G_2\, \mathbf{SF}^2 - x\, \mathbf{SF}^3 - y\, \mathbf{SF}^4, \tag{21}$$

where $\boldsymbol{L} := [L_{12}\ L_{13}\ L_{23}\ L_{24}\ L_{34}]$ is the vector of power flows L_{ij} for all lines. Hence, we can conclude that if

$$\max_{ij} \left\{ 0, \frac{L_{ij}^{\max} + x\mathrm{SF}_{ij}^3 + y\mathrm{SF}_{ij}^4}{\mathrm{SF}_{ij}^2} \right\} \leq \min_{ij} \left\{ x + y, \frac{-L_{ij}^{\max} + x\mathrm{SF}_{ij}^3 + y\mathrm{SF}_{ij}^4}{\mathrm{SF}_{ij}^2} \right\} \tag{22}$$

then

$$G_2^{\text{opt}} = \max_{ij} \left\{ 0, \frac{L_{ij}^{\max} + x\text{SF}_{ij}^3 + y\text{SF}_{ij}^4}{\text{SF}_{ij}^2} \right\}. \tag{23}$$

If (22) is not satisfied then, the network cannot support the load and the DC-OPF problem does not have a solution.

Clearly, $G_2 \neq 0$ only if there is congested transmission line. Let ij denote the congested line. If the load at bus k increases by one unit, the generation at bus 2 should be increased by $\Delta G_2 := -\text{SF}_{ij}^k/\text{SF}_{ij}^2$ in order to compensate for the variation of the power flow at the congested line ij. Hence, the increase in the cost of the system becomes

$$\begin{aligned} \text{LMP}_k &= c_1 + \Delta G_2(c_2 - c_1) \\ &= c_1 - \frac{\text{SF}_{ij}^k}{\text{SF}_{ij}^2}(c_2 - c_1). \end{aligned} \tag{24}$$

Using this approach, the LMPs of the two case studies can be formulated, given $c_1 = \$10$ and $c_2 = \$12$, as we explain next.

A. Proof of Theorem 1

Since the only bottleneck line is line 12, it is the only line that can be congested. Therefore, all other lines can be eliminated from the $\max$ and $\min$ operations in (22) and (23). Given the constraints $0 \leq x \leq E_3 = 300$ and $0 \leq y \leq E_4 = 200$, the following cases can be considered:

(a) The case $(3x + 4y)/8 < 0$, cannot occur for $x, y \geq 0$.

(b) If $(3x+4y)/8 \leq 80$, then $|L_{12}| < L_{12}^{\max}$. Thus there is no congestion and (21) gives the OPF with $G_1 = x + y$ and $G_2 = 0$. Since there is no congestion, in this case, we have $\text{LMP}_3 = c_1 = 10$\$/MWh and $\text{LMP}_4 = c_1 = 10$\$/MWh.

(c) If $(3x + 4y)/8 > 80$, then $L_{12} \geq L_{12}^{\max}$, making line 12 congested. In this case, (22) is satisfied and (23) gives

$$G_2 = \frac{3x + 4y}{5} - 128.$$

Using (24) and noting that the congested line is 12, we have $\text{LMP}_3 = 11.2$\$/MWh and $\text{LMP}_4 = 11.6$\$/MWh.

In summary, we can wrap up the LMP in the following form

$$\text{LMP}_3(x, y) = \begin{cases} 10 & \text{if } 3x + 4y \leq 640 \\ 11.2 & \text{if } 3x + 4y > 640 \end{cases}$$

$$\text{LMP}_4(x, y) = \begin{cases} 10 & \text{if } 3x + 4y \leq 640 \\ 11.6 & \text{if } 3x + 4y > 640. \end{cases}$$

From the definition of CLMP in Section II, the expressions in (4) and (5) are resulted and the proof is complete. ■

B. Proof of Theorem 2

Since the only bottleneck line is line 34, it is the only line that can be congested. Based on the values of $0 \leq x \leq E_3 = 300$ and $0 \leq y \leq E_4 = 200$, the following cases can be considered:

(a) If $(-x+4y)/8 < -30$, then $L_{34} < -L_{34}^{\max}$, indicating that the transmission line between buses 3 and 4 is congested. Since (22) is not satisfied in this case, there is no feasible power flow, and hence, the LMPs are not defined.

(b) If $|-x + 4y|/8 \leq 30$, then there is no congestion and (21) results in $G_1 = x+y$ and $G_2 = 0$. Consequently, we have $\text{LMP}_3 = 10$\$/MWh and $\text{LMP}_4 = 10$\$/MWh.

(c) If $(-x + 4y)/8 > 30$, then line 34 is congested. It is not hard to show that (22) is satisfied if and only if $(-2x + 3y)/8 \leq 30$. Thus, in this case, we have

$$G_2 = -x + 4y - 240.$$

Using (24) and noting that the congested line is 34, we have $\text{LMP}_3 = 8$\$/MWh and $\text{LMP}_4 = 18$\$/MWh. If (22) is not satisfied, then LMPs are not defined.

In summary, after rewording the terms, we have

$$\text{LMP}_3(x, y) = \begin{cases} 8 & \text{if } 3y - 2x \leq 240 < 4y - x \\ 10 & \text{if } -240 < 4y - x \leq 240 \\ \text{not defined} & \text{otherwise} \end{cases}$$

$$\text{LMP}_4(x, y) = \begin{cases} 18 & \text{if } 3y - 2x < 240 \leq 4y - x \\ 10 & \text{if } -240 \leq 4y - x < 240 \\ \text{not defined} & \text{otherwise.} \end{cases}$$

From the definition of CLMP in Section II, the expressions in (12) and (13) are resulted and the proof is complete. ■

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