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Langevin approach for the shot noise calculation in single-electron tunneling

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Abstract. – The Langevin method for the calculation of the shot noise in correlated single-electron tunneling is developed. Its equivalence to the existing Fokker-Plank-type approach is shown in the "orthodox" framework. The advantage of the Langevin method is a natural possibility to describe also the fluctuations in the high ("quantum") frequency range.

Correlated single-electron tunneling [1] has remained an attractive topic during last decade. Because in the systems of small-capacitance high-resistance tunnel junctions electrons tunnel almost as the classical particles, most experiments are well described within the framework of "orthodox" theory [1] based on the classical master equation.

The shot noise in single-electron tunneling is due to the randomness of tunneling events (for general review on the shot noise in mesoscopic physics see ref. [2]). The basic theory of the shot noise in single-electron transistor has been developed in refs. [3, 4] and independently in ref. [5]. Despite the classical description of the system, the current in this theory is treated as a kind of an operator because it is caused by tunneling events which change the charge state of the system. The shot noise in single-electron tunneling has been studied theoretically in a number of publications — see, *e.g.*, refs. [6-12] (while there is only one experiment [13] so far). For example, the case of Andreev reflection has been considered in ref. [9]. The shot noise in single-electron transistor with discrete energy levels has been studied in ref. [10]. The shot noise theory has also been applied to single-electron systems other than single-electron transistors [11, 12]. Besides the noise in the "orthodox" frequency range $\omega \sim W/e^2R$ (where W is a typical energy and R is a typical resistance), the noise in the "quantum" frequency range $\omega \sim W/\hbar$ has been studied for a particular system [11] and the matching of two limits has been proven; however, an approach unifying both frequency ranges in one formalism has not been found.

The existing formalism for the shot noise in single-electron tunneling is of the Fokker-Plank type and is based on the deterministic master equation. In the present letter we show that the alternative Langevin approach, in which the random term is introduced into the evolution equation, can also be applied (the method used is similar to that developed in ref. [14]).

Let us start with the generalization of the existing Fokker-Plank–type method to an arbitrary system consisting of voltage sources, capacitances and tunnel junctions with sufficiently large resistances ($R_j \gg R_K = h/e^2 \simeq 26$ kΩ). In this case the dynamics is Markovian and is governed by the matrix master equation

$$\dot{\sigma} = \mathbf{\Gamma}\sigma\,, \tag{1}$$

where the element σ_n of the vector σ is the probability to find the system in the charge state $n \equiv \{n_1, ...n_L\}$ (which is characterized by the numbers n_i of excess electrons in each of L internal nodes of the system) and

$$\Gamma_{mn} = \Gamma_{m\leftarrow n} - \delta_{mn}\sum_k \Gamma_{k\leftarrow n},\;\; \Gamma_{m\leftarrow n} = \sum_j \Gamma^j_{m\leftarrow n}, \tag{2}$$

where $\Gamma^j_{m\leftarrow n}$ are the corresponding tunneling rates and the summation over the junction number j is necessary when an electron can tunnel to (from) an internal node from (to) different external electrodes.

To find the mutual spectral density for two processes $X(t)$ and $Y(t)$ we can calculate first the correlation function $K_{XY}(\tau) = \langle X(t+\tau)Y(t)\rangle - \langle X\rangle\langle Y\rangle$ (brackets denote the averaging over time) and then take the Fourier transform $S_{XY}(\omega) = 2\int_{-\infty}^{+\infty} K_{XY}(\tau)\exp[i\omega\tau]\,\mathrm{d}\tau$. If both X and Y are functions of the charge state n (for example, potential of a node) then the correlation function is given by the simple expression

$$\begin{aligned} K_{XY}(\tau) = \theta(\tau)\sum_{m,n} X(m)\,\sigma(\tau, m|n)\,Y(n)\,\sigma^{\mathrm{st}}_n + \\ +\theta(-\tau)\sum_{m,n} Y(m)\,\sigma(m, -\tau|n)\,X(n)\,\sigma^{\mathrm{st}}_n - \langle X\rangle\langle Y\rangle, \end{aligned} \tag{3}$$

where $\sigma(\tau, m|n)$ is the retarded Green's function of eq. (1) being the probability to find the system in the state m at $t = \tau > 0$ if at $t = 0$ it was in the state n, $\langle X\rangle = \sum_n X(n)\sigma^{\mathrm{st}}_n$, and σ^{st}_n is the stationary distribution, $\mathbf{\Gamma}\sigma^{\mathrm{st}} = 0$, $\sum_n \sigma^{\mathrm{st}}_n = 1$. (Notice that X and Y are classical variables, and their commutator is zero.)

However, if X and/or Y represent the current through a tunnel junction or in an external lead, eq. (3) should be modified. For example, if $X(t)$ is the current contribution corresponding to tunneling events $\Gamma^j_{m\leftarrow n}$ while $Y(t)$ corresponds to $\Gamma^{j'}_{m'\leftarrow n'}$, then (similar to refs. [3,4])

$$\begin{aligned} K_{XY}(\tau)/\tilde{e}^j_\pm\tilde{e}^{j'}_\pm = \theta(\tau)\,\Gamma^j_{m\leftarrow n}\,\sigma(\tau, n|m')\,\Gamma^{j'}_{m'\leftarrow n'}\sigma^{\mathrm{st}}_{n'} + \\ +\theta(-\tau)\,\Gamma^{j'}_{m'\leftarrow n'}\sigma(n', -\tau|m)\,\Gamma^j_{m\leftarrow n}\,\sigma^{\mathrm{st}}_n - \\ -\Gamma^j_{m\leftarrow n}\sigma^{\mathrm{st}}_n\,\Gamma^{j'}_{m'\leftarrow n'}\sigma^{\mathrm{st}}_{n'} + \delta_{mm'}\delta_{nn'}\delta_{jj'}\delta(\tau)\,\Gamma^j_{m\leftarrow n}\sigma^{\mathrm{st}}_n\,. \end{aligned} \tag{4}$$

Here the last term is responsible for the high-frequency limit. The effective charges $\tilde{e}^j_\pm$ and $\tilde{e}^{j'}_\pm$ are determined by the direction of electron tunneling, $\tilde{e}^j_+ = -\tilde{e}^j_-$, and by the circuit capacitances [3,4] (so that $\tilde{e}^j = e$ only if the current through junction j is measured). Any current-current correlation function can be written as a sum of $K_{XY}(\tau)$ given by eq. (4) over all possible transitions between charge states (such a sum is a counterpart of eq. (3) in which the sum is written explicitly).

For the correlation functions when X is a current and Y is a function of the charge state (or vice versa), the recipe is the "combination" of eqs. (3) and (4) while the term proportional to $\delta(\tau)$ is absent.

The expressions for spectral densities directly follow from eqs. (3) and (4) because the Fourier transformation affects only the evolution operator $\sigma(\tau, m|n)$ and the corresponding

Green's function in the frequency representation is simply obtained from eq. (1): $\sigma(\omega, m|n) = [(-i\omega\mathbf{1} - \mathbf{\Gamma})^{-1}]_{mn}$, where $\mathbf{1}$ is the unity matrix. For example, eq. (4) leads to the following spectral density:

$$\begin{aligned} S_{XY}(\omega)/\tilde{e}^j_{\pm}\tilde{e}^{j'}_{\pm} = {} & 2\,\Gamma^j_{m\leftarrow n}\left[(-i\omega\mathbf{1} - \mathbf{\Gamma})^{-1}\right]_{nm'}\Gamma^{j'}_{m'\leftarrow n'}\sigma^{\rm st}_{n'} + \\ & + 2\,\Gamma^{j'}_{m'\leftarrow n'}\left[(i\omega\mathbf{1} - \mathbf{\Gamma})^{-1}\right]_{n'm}\Gamma^j_{m\leftarrow n}\sigma^{\rm st}_n + \\ & + 2\,\delta_{nn'}\delta_{mm'}\delta_{jj'}\,\Gamma^j_{m\leftarrow n}\,\sigma^{\rm st}_n. \end{aligned} \tag{5}$$

This method allows to calculate all spectral densities within the framework of "orthodox" theory, and at least for the single-electron transistor the numerical procedure is rather trivial [3, 4] because the matrix $\mathbf{\Gamma}$ is three-diagonal and the matrix inversion is straightforward [15].

Now let us develop the Langevin-type approach. Because of the discrete nature of the charge states, the random term cannot be simply added (in analogy with the standard Langevin method) to some evolution equation for the "cordinate" $n(t)$, but should be introduced into the master equation (similar to ref. [14]). The derivation of the formalism can be understood in the following way. Let us consider the ensemble of M ($M \gg 1$) independent similar circuits, and let us average all magnitudes over this ensemble. Then the average (over time) currents and voltages will not change (due to ergodicity), but the spectral densities of fluctuations (second-order magnitudes) will decrease M times. Hence, to calculate the spectral densities of the initial system, we can take the leading ($\sim M^{-1}$, $M \to \infty$) order of the spectral density of magnitudes averaged over the ensemble.

In contrast to the single system, the dynamics of the large ensemble is easily described using the Langevin approach. At any moment of time t the ensemble can be characterized by "coordinates" $M\sigma_n(t)$ which represent the numbers of participants being in different charge states n (notice that now $\sigma_n(t)$ is not a probability but the fluctuating coordinate). While in the stationary state the average number of transitions from state m to state n during small time Δt is given by $M\Gamma_{m\leftarrow n}\sigma^{\rm st}_n\Delta t$, the r.m.s. of this number is obviously $(M\Gamma_{m\leftarrow n}\sigma^{\rm st}_n\Delta t)^{1/2}$. Hence, the recipe is the following [14]: for each average flux $M\Gamma^j_{m\leftarrow n}\sigma^{\rm st}_n$ in the space of charge states, we should add in the master equation the random δ-correlated (white) flux $\xi^j_{m\leftarrow n}(t)$ with the corresponding "seed" spectral density given by the Schottky-like formula

$$\dot{\sigma}_m(t) = \sum_n \Gamma_{mn}\sigma_n(t) + \xi_m(t), \quad \xi_m(t) = \sum_{n,j}\xi^j_{m\leftarrow n}(t) - \xi^j_{n\leftarrow m}(t), \tag{6}$$

$$S_{\xi^j_{m\leftarrow n}\xi^{j'}_{m'\leftarrow n'}}(\omega) = 2M^{-1}\delta_{mm'}\delta_{nn'}\delta_{jj'}\Gamma^j_{m\leftarrow n}\sigma^{\rm st}_n. \tag{7}$$

For the fluxes in opposite directions ($m \leftarrow n$ and $n \leftarrow m$) we should apply $\xi(t)$ for each direction, so that the random flux does not vanish even if the net average flux is zero.

Because of the linearity of eqs. (6)-(7) the final spectral densities of the averaged (over M) magnitudes are obviously proportional to $1/M$. Hence, rescaling to the single system can be done formally assuming $M = 1$ in eqs. (6)-(7). So, instead of keeping M and rescaling at the final stage, we will use $M = 1$ in all the equations below.

Using the standard procedure we find the Fourier transform

$$\sigma_m(\omega) = \left[(-i\omega\mathbf{1} - \mathbf{\Gamma})^{-1}\right]_{mn}\xi_n(\omega). \tag{8}$$

Then for the occupation-occupation spectral density we obtain the expression

$$\begin{aligned} S_{\sigma_m\sigma_n} = {} & \sum_{m'n'}\left[(-i\omega\mathbf{1} - \mathbf{\Gamma})^{-1}\right]_{mm'}\left[(i\omega\mathbf{1} - \mathbf{\Gamma})^{-1}\right]_{nn'}S_{\xi_{m'}\xi_{n'}} = \\ = {} & 2\left[(-i\omega\mathbf{1} - \mathbf{\Gamma})^{-1}\right]_{mn}\sigma^{\rm st}_n + 2\left[(i\omega\mathbf{1} - \mathbf{\Gamma})^{-1}\right]_{nm}\sigma^{\rm st}_m, \end{aligned} \tag{9}$$

which coincides with the result of the Fokker-Plank approach (Fourier transform of eq. (3) without X and Y factors).

The technique is similar for the current-current fluctuations. The case of eqs. (4) and (5) corresponds to currents

$$X(t) = \tilde{e}^j_{\pm}\left[\Gamma^j_{m\leftarrow n}\sigma_n(t) + \xi^j_{m\leftarrow n}(t)\right], \quad Y(t) = \tilde{e}^{j'}_{\pm}\left[\Gamma^{j'}_{m'\leftarrow n'}\sigma_{n'}(t) + \xi^{j'}_{m'\leftarrow n'}(t)\right], \tag{10}$$

and the straightforward (though rather lengthy) calculations using eqs. (7) and (8) lead to eq. (5). The final expression for the current-occupation spectral density is

$$S_{X\sigma_k}(\omega)/\tilde{e}^j_{\pm} = 2\Gamma^j_{m\leftarrow n}\left[(-i\omega\mathbf{1} - \mathbf{\Gamma})^{-1}\right]_{nk}\sigma^{\rm st}_k + 2\left[(i\omega\mathbf{1} - \mathbf{\Gamma})^{-1}\right]_{km}\Gamma^j_{m\leftarrow n}\sigma^{\rm st}_n, \tag{11}$$

and it also coincides with the corresponding expression obtained in the Fokker-Plank technique.

Thus, we have proven that the Fokker-Plank method is equivalent to the Langevin method within the "orthodox" framework. However, in contrast to the former approach, the Langevin method easily allows phenomenological generalization for the fluctuations in the "quantum" frequency range.

Let us remind that in "orthodox" theory [1] the tunneling rate $\Gamma = I_0(W/e)/e[1 - \exp[-W/T]]$ is determined by the energy gain due to tunneling $W = eV_b - e^2/2C_{\rm eff}$, where $I_0(v)$ is the "seed" I-V curve of the junction (in the linear case $I_0(v) = v/R_j$), V_b is the voltage across the junction before the tunneling, and $C_{\rm eff}$ is the effective junction capacitance (which also accounts for the environment). The generalization of the Langevin method is the substitution of eq. (7) by the equation (see ref. [11])

$$\begin{aligned} S_{\xi^j_{m\leftarrow n}\xi^{j'}_{m'\leftarrow n'}}(\omega) &= \delta_{mm'}\delta_{nn'}\delta_{jj'}[\tilde{\Gamma}^+ + \tilde{\Gamma}^-]\,\sigma^{\rm st}_n, \\ \tilde{\Gamma}^{\pm} &= \frac{I_{0,j}(W^j_{m\leftarrow n}/e \pm \hbar\omega/e)}{e\left[1 - \exp\left[-(W^j_{m\leftarrow n} \pm \hbar\omega)/T\right]\right]}, \end{aligned} \tag{12}$$

which is derived for the individual tunneling event within the standard tunneling Hamiltonian technique averaging the quantum current-current correlator and then taking the Fourier transform. Equation (12) can be considered as a generalization of the fluctuation-dissipation theorem and equations of ref. [16] for the case of single-electron tunneling. (Actually, the only difference is that we separate fluctuations corresponding to two directions of tunneling. In the absence of the Coulomb blockade they could be summed together leading to the standard factor [16] $\coth((W \pm \hbar\omega)/2T)$ instead of the denominator of eq. (12).)

Equations (6) and (12) represent a phenomenological generalization in which the low-frequency behavior is treated by the master equation while high-frequency properties are taken into account for individual tunneling events. At high frequencies, $\omega \gg \Gamma$, the occupation-occupation and occupation-current spectral densities vanish, while for the current-current spectral density instead of eq. (5) we get

$$S_{XY}(\omega) = \tilde{e}^j_{\pm}\tilde{e}^{j'}_{\pm}S_{\xi^j_{m\leftarrow n}\xi^{j'}_{m'\leftarrow n'}}(\omega), \tag{13}$$

because the first terms of eq. (10) are too slow to give a contribution. This result coincides with the result of ref. [11]. The advantage of the Langevin approach is the possibility to obtain spectral densities in the "orthodox" and "quantum" frequency ranges using the same formalism while before they were necessarily treated on different footing.

For very high frequencies, $\hbar\omega \gg \max(W, T)$, we get $\tilde{\Gamma}^+ = \hbar\omega/e^2R_j$, $\tilde{\Gamma}^- = 0$ in eq. (12) independently of the charge state (the linear case is assumed for simplicity). Hence, each tunnel junction can be considered as a source of current fluctuations with spectral density

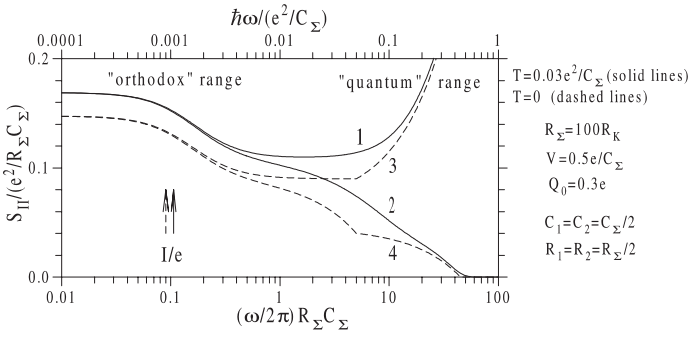

Fig. 1. – Curve 1 shows the spectral density of the current (in the external lead) through the symmetric single-electron transistor at $T = 0.03e^2/C_\Sigma$ as a function of frequency. Curve 2 shows the spectral density corresponding to the available power with zero-point cotribution subtracted ($T_\mathrm{r} = 0$). The dashed curves 3 and 4 demonstrate similar results for $T = 0$. The average current $I = 0.106e/R_\Sigma C_\Sigma$ for $T = 0.03e^2/C_\Sigma$ while $I = 0.09e/R_\Sigma C_\Sigma$ for $T = 0$.

$2\hbar\omega/R_j$ (the factor 2 accounts for two directions of tunneling). If we calculate, for example, the current fluctuations in the external lead of the single-electron transistor, then the effective charges are $|\tilde{e}^1_\pm| = e(C_2 + C_\mathrm{g})/C_\Sigma$ and $|\tilde{e}^2_\pm| = eC_1/C_\Sigma$ (we chose the lead close to the first junction), where $C_\Sigma = C_1 + C_2 + C_\mathrm{g}$ is the sum of junction and gate capacitances. The high-frequency asymptote in this case is $S_{II}(\omega) = 2\hbar\omega[(C_2 + C_\mathrm{g})^2/C_\Sigma^2 R_1 + C_1^2/C_\Sigma^2 R_2]$.

In the "quantum" frequency range the current spectral density does not correspond directly to the available power because of the contribution from zero-point oscillations. The spectral density of the current calculated above can be considered as the power (within the unit bandwidth) going from the system to a small external resistance r (divided by r and the coupling factor α). To obtain the available power, we should subtract the power flow in the opposite direction which is the product of the voltage spectral density of the resistance $2\hbar\omega r \coth(\hbar\omega/2T_\mathrm{r})$ (the "receiver" temperature T_r can differ from T), the factor α, and the active conductance of the fluctuation source, which in the case corresponding to eqs. (12)-(13) is given by

$$\mathrm{Re}G(\omega) = \left((\tilde{e}^j_\pm)^2/2\hbar\omega\right)[\tilde{\Gamma}^+ - \tilde{\Gamma}^-] \tag{14}$$

(this expression obviously corresponds to the lowest order of photon-assisted tunneling). Hence, the contribution to be subtracted from $S_{\xi^j_{m\leftarrow n}}$ is equal to $2\hbar\omega\coth(\hbar\omega/2T_\mathrm{r})\mathrm{Re}G(\omega)$. Traditionally this contribution is called zero-point for $T_\mathrm{r} = 0$ (while for finite temperature T another natural choice would be $T_\mathrm{r} = T$). Notice that to get the total conductance $\mathrm{Re}G_t(\omega)$, eq. (14) should be summed over all kinds of tunneling events.

The solid line 1 in fig. 1 shows the numerical result for the spectral density $S_{II}(\omega)$ of the current in the external lead of the single-electron transistor consisting of two tunnel junctions with similar capacitances $C_1 = C_2$ ($C_\mathrm{g} \ll C_\Sigma$) and resistances $R_1 = R_2$ for $V = 0.5e/C_\Sigma$, $Q_0 = 0.3e$, and $T = 0.03e^2/C_\Sigma$ [17]. The frequency dependence in the "orthodox" frequency range is important at $f = \omega/2\pi \sim I/e$ while the "quantum" frequency dependence occurs at $\omega \sim e^2/C_\Sigma\hbar$. Because the junction resistances are chosen sufficiently large, $R_\Sigma = 100R_K$, two frequency ranges are far apart from each other (the ratio of typical frequencies is on the order of R/R_K). The curve 2 shows the spectral density corresponding to the available power (zero-point contribution is subtracted, $T_\mathrm{r} = 0$). The dashed lines 3 and 4 show the results for $T = 0$. Notice the features at $\hbar\omega = 0.05e^2/C_\Sigma$ (this energy corresponds to the

"first" tunneling event) and the abrupt vanishing of the available power above the frequency $\omega = W_{\max}/\hbar$ ($\omega = 0.45e^2/C_\Sigma\hbar$ in the figure), where $W_{\max}$ is the maximum energy gain among all possible tunneling events.

In the case $R_j \gg R_K$ the "orthodox" and "quantum" frequency ranges are far apart from each other because $\Gamma \sim W/eR_j \ll W/\hbar$. The two ranges would overlap when $R_j \sim R_K$. In this case the developed formalism does not work well because of the strong cotunneling which cannot be described by a master equation—see, *e.g.*, ref. [18].

In conclusion, we developed a Langevin-type method for the calculation of the fluctuations in single-electron tunneling. In the "orthodox" frequency range the results coincide with that of the Fokker-Plank approach. The advantage of the Langevin method is the natural generalization for the "quantum" frequency range.

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