Computation of the Quaternion from a Rotation Matrix

Jay A. Farrell
University of California, Riverside

November 30, 2015

0.1 Purpose

Let $\mathbf{b}$ represent the unit quaternion for the rotation from a-frame to b-frame. The rotation matrix corresponding to the quaternion $\mathbf{b}$ is

$$\mathbf{R}(\mathbf{b}) = \begin{bmatrix} b_1^2+b_2^2-b_3^2-b_4^2 & 2(b_2b_3-b_1b_4) & 2(b_1b_3+b_2b_4) \\ 2(b_2b_3+b_1b_4) & b_1^2-b_2^2+b_3^2-b_4^2 & 2(b_3b_4-b_1b_2) \\ 2(b_2b_4-b_1b_3) & 2(b_1b_2+b_3b_4) & b_1^2-b_2^2-b_3^2+b_4^2 \end{bmatrix} \tag{1}$$

where $\mathbf{R}(\mathbf{b})$ is used as a shorthand for $\mathbf{R}_a^b(\mathbf{b}_a^b)$ to simplify the notation below.

The purpose of this document is to derive and present the equations for computing the quaternion representation $\mathbf{b}$ from the given rotation matrix $\mathbf{R}$. The document uses the notation defined in Appendix D of [1]. The approach follows that summarized in eqns. (166-168) of [2].

0.1.1 Direction Cosine to Quaternion.

The diagonal of eqn. (1) can be solve for any of the components of $\mathbf{b} = [b_1, b_2, b_3, b_4]^\top$ as shown in the following four subsections.

0.1.2 Solution of diagonal for b_1.

Due to the fact that

$$b_1^2 + b_2^2 + b_3^2 + b_4 = 1, \tag{2}$$

it is straightforward to show that

$$4b_1^2 = 1 + R_{11} + R_{22} + R_{33}$$

which is easily solved for b_1. Given b_1 and eqn. (1), it is then straightforward to find the remaining elements of $\mathbf{b}$:

$$\mathbf{b} = \begin{bmatrix} \frac{1}{2}\sqrt{1+\mathbf{R}_{11}+\mathbf{R}_{22}+\mathbf{R}_{33}} \\ \frac{\mathbf{R}_{32}-\mathbf{R}_{23}}{4b_1} \\ \frac{\mathbf{R}_{13}-\mathbf{R}_{31}}{4b_1} \\ \frac{\mathbf{R}_{21}-\mathbf{R}_{12}}{4b_1} \end{bmatrix}. \tag{3}$$

0.1.3 Solution of diagonal for b_2.

Using eqn. (2), we have that

$$R_{11} - R_{22} - R_{33} = 4b_2^2 - 1$$

which is easily solved for b_2. Given b_2 and eqn. (1), it is then straightforward to find the remaining elements of $\mathbf{b}$:

$$\mathbf{b} = \begin{bmatrix} \frac{\mathbf{R}_{32}-\mathbf{R}_{23}}{4b_2} \\ \frac{1}{2}\sqrt{1+\mathbf{R}_{11}-\mathbf{R}_{22}-\mathbf{R}_{33}} \\ \frac{\mathbf{R}_{12}+\mathbf{R}_{21}}{4b_2} \\ \frac{\mathbf{R}_{13}+\mathbf{R}_{31}}{4b_2} \end{bmatrix}. \tag{4}$$

0.1.4 Solution of diagonal for b_3.

Using eqn. (2), we have that

$$-R_{11} + R_{22} - R_{33} = 4b_3^2 - 1$$

which is easily solved for b_3. Given b_3 and eqn. (1), it is then straightforward to find the remaining elements of $\mathbf{b}$:

$$\mathbf{b} = \begin{bmatrix} \frac{\mathbf{R}_{13}-\mathbf{R}_{31}}{4b_3} \\ \frac{\mathbf{R}_{12}+\mathbf{R}_{21}}{4b_3} \\ \frac{1}{2}\sqrt{1-\mathbf{R}_{11}+\mathbf{R}_{22}-\mathbf{R}_{33}} \\ \frac{\mathbf{R}_{23}+\mathbf{R}_{32}}{4b_3} \end{bmatrix}. \tag{5}$$

0.1.5 Solution of diagonal for b_4.

Using eqn. (2), we have that

$$-R_{11} - R_{22} + R_{33} = 4b_4^2 - 1$$

which is easily solved for b_4). Given b_4 and eqn. (1, it is then straightforward to find the remaining elements of $\mathbf{b}$:

$$\mathbf{b} = \begin{bmatrix} \frac{\mathbf{R}_{21}-\mathbf{R}_{12}}{4b_4} \\ \frac{\mathbf{R}_{13}+\mathbf{R}_{31}}{4b_4} \\ \frac{\mathbf{R}_{23}+\mathbf{R}_{32}}{4b_4} \\ \frac{1}{2}\sqrt{1-\mathbf{R}_{11}-\mathbf{R}_{22}+\mathbf{R}_{33}} \end{bmatrix}. \tag{6}$$

0.2 Summary.

Note the following:

1. Each of the quaternions involves a sign ambiguity due to the fact that either the positive or negative square root could have been selected. This document has selected the positive square root throughout. If the negative square root is selected, then the direction of the vector portion of the quaternion will also be reversed. This results in the same rotation matrix.

2. This document presented four sets of equations for computation of the quaternion representation of a given rotation matrix. Theoretically all the approaches are identical. Numerically they are not. The slope of the square root function approaches infinity as its argument approaches zero. Therefore, it is prudent to select between the four approaches based on the magnitude of the argument of the square root function.

Bibliography

[1] J. A. Farrell. *Aided Navigation: GPS and High Rate Sensors*. McGraw-Hill, New York, NY, 2008.

[2] M. D. Shuster. Survey of attitude representations. *Journal of the Astronautical Sciences*, 41(4):439–517, 1993.