Non-projective quantum measurement of solid-state qubits: Bayesian formalism (what is “inside” collapse)

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Outline:
• Very long introduction (incl. EPR, solid-state qubits)
• Basic Bayesian formalism for quantum measurement and its derivations
• Non-ideal detectors
• Bayesian formalism in circuit QED setup

Acknowledgements
Many useful discussions and collaborations
Quantum mechanics is weird…

Niels Bohr:  
“If you are not confused by quantum physics then you haven’t really understood it”

Richard Feynman:  
“I think I can safely say that nobody understands quantum mechanics”

Weirdest part is quantum measurement
Quantum mechanics =
Schrödinger equation (evolution) +
collapse postulate (measurement)

1) Probability of measurement result \( p_r = |\langle \psi | \psi_r \rangle|^2 \)

2) Wavefunction after measurement = \( \psi_r \)

- State collapse follows from common sense
- Does not follow from Schrödinger equation (contradicts, random vs. deterministic, “philosophy”)

Collapse postulate is controversial since 1920s (needs an observer, contradicts causality)

Our focus: what is “inside” collapse, but first discuss EPR
Einstein-Podolsky-Rosen (EPR) paradox

Phys. Rev., 1935

In a complete theory there is an element corresponding to each element of reality. A sufficient condition for the reality of a physical quantity is the possibility of predicting it with certainty, without disturbing the system.

\[ \psi(x_1, x_2) = \sum_n \psi_n(x_2) u_n(x_1) \] (nowadays we call it entangled state)

\[ \psi(x_1, x_2) = \int_{-\infty}^{\infty} \exp\left[\left(\frac{i}{\hbar}\right)(x_1 - x_2)p\right] dp \sim \delta(x_1 - x_2) \]

Measurement of particle 1 cannot affect particle 2, while QM says it affects (contradicts causality)

=> Quantum mechanics is incomplete

Bohr’s reply (Phys. Rev., 1935) (seven pages, one formula: \( \Delta p \Delta q \sim \hbar \)) (except in footnotes)

It is shown that a certain “criterion of physical reality” formulated … by A. Einstein, B. Podolsky and N. Rosen contains an essential ambiguity when it is applied to quantum phenomena.

Very crudely: No need to understand QM, just use the result
Bell’s inequality (John Bell, 1964)

\[ \psi = \frac{1}{\sqrt{2}} (\uparrow_1 \downarrow_2 - \downarrow_1 \uparrow_2) \]

Perfect anticorrelation of results for same meas. directions, \( \vec{a} = \vec{b} \)

Is it possible to explain the QM result assuming local realism and hidden variables (without superluminal collapse)? **No!!!**

**Assume:** \( A(\vec{a}, \lambda) = \pm 1, \quad B(\vec{b}, \lambda) = \pm 1, \quad \) (perfect anticorr. for \((\vec{a}, \vec{a})\))

**Then:** \[ |P(\vec{a}, \vec{b}) - P(\vec{a}, \vec{c})| \leq 1 + P(\vec{b}, \vec{c}) \]

where \( P \equiv P(++) + P(--) - P(+-) - P(-+) \)

**QM:** \( P(\vec{a}, \vec{b}) = -\vec{a} \cdot \vec{b} \)  
For \( 0^\circ, 90^\circ, \) and \( 45^\circ: \quad 0.71 \not\leq 1 - 0.71 \) violation!

**Experiment** (Aspect et al., 1982; photons instead of spins, CHSH):

yes, “spooky action-at-a-distance”
CHSH paper (Clauser, Horne, Shimony, Holt, 1969)

Problem with original Bell ineq.: need **perfect** anticorrelation for same directions ⇒ not practical!

In CHSH perfect anticorrelation not required ⇒ practical

\[ |S| \leq 2, \text{ where } S = P(a,b) - P(a,b') + P(a',b) + P(a',b') \]

(Apsect’s version)

\[ P \equiv p(++) + p(--) - p(+-) - p(-+) \]

Maximum violation by QM:

\[ S = \pm 2\sqrt{2} \]

\[ \begin{align*}
P(a,b) &= -\cos(a,b) \\
S &= 2\sqrt{2} \quad & a=0^\circ, \quad a'=270^\circ, \\
S &= -2\sqrt{2} \quad & a=0^\circ, \quad a'=90^\circ,
\end{align*} \]

\begin{align*}
b &= 135^\circ, \quad b'=45^\circ \\
b &= 45^\circ, \quad b'=135^\circ
\end{align*}
What about causality?

Actually, not too bad: you cannot transmit your own information choosing a particular measurement direction $a$

Result of the other measurement does not depend on direction $a$

Randomness saves causality

Collapse is still instantaneous: OK, just our recipe, not an “objective reality”, not a “physical” process

Consequence of causality: **No-cloning theorem**

Wootters-Zurek, 1982; Dieks, 1982; Yurke

*You cannot copy an unknown quantum state*

**Proof:** Otherwise get information on direction $a$ (and causality violated)

**Application:** quantum cryptography

Information is an important concept in quantum mechanics
Quantum measurement in solid-state systems

No violation of locality – too small distances

However, interesting issue of continuous measurement (weak coupling, noise $\Rightarrow$ gradual collapse)

Same origin of paradoxes as in EPR (Schr. Eq. not enough)

Starting point:

What happens to a solid-state qubit (two-level system) during its continuous measurement by a detector?
Superconducting “charge” qubit


\[ \hat{H} = \frac{(2e)^2}{2C} (\hat{n} - n_g)^2 - \frac{E_J}{2} (|n\rangle\langle n+1| + |n+1\rangle\langle n|) \]

Vion et al. (Devoret’s group); Science, 2002
Q-factor of Ramsey oscillations = 25,000

Quantum coherent (Rabi) oscillations

Vion et al. (Devoret’s group); Science, 2002
Q-factor of Ramsey oscillations = 25,000

n
n+1

\( E_J \)

n: number of Cooper pairs on the island
Charge qubits with SET readout

Duty, Gunnarsson, Bladh, Delsing, PRB 2004
Guillaume et al. (Echternach’s group), PRB 2004

Cooper-pair box measured by single-electron transistor (rf-SET)

Setup can be used for continuous measurements

All results are averaged over many measurements (not “single-shot”)
Some other superconducting qubits

**Flux qubit**
Mooij et al. (Delft)

**Phase qubit**
J. Martinis et al. (UCSB and NIST)

**Charge qubit with circuit QED**
R. Schoelkopf et al. (Yale)
Some other superconducting qubits

**Flux qubit**

J. Clarke et al. (Berkeley)

![Flux qubit circuit diagram](image)

(a)

Scaled switching probability

- Scaled switching probability vs. pulse width (ns)

<table>
<thead>
<tr>
<th>Pulse width (ns)</th>
<th>0</th>
<th>20</th>
<th>40</th>
<th>60</th>
<th>80</th>
<th>100</th>
<th>120</th>
<th>140</th>
<th>160</th>
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<tbody>
<tr>
<td>Probability</td>
<td>0.2</td>
<td>0.3</td>
<td>0.4</td>
<td>0.5</td>
<td>0.6</td>
<td>0.7</td>
<td>0.8</td>
<td>0.9</td>
<td>1.0</td>
</tr>
</tbody>
</table>

**“Quantronium” qubit**

I. Siddiqi, R. Schoelkopf, M. Devoret, et al. (Yale)

![Quantronium qubit diagram](image)

![Graph](image)

- [Graph showing](image) - $P_{\text{switch}}$ vs. $\tau$ (ns)

0.2 | 0.4 | 0.6 | 0.8
0 | 100 | 200 | 300 | 400 | 500

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Semiconductor (double-dot) qubit

T. Hayashi et al. (NTT), PRL 2003

Detector is not separated from qubit, also possible to use a separate detector
Some other semiconductor qubits

**Spin qubit (QPC meas.)**
C. Marcus et al. (Harvard)

**Spin qubit**
L. Kouwenhoven et al. (Delft)

**Double-dot qubit**
Gorman, Hasko, Williams (Cambridge)
“Which-path detector” experiment


Dephasing rate: \[ \Gamma = \frac{eV}{h} \frac{(\Delta T)^2}{T(1-T)} = \frac{(\Delta I)^2}{4S_I} \]

\(\Delta I\) – detector response, \(S_I\) – shot noise

The larger noise, the smaller dephasing!!!

\[(\Delta I)^2/4S_I \sim \text{rate of “information flow”}\]
What is the evolution due to measurement? (What is “inside” collapse?)

• controversial for last 80 years, many wrong answers, many correct answers
• solid-state systems are more natural to answer this question

Various approaches to non-projective (weak, continuous, partial, generalized, etc.) quantum measurements

Names: Davies, Kraus, Holevo, Mensky, Caves, Knight, Plenio, Walls, Carmichael, Milburn, Wiseman, Gisin, Percival, Belavkin, etc. (very incomplete list)

Key words: POVM, restricted path integral, quantum trajectories, quantum filtering, quantum jumps, stochastic master equation, etc.

Our limited scope: (simplest system, experimental setups)

solid-state qubit

detector

$I(t)$, noise $S$
“Typical” setup: double-quantum-dot (DQD) qubit + quantum point contact (QPC) detector

$H = H_{QB} + H_{DET} + H_{INT}$

$H_{QB} = \frac{\varepsilon}{2} \sigma_z + H \sigma_x$

$I(t) = I_0 + \frac{\Delta I}{2} z(t) + \xi(t)$

const + signal + noise

Two levels of average detector current: $I_1$ for qubit state $|1\rangle$, $I_2$ for $|2\rangle$

Response: $\Delta I = I_1 - I_2$

Detector noise: white, spectral density $S_I$

For low-transparency QPC

$H_{DET} = \sum_l E_l a_l^\dagger a_l + \sum_r E_r a_r^\dagger a_r + \sum_{l,r} T(a_r^\dagger a_l + a_l^\dagger a_r)$

$H_{INT} = \sum_{l,r} \Delta T (c_1^\dagger c_1^\dagger - c_2^\dagger c_2^\dagger)(a_r^\dagger a_l + a_l^\dagger a_r)$

$S_I = 2eI$
What happens to a qubit state during measurement?

Start with density matrix evolution due to measurement only ($H = \varepsilon = 0$)

```
\begin{pmatrix}
1 & 1 \\
2 & 2 \\
1 & 1 \\
2 & 2
\end{pmatrix} \xrightarrow{\text{“Orthodox” answer}} \begin{pmatrix}
1 & 0 \\
0 & 0 \\
1 & 2 \\
0 & 1
\end{pmatrix}
```

```
\begin{pmatrix}
1 & 1 \\
2 & 2 \\
1 & 1 \\
2 & 2
\end{pmatrix} \xrightarrow{\text{“Decoherence” answer}} \begin{pmatrix}
1 & \exp(-\Gamma t) \\
2 & \exp(-\Gamma t) \\
1 & 2 \\
0 & 1
\end{pmatrix} \rightarrow \begin{pmatrix}
1 & 0 \\
2 & 0 \\
1 & 1 \\
2 & 0
\end{pmatrix}
```

|1> or |2>, depending on the result  

no measurement result! (ensemble averaged)

**Decoherence has nothing to do with collapse!**

<table>
<thead>
<tr>
<th>applicable for:</th>
<th>single quant. system</th>
<th>continuous meas.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Orthodox</td>
<td>yes</td>
<td>no</td>
</tr>
<tr>
<td>Decoherence (ensemble)</td>
<td>no</td>
<td>yes</td>
</tr>
<tr>
<td>Bayesian, POVM, quant. traject., etc.</td>
<td>yes</td>
<td>yes</td>
</tr>
</tbody>
</table>

Bayesian (POVM, quant. traj., etc.) formalism describes gradual collapse of a single quantum system, **taking into account measurement result**
Bayesian formalism for DQD-QPC system

\[ H_{QB} = 0 \]

Qubit evolution due to measurement (quantum back-action):

\[ \psi(t) = \alpha(t) |1\rangle + \beta(t) |2\rangle \quad \text{or} \quad \rho_{ij}(t) \]

1) \(|\alpha(t)|^2\) and \(|\beta(t)|^2\) evolve as probabilities, i.e. according to the Bayes rule (same for \(\rho_{ii}\))

2) phases of \(\alpha(t)\) and \(\beta(t)\) do not change (no dephasing!), \(\rho_{ij}/(\rho_{ii}\rho_{jj})^{1/2} = \text{const}\)

Bayes rule (1763, Laplace-1812):

\[
P(A_i | \text{res}) = \frac{P(A_i) P(\text{res} | A_i)}{\sum_k P(A_k) P(\text{res} | A_k)}
\]

\[
\frac{1}{\tau} \int_0^\tau I(t) \, dt
\]

So simple because:

1) no entanglement at large QPC voltage
2) QPC happens to be an ideal detector
3) no Hamiltonian evolution of the qubit
“Quantum Bayes theorem“ (ideal detector assumed)

\[ |1\rangle \xrightarrow{H} |2\rangle \]

\[ H = \varepsilon = 0 \]  
("frozen" qubit)

Initial state:
\[
\begin{pmatrix}
\rho_{11}(0) & \rho_{12}(0) \\
\rho_{21}(0) & \rho_{22}(0)
\end{pmatrix}
\]

Measurement (during time \( \tau \)):

\[ I(t) \]

\[ \bar{I} = \frac{1}{\tau} \int_{0}^{\tau} I(t) \, dt \]

\[ P(\bar{I}, \tau) = \rho_{11}(0) P_1(\bar{I}, \tau) + \rho_{22}(0) P_2(\bar{I}, \tau) \]

\[ P_i(\bar{I}, \tau) = \frac{1}{\sqrt{2\pi D}} \exp\left[-(\bar{I} - I_i)^2 / 2D\right], \]

\[ D = S_I / 2\tau, \quad |I_1 - I_2| \ll I_i, \quad \tau \gg S_I / I_i^2 \]

After the measurement during time \( \tau \), the probabilities should be updated using the standard Bayes formula:

\[ P(B_i \mid A) = \frac{P(B_i) P(A \mid B_i)}{\sum_k P(B_k) P(A \mid B_k)} \]

Quantum Bayes formulas:

\[ \rho_{11}(\tau) = \frac{\rho_{11}(0) \exp[-(\bar{I} - I_1)^2 / 2D]}{\rho_{11}(0) \exp[-(\bar{I} - I_1)^2 / 2D] + \rho_{22}(0) \exp[-(\bar{I} - I_2)^2 / 2D]} \]

\[ \rho_{12}(\tau) = \frac{\rho_{12}(0)}{[\rho_{11}(\tau) \rho_{22}(\tau)]^{1/2}} \], \quad \rho_{22}(\tau) = 1 - \rho_{11}(\tau) \]
Bayesian formalism for a single qubit

- Time derivative of the quantum Bayes rule
- Add unitary evolution of the qubit
- Add decoherence $\gamma$ (if any)

\[
\begin{align*}
\dot{\rho}_{11} &= -\rho_{22} = -2 \frac{H}{\hbar} \text{Im} \rho_{12} + \rho_{11} \rho_{22} \frac{2\Delta I}{S_I} [I(t) - I_0] \\
\dot{\rho}_{12} &= i \varepsilon \rho_{12} + i \frac{H}{\hbar} (\rho_{11} - \rho_{22}) + \rho_{12} (\rho_{11} - \rho_{22}) \frac{\Delta I}{S_I} [I(t) - I_0] - \gamma \rho_{12}
\end{align*}
\]

$\Delta I = I_1 - I_2$, $I_0 = (I_1 + I_2)/2$, $S_I$ – detector noise

$\gamma = 0$ for QPC

For simulations: $I = I_0 + \frac{\Delta I}{2} (\rho_{11} - \rho_{22}) + \xi$

Evolution of qubit wavefunction can be monitored if $\gamma=0$ (quantum-limited)

Natural generalizations: • add classical back-action
• entangled qubits

(A.K., 1998)
Relation to “conventional” master equation

\[
\rho_{11} = -\rho_{22} = -2H \text{Im} \rho_{12} + \rho_{11} \rho_{22} \frac{2\Delta I}{S_I} [I(t) - I_0]
\]

\[
\rho_{12} = i\varepsilon \rho_{12} + iH(\rho_{11} - \rho_{22}) + \rho_{12}(\rho_{11} - \rho_{22}) \frac{\Delta I}{S_I} [I(t) - I_0] - \gamma \rho_{12}
\]

$\Delta I$ – detector response, $S_I$ – detector noise

Averaging over result $I(t)$ leads to conventional master equation:

\[
\dot{\rho}_{11} = \frac{-\rho_{22}}{dt} = -2H \text{Im} \rho_{12}
\]

\[
\dot{\rho}_{12} = i\varepsilon \rho_{12} + iH(\rho_{11} - \rho_{22}) - \Gamma \rho_{12}
\]

$\Gamma$ – ensemble decoherence, $\Gamma = \gamma + (\Delta I)^2 / 4S_I$

Ensemble averaging includes averaging over measurement result

Quantum efficiency:

\[
\eta = \frac{(\Delta I)^2 / 4S_I}{\Gamma} = \frac{\text{quantum}}{\text{total}} = 1 - \frac{\text{decoherence}}{\text{total}}
\]
Assumptions needed for the Bayesian formalism:

- Detector voltage is much larger than the qubit energies involved 
  $$eV \gg \hbar\Omega, \ eV \gg \hbar\Gamma, \ \hbar/eV \ll (1/\Omega, 1/\Gamma),$$
  $$\Omega=(4H^2+\epsilon^2)^{1/2}$$
  (no coherence in the detector, classical output, Markovian approximation)

- Simpler if weak response, $$|\Delta I| \ll I_0, \ (coupling \ C \sim \Gamma/\Omega \ \text{is arbitrary})$$

Derivations:

1) “logical”: via correspondence principle and comparison with decoherence approach (A.K., 1998)

2) “microscopic”: Schr. eq. + collapse of the detector (A.K., 2000)

3) from “quantum trajectory” formalism developed for quantum optics
   (Goan-Milburn, 2001; also: Wiseman, Sun, Oxtoby, etc.)

4) from POVM formalism (Jordan-A.K., 2006)

5) from Keldysh formalism (Wei-Nazarov, 2007)
“Informational” derivation of the Bayesian formalism

(A.K., 1998)

**Step 1.** Assume $H = \varepsilon = 0$ (“frozen” qubit).
Since $\rho_{12}$ is not involved, evolution of $\rho_{11}$ and $\rho_{22}$ should be the same as in the classical case, i.e. Bayes formula (correspondence principle).

**Step 2.** Assume $H = \varepsilon = 0$ and pure initial state: $\rho_{12}(0) = [\rho_{11}(0) \rho_{22}(0)]^{1/2}$.
For any realization $|\rho_{12}(t)| \leq [\rho_{11}(t) \rho_{22}(t)]^{1/2}$. Then averaging over realizations gives $|\rho_{12}^{av}(t)| \leq \rho_{12}^{av}(0) \exp[-(\Delta I^2/4S_I) t]$. Compare with conventional (ensemble) result (Gurvitz-1997, Aleiner et al.-97) for QPC: $\rho_{12}^{av}(t) = \rho_{12}^{av}(0) \exp[-(\Delta I^2/4S_I) t]$. Exactly the upper bound! **Therefore, pure state remains pure**: $\rho_{12}(t) = [\rho_{11}(t) \rho_{22}(t)]^{1/2}$.

**Step 3.** Account of a mixed initial state
Result: the degree of purity $\rho_{12}(t) / [\rho_{11}(t) \rho_{22}(t)]^{1/2}$ is conserved.

**Step 4.** Add qubit evolution due to $H$ and $\varepsilon$.

**Step 5.** Add extra dephasing due to detector nonideality (i.e., for SET).
“Microscopic” derivation of the Bayesian formalism

(A.K., 2000)

Schrödinger evolution of “qubit + detector” for a low-\( T \) QPC as a detector (Gurvitz, 1997)

\[
\frac{d}{dt} \rho_{11}^n = -\frac{I_1}{e} \rho_{11}^n + \frac{I_1}{e} \rho_{11}^{n-1} - 2 \frac{H}{\hbar} \text{Im} \rho_{12}^n
\]

\[
\frac{d}{dt} \rho_{22}^n = -\frac{I_2}{e} \rho_{22}^n + \frac{I_2}{e} \rho_{22}^{n-1} + 2 \frac{H}{\hbar} \text{Im} \rho_{12}^n
\]

\[
\frac{d}{dt} \rho_{12}^n = i \frac{\varepsilon}{\hbar} \rho_{12}^n + i \frac{H}{\hbar} (\rho_{11}^n - \rho_{22}^n) - \frac{I_1 + I_2}{2e} \rho_{12}^n + \frac{\sqrt{I_1 I_2}}{e} \rho_{12}^{n-1}
\]

If \( H = \varepsilon = 0 \), this leads to

\[
\rho_{11}(t) = \frac{\rho_{11}(0) P_1(n)}{\rho_{11}(0) P_1(n) + \rho_{22}(0) P_2(n)}, \quad \rho_{22}(t) = \frac{\rho_{22}(0) P_2(n)}{\rho_{11}(0) P_1(n) + \rho_{22}(0) P_2(n)}
\]

\[
\rho_{12}(t) = \rho_{12}(0) \left| \frac{\rho_{11}(t) \rho_{22}(t)}{\rho_{11}(0) \rho_{22}(0)} \right|^{1/2}, \quad P_i(n) = \frac{(I_i t / e)^n}{n!} \exp(-I_i t / e),
\]

which are exactly quantum Bayes formulas
**Derivation via POVM**

*(Jordan, A.K., 2006)*

Quantum measurement in POVM formalism *(Nielsen-Chuang, p. 85,100):*

Measurement (Kraus) operator $M_r$ (any linear operator in H.S.):

$$\psi \rightarrow \frac{M_r \psi}{\|M_r \psi\|} \quad \text{or} \quad \rho \rightarrow \frac{M_r \rho M_r^\dagger}{\text{Tr}(M_r \rho M_r^\dagger)}$$

Probability: $P_r = \|M_r \psi\|^2$ or $P_r = \text{Tr}(M_r \rho M_r^\dagger)$

Completeness: $\sum_r M_r^\dagger M_r = 1$ (People often prefer linear evolution and non-normalized states)

For each incident electron:

- $|\text{in}\rangle (\alpha |1\rangle + \beta |2\rangle) \rightarrow \alpha (r_1 |L\rangle + t_1 |R\rangle) |1\rangle$
  
- $+ \beta (r_2 |L\rangle + t_2 |R\rangle) |2\rangle$

$$M_{\text{refl}} = \begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix}, \quad M_{\text{trans}} = \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix}$$

For many incident electrons $\Rightarrow$ Bayesian formalism

Relation between POVM and quantum Bayesian formalism:

*decomposition* $M_r = U_r \sqrt{M_r^\dagger M_r}$

(almost equivalent) unitary Bayes

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Where POVM measurement comes from

Initial state
$$|\psi_{in}\rangle = \sum_{k} c_k |k\rangle$$

Interaction with ancilla
$$|k\rangle |0\rangle \rightarrow \sum_{l,a} U_{k,la} |l\rangle |a\rangle$$

$$|\psi_{in}\rangle |0\rangle \rightarrow \sum_{k,l,a} c_k U_{k,la} |l\rangle |a\rangle = \sum_{l} |l\rangle \left( \sum_{k,a} c_k U_{k,la} |a\rangle \right)$$

Project ancilla onto
$$|r\rangle = \sum_{a} r_a |a\rangle$$

$$|\psi_{in}\rangle |0\rangle \rightarrow \frac{1}{\text{Norm}} \sum_{l} |l\rangle \left( \sum_{k,a} c_k U_{k,la} \langle r|a\rangle \right) |r\rangle = \frac{1}{\text{Norm}} \sum_{l} \sum_{k} c_k \left( \sum_{a} r_a^* U_{k,la} \right) |r\rangle$$

So, as a result:

ancilla
$$|0\rangle \rightarrow |r\rangle$$

system
$$\sum_{k} c_k |k\rangle \rightarrow \frac{\sum_{k,l} M_{r,lk} c_k |l\rangle}{\text{Norm}}$$

i.e.
$$|\psi_{in}\rangle \rightarrow \frac{M_r |\psi_{in}\rangle}{\text{Norm}}$$
Quantum trajectory formalism for the same system

Goan, Milburn, Wiseman, Sun, 2000
Goan, Milburn, 2001

Looks different, but equivalent to Bayesian formalism

\[
\dot{\rho}_c(t) = -\frac{i}{\hbar} [H_{CQD}, \rho_c(t)] + \mathcal{D}[T + \lambda n_1] \rho_c(t) + \xi(t) \frac{\sqrt{\xi}}{T} [T^* \lambda n_1 \rho_c(t) + \lambda^* T \rho_c(t) n_1] - 2 \text{Re}(T^* \lambda \langle n_1 \rangle_c(t) \rho_c(t)).
\]

\[
\mathcal{D}[B] \rho = \mathcal{J}[B] \rho - \mathcal{A}[B] \rho,
\]
\[
\mathcal{J}[B] \rho = B \rho B^\dagger,
\]
\[
\mathcal{A}[B] \rho = (B^\dagger B \rho + \rho B^\dagger B)/2.
\]

\[
|T_\pm|^2 = D_\pm = 2 \pi e |T_{00}|^2 g_{LR} V_\pm /\hbar,
\]
\[
|T_\pm + \chi_\pm|^2 = D'_\pm = 2 \pi e |T_{00} + \chi_{00}|^2 g_{LR} V_\pm /\hbar.
\]

\[
\dot{\rho}_{aa}(t) = i \Omega [\rho_{ab}(t) - \rho_{ba}(t)] - \sqrt{8 \Gamma} d \rho_{aa}(t) \rho_{bb}(t) \xi(t),
\]
\[
\dot{\rho}_{ab}(t) = i \epsilon \rho_{ab}(t) + i \Omega [\rho_{aa}(t) - \rho_{bb}(t)] - \Gamma d \rho_{ab}(t)
\]
\[
+ \sqrt{2} \Gamma d \rho_{ab}(t) [\rho_{aa}(t) - \rho_{bb}(t)] \xi(t).
\]

\[\xi(t)\] - white noise
Stratonovich and Ito forms for nonlinear stochastic differential equations

Definitions of the derivative:

\[
\frac{df(t)}{dt} = \lim_{\Delta t \to 0} \frac{f(t + \Delta t/2) - f(t - \Delta t/2)}{\Delta t} \quad \text{(Stratonovich)}
\]

\[
\frac{df(t)}{dt} = \lim_{\Delta t \to 0} \frac{f(t + \Delta t) - f(t)}{\Delta t} \quad \text{(Ito)}
\]

Why matters? Usually \((f + df)^2 \approx f^2 + 2f df,\ (df)^2 \ll df\)

But if \(df = \xi dt\) (white noise \(\xi\)), then \((df)^2 = \xi^2 dt^2 \approx \frac{S_\xi}{2} dt\)

Simple translation rule:

\[
\frac{d}{dt} x_i(t) = G_i(\vec{x}, t) + F_i(\vec{x}, t) \xi(t) \quad \text{(Stratonovich)}
\]

\[
\frac{d}{dt} x_i(t) = G_i(\vec{x}, t) + F_i(\vec{x}, t) \xi(t) + \frac{S_\xi}{4} \sum_k \frac{\partial F_i(\vec{x}, t)}{dx_k} F_k(\vec{x}, t) \quad \text{(Ito)}
\]

Advantage of Stratonovich: usual calculus rules (intuition)
Advantage of Ito: simple averaging
Methods for calculations

Monte Carlo

- “Ideologically” simplest
- In many cases most efficient

Idea:
- use finite time step $\Delta t$
  - find probability distribution for $\bar{I}(\Delta t)$
  - pick a random number for $\bar{I}(\Delta t)$
  - do quantum Bayesian update

Analytics (or non-random numerics)

- Be very careful about Ito-Stratonovich issue
- Use Stratonovich form for derivations (derivatives, etc.)
- Convert into Ito for averaging over noise
- Very good idea to compare with Monte Carlo and/or check second order terms in $dt$
Fundamental limit for ensemble decoherence

\[ \Gamma = \frac{(\Delta I)^2}{4S_I} + \gamma \]

ensemble decoherence rate

\[ \gamma \geq 0 \Rightarrow \Gamma \geq \frac{(\Delta I)^2}{4S_I} \]

\[ \eta = 1 - \frac{\gamma}{\Gamma} = \frac{(\Delta I)^2}{4S_I} \frac{\Gamma}{\Gamma} \]

detector ideality (quantum efficiency)

\[ \eta \leq 100\% \]

\[ \Gamma \tau_m \geq \frac{1}{2} \]

"measurement time" (S/N=1)

S. Pilgram et al., 2002
A. Clerk et al., 2002
D. Averin, 2000, 2003

\[ \Delta I \sim \text{information flow [bit/s]} \]

\[ \tau_m = 2S_I/(\Delta I)^2 \]

\[ (\varepsilon_O \varepsilon_{BA})^{1/2} \geq \hbar/2 \]

Danilov, Likharev, Zorin, 1983

\[ \varepsilon_O, \varepsilon_{BA}: \text{sensitivities [J/Hz] limited by output noise and back-action} \]

Known since 1980s (Caves, Clarke, Likharev, Zorin, Vorontsov, Khalili, etc.)

\[ (\varepsilon_O \varepsilon_{BA} - \varepsilon_{O,BA}^2)^{1/2} \geq \hbar/2 \Leftrightarrow \Gamma \geq \frac{(\Delta I)^2}{4S_I} + K^2S_I/4 \]
Two ways to think about a non-ideal detector ($\eta<1$)

$$\Gamma_\Sigma = \frac{(\Delta I)^2}{4S_0}$$

$$\Gamma_\Sigma = \Gamma_0 + \Gamma_1$$

$$\Gamma_0 = \frac{(\Delta I)^2}{4S_\Sigma}$$

$$\Gamma_1$$

$$\Gamma_\Sigma = S_0 + S_1$$

$$\eta = \frac{(\Delta I)^2 / 4S_\Sigma}{\Gamma_\Sigma}$$

These ways are equivalent (same results for any expt.) ➞ matter of convenience
Nonideal detectors with input-output noise correlation

A.K., 2002

$$K = \frac{AS_1 + \theta S_0}{\hbar S_I}, \quad S_I = S_0 + S_1$$

$$\mathbf{K}$$ – correlation between output and $\varepsilon$–backaction noises

$$\frac{d}{dt} \rho_{11} = - \frac{d}{dt} \rho_{22} = -2H \text{Im} \rho_{12} + \rho_{11} \rho_{22} \frac{2 \Delta I}{S_I} [I(t) - I_0]$$

$$\frac{d}{dt} \rho_{12} = i \varepsilon \rho_{12} + i H (\rho_{11} - \rho_{22}) + \rho_{12} (\rho_{11} - \rho_{22}) \frac{\Delta I}{S_I} [I(t) - I_0] + i K [I(t) - I_0] \rho_{12} - \gamma \rho_{12}$$

Quantum efficiency:

$$\tilde{\eta} = 1 - \frac{\tilde{\gamma}}{\Gamma} = \frac{(\Delta I)^2}{4S_I} + \frac{K^2 S_I}{4}$$

or

$$\eta = \frac{(\Delta I)^2}{4S_I}$$
A simple general form for a broadband linear detector (QPC, SET, etc.)

\[
\begin{align*}
|1\rangle & \quad H_{qb} = 0 \\
|2\rangle & \quad \Delta I = I_1 - I_2 \\
\end{align*}
\]

\[
I(t) \\
\Delta I = I_1 - I_2 \\
\text{noise } S_I
\]

\[
\bar{I} = \frac{1}{\tau} \int_0^\tau I(t) \, dt
\]

\[
D = S_I / 2\tau
\]

\[
|1\rangle \otimes |2\rangle \\
|1\rangle \\
|2\rangle
\]

\[
\begin{align*}
\rho_{11}(\tau) &= \rho_{11}(0) \exp[-(\bar{I} - I_1)^2 / 2D] \\
\rho_{22}(\tau) &= \rho_{22}(0) \exp[-(\bar{I} - I_2)^2 / 2D] \\
\rho_{12}(\tau) &= \rho_{12}(0) \exp(iK\bar{I}\tau) \exp(-\gamma\tau) \\
\end{align*}
\]

Example of classical (“physical”) backaction:
Each electron passed through QPC rotates qubit (sensitivity of tunneling phase for an asymmetric barrier)

\[
\text{det}
\]

\[
\text{INT}
\]

\[
\text{classical backaction (unitary)}
\]

\[
\text{decoherence}
\]

\[
\text{quantum backaction (non-unitary, “spooky”, “unphysical”)}
\]

\[
\text{no self-evolution of qubit assumed}
\]

\[
\text{arg}(T^* \Delta T) \neq 0
\]

\[
H_{DET} = \sum_l E_l a_l^\dagger a_l + \sum_r E_r a_r^\dagger a_r + \sum_{l,r} T(a_r^\dagger a_l + a_l^\dagger a_r)
\]

\[
H_{INT} = \sum_{l,r} \Delta T (c_1^\dagger c_1 - c_2^\dagger c_2)(a_r^\dagger a_l + a_l^\dagger a_r)
\]
A simple general form for a broadband linear detector (QPC, SET, etc.)

\[ H_{qb} = 0 \]

\[ \rho_{11}(\tau) = \rho_{11}(0) \exp[-(\bar{I} - I_1)^2 / 2D] \]

\[ \rho_{22}(\tau) = \rho_{22}(0) \exp[-(\bar{I} - I_2)^2 / 2D] \]

\[ \rho_{12}(\tau) = \rho_{12}(0) \frac{\rho_{11}(\tau) \rho_{22}(\tau)}{\rho_{11}(0) \rho_{22}(0)} \exp(iK\bar{I}\tau) \exp(-\gamma\tau) \]

\[ \Delta I = I_1 - I_2 \]

\[ \bar{I} = \frac{1}{\tau} \int_{0}^{\tau} I(t) \, dt \]

\[ D = S_I / 2\tau \]

\[ \rho = \] no self-evolution of qubit assumed

\[ \text{decoherence} \]

\[ \text{classical backaction (unitary)} \]

\[ \text{quantum backaction (non-unitary, "spooky", "unphysical")} \]

Another example of classical backaction:

Correlation between voltage and current noises in SET

\[ S_{I\varphi} \neq 0 \]

\[ \frac{S_{I\varphi}(0)}{\sqrt{S_{II}(0)S_{\varphi\varphi}(0)}} = \frac{\Gamma_L - \Gamma_R}{\sqrt{2(\Gamma_L^2 + \Gamma_R^2)}} \]

A.K., 1994

(easy to understand when \( \Gamma_L << \Gamma_R \))
Narrowband linear measurement

Difference from broadband: two quadratures

System: qubit in cQED setup + parametric amplifier

Paramp traditionally discussed in terms of noise temperature

\[ \theta \geq 0 \] for phase-sensitive (degenerate, homodyne) paramp

\[ \theta \geq \frac{\hbar \omega}{2} \] for phase-preserving (non-degenerate, heterodyne) paramp

Haus, Mullen, 1962

Giffard, 1976

Ackn.: Likharev, Devoret

We will discuss it in terms of qubit evolution due to measurement
Simplest case

\[
H = \frac{\hbar \tilde{\omega}_q}{2} \sigma_z + \hbar \omega_r a^\dagger a + \hbar \chi a^\dagger a \sigma_z \quad \text{(dispersive)}
\]

\[
\frac{\omega_r}{Q} = \kappa \gg \max(\Gamma, \Omega_R) \quad \text{(Markovian, “bad cavity”)}
\]

\[
\kappa_{out} = \kappa \quad \text{(everything collected; i.e. reflection)}
\]

\[
\chi \ll \kappa \quad \text{(weak response)}
\]

\[
\omega_d = \omega_r \quad \text{(center of resonance, only phase change if transmission)}
\]

assume everything most ideal

Blais et al., 2004
Gambetta et al., 2006, 2008

\[
\cos(\omega_d t)
\]

\[
\sin(\omega_d t)
\]
Phase-sensitive (degenerate) paramp

pumps $\omega_a + \omega_b = 2\omega_d$

quadrature $\cos(\omega_d t + \varphi)$ is amplified,
quadrature $\sin(\omega_d t + \varphi)$ is suppressed

Assume $I(t)$ measures $\cos(\omega_d t + \varphi)$, then $Q(t)$ not needed

get some information ($\sim \cos^2 \varphi$) about qubit state and
some information ($\sim \sin^2 \varphi$) about photon fluctuations

\[
\left\{ \begin{array}{l}
\rho_{gg}(\tau) = \rho_{gg}(0) \exp\left[-\left(\bar{I} - I_g\right)^2 / 2D\right]
\\
\rho_{ee}(\tau) = \rho_{ee}(0) \exp\left[-\left(\bar{I} - I_e\right)^2 / 2D\right]
\\
\rho_{ge}(\tau) = \rho_{ge}(0) \sqrt{\frac{\rho_{gg}(\tau) \rho_{ee}(\tau)}{\rho_{gg}(0) \rho_{ee}(0)}} \exp(iK\bar{I} \tau)
\end{array} \right.
\]

(rotating frame)

\[
\bar{I} \equiv \frac{1}{\tau} \int_0^\tau I(t) \, dt \quad D = S_I / 2\tau
\]

 Same as for QPC/SET, but trade-off ($\varphi$)
between quantum & classical back-actions

$K = \frac{\Delta I}{S_I} \sin \varphi$

$\Gamma = \frac{(\Delta I \cos \varphi)^2}{4 S_I} + K^2 \frac{S_I}{4} = \frac{\Delta I^2}{4 S_I} = \frac{8 \chi^2 \bar{n}}{\kappa}$
Phase-preserving (nondegenerate) paramp

pumps $\omega_a + \omega_b = 2(\omega_d + \delta \omega)$ \quad $\varphi = \delta \omega t$

Now information in both $I(t)$ and $Q(t)$.

Small $\delta \omega \Rightarrow$ can follow $\varphi(t)$
Large $\delta \omega (>> \Gamma) \Rightarrow$ averaging over $\varphi$ (phase-preserving)

\[
\frac{\rho_{gg}(\tau)}{\rho_{ee}(\tau)} = \frac{\rho_{gg}(0)}{\rho_{ee}(0)} \exp[-(\bar{I} - I_g)^2/2D] \\
\frac{\rho_{ge}(\tau)}{\rho_{ge}(0)} = \sqrt{\frac{\rho_{gg}(\tau) \rho_{ee}(\tau)}{\rho_{gg}(0) \rho_{ee}(0)}} \exp(iKQ\tau)
\]

(rotating frame)

Equal contributions to ensemble dephasing from quantum & classical back-actions

$\bar{I} = \frac{1}{\tau} \int_0^\tau I(t) \, dt$ \quad $\bar{Q} = \frac{1}{\tau} \int_0^\tau Q(t) \, dt$ \quad $D = \frac{S_I}{2\tau}$

$I_g - I_e = \frac{\Delta I}{\sqrt{2}}$ \quad $K = \frac{\Delta I}{\sqrt{2}S_I}$

$\Gamma = \frac{\Delta I^2}{8S_I} + \frac{\Delta I^2}{8S_I} = \frac{8\chi^2 n}{\kappa}$
Why not just use Schrödinger equation for the whole system?

Impossible in principle!

Technical reason: Outgoing information (measurement result) makes it an open system

Philosophical reason: Random measurement result, but deterministic Schrödinger equation

Einstein: God does not play dice
Heisenberg: unavoidable quantum-classical boundary
Measurement vs. decoherence

Widely accepted point of view:

measurement = decoherence (environment)

Is it true?

• Yes, if not interested in information from detector (ensemble-averaged evolution)

• No, if take into account measurement result (single quantum system)

Measurement result obviously gives us more information about the measured system, so we know its quantum state better (ideally, a pure state instead of a mixed state).
Can we verify the Bayesian formalism experimentally?

Direct way:

- prepare
- partial measur.
- control (rotation)
- projective measur.

However, difficult: bandwidth, control, efficiency (expt. realized only for supercond. phase qubits)

Tricks are needed for real experiments

A.K., 1998
Experimental predictions and proposals from Bayesian formalism

- Direct experimental verification (1998)
- Bell-type correlation experiment (2000)
- Entanglement by measurement (2002)
- Violation of Leggett-Garg inequality (2005)
- Partial collapse of a phase qubit (2005)
- Undoing of a weak measurement (2006, 2008)
- Decoherence suppression by uncollapsing (2010)
- Persistent Rabi revealed in noise (2010)

3 solid-state experiments realized so far
Conclusions

- Quantum measurement is the most controversial and fascinating part of quantum mechanics.
- It is easy to see what is “inside” collapse: simple Bayesian formalism works for many solid-state setups.
- Classical information plays a very important part in quantum measurement.
- Measurement backaction necessarily has a “spooky” part (“unphysical”, informational, without a mechanism); it may also have a “classical” part (with a physically understandable mechanism).